On the self-adjointness of the Lorentz generator for

\( (: \phi^4 :)_1 + 1 \)

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An alternative proof to that provided by Jaffe and Cannon of the self-adjointness of the local Lorentz generator for the \( (: \phi^4 :)_1 \) quantum field theory is given. The proof avoids the use of second-order estimates and a singular perturbation theory.

In this brief note, we establish the self-adjointness of the local Lorentz generator for the two-dimensional \( \phi^4 \) interaction by the method of Ref. 1. This result has been previously obtained by Cannon and Jaffe\(^3\) using first- and second-order estimates, and a singular perturbation theory. Here we avoid the use of second-order estimate and the Glimm–Jaffe singular perturbation theory.\(^2\) It is hoped that a new proof may lead to some new results and insights.

The \( (: \phi^4 :)_1 \) quantum field theory has been brought to a very satisfactory stage mainly by the work of Glimm and Jaffe.\(^4\) On the Fock space, they constructed a densely defined bilinear form \( \varphi(x, t) \), continuous in \( x \) and \( t \), which gives rise to a unique self-adjoint operator

\[
\varphi(f) = \int dx dt \varphi(x, t) f(x, t)
\]

for a real function \( f \in C^\infty_0(R^4) \). The \( C^* \)-algebra of local observables is defined as the norm closure

\[
= \{ (\cdot) (B) \}^*.
\]

Here the union is taken over bounded regions \( B \) of space–time and \( (\cdot) \) is the weakly closed \( \mathcal{A} \) product of free field operators. The \( C^* \)-algebra of local observables is defined as the norm closure

\[
\mathcal{A} = \{ (\cdot) (B) \}^*.
\]

Poincaré covariance means that there exists a representation

\[
\sigma_{\{a, \Lambda\}}(\cdot)(B) = \{ (a, \Lambda)(B) \}
\]

for all bounded open sets \( B \) and all \( \{ a, \Lambda \} \in P \). The covariance of the local algebras ensures the covariance of the field operators, namely

\[
\sigma_{\{a, \Lambda\}}(\varphi(f)) = \varphi(f_{\{a, \Lambda\}})
\]

with

\[
f_{\{a, \Lambda\}}(x, t) = f((a, \Lambda)x, t).
\]

where \( H(g) \) is the Hamiltonian with a space cutoff \( g(x) \in C^\infty_0(R) \), \( g(x) = 1 \) on a sufficiently large set depending on \( B \). The space translation is implemented by \( \exp(-ixP) \), where \( P \) is the free field momentum operator.

The pure Lorentz transformation is locally implemented by a unitary operator \( U(\Lambda_2; B) \), i.e.,

\[
\sigma_{\Lambda_2}(\cdot)(B) = U(\Lambda_2; B)(\cdot)U^{-1}(\Lambda_2; B).
\]

The formal infinitesimal generator of Lorentz transformations in a region \( B \) is

\[
M(g) = M_0 + M_1(g)
\]

where the space cutoff function \( g \) is 1 on a sufficiently large interval. Here, \( H(x) = H_0(x) + H_1(x) \) is the energy density. Using space–time covariance, Cannon and Jaffe showed that it suffices to consider region \( B \) of space–time in the domain \( x > 0 \). Also, it is technically convenient to use different spatial cutoffs in the free and the interaction part of \( M \). Thus, for a region \( B \) in \( x > 0 \), we take

\[
M = M_0 + M_1(g)
\]

where \( M_0 = \sigma H_0, \) and \( M_1 = \sigma H_1(xg_2) + H_1(xg_2), \)

\[
[\sigma, \zeta(x)g(x)g(x_1)] = \alpha x g_1(x) - \alpha x g_2(x)
\]

for \( x \) in a sufficiently large interval of the positive \( x \) axis. Here we have defined \( g_1(x) = xg_1(x), \) and \( g_2(x) = xg_2(x). \)

The first step toward proving that \( M = M_0 + M_1(g) \) is the infinitesimal generator for local Lorentz rotations, is to prove the self-adjointness of \( M \).

We write

\[
M = \sigma H_0 + H_0\zeta(g) + \alpha H_1 - H_1,\zeta(g) + \sigma H_0 - H_0\zeta(g)
\]

\[
+ [\sigma H_1, \zeta(g)]
\]

where as usual \( \zeta \) is an upper momentum cutoff. We first estimate each term in (15). By undoing the Wick ordering we obtain

\[
H_0\zeta(g) \geq -c_1 \zeta^2;
\]

\[
H_1,\zeta(g) \geq -c_2 \zeta^2
\]

where \( c_1, c_2 \) are positive constants independent of \( \zeta \). By a standard \( N_\zeta \) estimate

\[
\| (N + 1)^{\zeta} (H_0(g) - H_1,\zeta(g))(N + 1)^{\zeta} \| \leq c_3 \zeta^{1/2},
\]

\[
XC_3 > 0.
\]

To estimate the difference \( H_0(g) - H_0\zeta(g) \), we write
with
\[ H_0^{(1)}(g_0) = \frac{1}{2(2\pi)} \int \, dk_1, dk_2, n_0 |k_1, k_2, n_0 - k_1, k_2, n_0 - k_1| \frac{\mu(k_1)\mu(k_2) + K_1k_1 + K_2k_2 + n_0^2}{\mu(k_1)\mu(k_2)} \frac{d_1 d_2}{d_{12}} \times \alpha(k) \alpha(k). \]
(20)

Similarly, for some constant \( \alpha^{*} \), we quickly obtain
\[ \alpha^{*}(k) a^{*}(-k) \alpha(k). \]
(21)

The limit
\[ \lim_{(N+1)^{-1/2}} H^{(1)}(g_0) = H^{(1)}(g_0) \]
(22)

Finally, we estimate the free term \( \alpha H_0 \) by
\[ \alpha H_0 \Rightarrow \alpha n_N. \]
(23)

Let \( P_n \) be the projection onto states with numbers of particles in the range
\[ n^6 < N < (n + 2)^6, \quad \beta > 4. \]
(24)

We note
\[ \sum_{n=\text{odd}} P_n = \sum_{n=\text{even}} P_n = I. \]
(24')

Picking \( \kappa = \text{exp} \left( [1/c_1] n^3/2 \right) \), and using (16), (17), (18), (22) and (23), we quickly obtain
\[ P_n \alpha H_0 P_n \to \alpha n_N n N P_n \to \alpha n_N, \]
(25)

\[ P_n H_0^{(1)}(g_0) P_n \to -C \exp(2/c_1), \quad n^6/2, \]
(26)

\[ P_n H_1^{(1)}(g_0) P_n \to -n^6, \]
(27)

\[ P_n H_0^{(1)}(g_0) - H^{(1)}(g_0) P_n \to 0, \]
(28)

\[ || P_n H_1^{(1)}(g_0) - H_1^{(1)}(g) P_n || \leq d_1 \exp(-d_1 n^6/2), \quad d_1, d_1 > 0, \]
(29)

\[ || P_n H_0^{(1)}(g_0) - H_0^{(1)}(g_0) || \leq d_2 \exp(-d_2 n^6/2), \quad d_2 > 0. \]
(30)

Using (25) through (30) and choosing an appropriate \( \alpha \), we get
\[ M_n = P_n M P_n \geq d \beta n^6, \]
(31)

where \( d \) is a positive constant. For \( d \) large enough we get
\[ b + M_n \geq d \beta n^6. \]
(32)

Let \( M' \) be obtained from \( M \) by replacing \( \alpha H_0 \) by \( \alpha n_N \), a multiple of the particle number operator. Then
\[ b + M_n \geq b + M_n = b + P_n \alpha n_N + H_0^{(1)}(g_0) + H_1^{(1)}(g) P_n. \]
(33)

By a standard \( N \), estimate
\[ || b + M_n || \leq d N^{28} \]
(34)

for some constant \( d' \).

Following Ref. 7, we define \( P_n \) and \( P_d \) as the projection operators onto states with number of particles in the ranges
\[ \sum_{n=\text{even, even}} (\beta^6 - 4 < N < \beta^6 + 4), \]
(35)

and
\[ \sum_{n=\text{odd, odd}} (\beta^6 - 4 < N < \beta^6 + 4), \]
(36)

respectively. We define \( M_n \) and \( M_d \) by
\[ M_n = \sum_{n=\text{even}} M_n, \]
(37)

\[ M_d = \sum_{n=\text{odd}} M_n. \]
(38)

We write \( M \) in two different forms
\[ M = M_n + L_n = M_d + L_d, \]
(39a)

\[ L_n = M_n - \sum_{n=\text{odd}} P_n M P_n. \]
(39b)

\[ L_d = M_d - \sum_{n=\text{odd}} P_n M P_n, \]
(39c)

where the ranges (35) and (36) have been chosen so that
\[ P_n L_n P_n = L_n P_n = P_L P_n = L_n, \]
(40)

\[ P_n L_n P_n = L_n P_n = P_L P_n = L_n. \]
(41)

Theorem 1: Let \( g_0, g \) satisfy (14), with \( \varepsilon_0, \varepsilon > 0 \), \( \varepsilon_0, \varepsilon \in C_0^\infty (R^7) \). Then there is a finite constant \( C \) such that (42) converges in the uniform operator topology for \( \beta > 0 \). The limit \( R(-b) \) is the resolvent of a self-adjoint operator \( M \) such that \( M > \delta \).

The proof of this theorem follows from the following three lemmas:

Lemma 1: For \( n \) large enough, there exist positive constants \( c_1, c_2 \), independent of \( n \) such that
\[ || P_n P_n R(-b; M_n) P_n || \leq c_1 \exp(-c_2 \beta^6/2). \]
(43)

Proof: Since \( \alpha H_0 \to \alpha n_N \), it is enough to prove (43) with \( M_n \) replacing \( M_n \). Estimates (32) and (34) permit us to apply Theorem 2.4 of Ref. 1, with \( |a| = P_n P_n |a|, |b| = P_n P_n |b|, \mu_0 = d \beta n^6, \beta = b + M_n - d \beta n^6 \) (this corresponds to \( M \) in the notation of Ref. 1), and \( N \leq [(n + 1)^6 - 4 - (\beta^6 + 4)]/4 \). Thus we obtain (43).

Lemma 2: Let \( \epsilon > 0 \) be small enough. Then there exist constant \( c(\epsilon) \) such that

\[ || P_n P_n R(-b; M_n) P_n || \leq c(\epsilon) \exp(-c(\epsilon) \beta^6/2). \]
(43)
\[ \begin{align*}
\| R(-b;M_2) \|, \| R(-b;M_4) \| & \leq 1/b, \quad (44) \\
\| L_4 P_4 R(-b;M_2) P_4 \|, \| L_4 P_4 R(-b;M_4) P_4 \| & \leq \frac{1}{2}, \quad (45) \\
\| R(-b;M_2) L_4 R(-b;M_4) \| & \leq c\psi b^{1-\alpha}, \quad (46) \\
\| L_4 R(-b;M_2) L_4 R(-b;M_4) \| & \leq c\psi b^{1-\alpha}. \quad (47)
\end{align*} \]

**Proof:** Inequality (44) is an easy consequence of estimate (31). Let \( |a \rangle \) and \( |b \rangle \) be two normalized states in the Fock space. To prove (45), we consider
\[
\langle a | L_4 P_4 R(-b;M_2) P_4 | b \rangle = \sum_{n=\text{odd}} \langle a | L_4 P_4 R(-b;M_2) P_4 | b \rangle \\
= \sum_{n=\text{odd}} \langle a | L_4 P_4 R(-b;M_4) P_4 | b \rangle \\
= \sum_{n=\text{odd}} \langle a | (1-P_4)M_2 P_4 \rangle P_n \times R(-b;M_4) P_4 | b \rangle \\
\leq \sum_{n=\text{odd}} c_{\psi} (n+2)^{2\alpha} \exp(-c_{\psi} b^{\alpha/2}) < \frac{1}{2}.
\]

In the last step above, we have used estimate (43) and a standard \( N_\psi \) estimate. Similar arguments establish estimate (46) and (47).

**Lemma 3:** For \( b \) large enough, the series (42) converges uniformly to an operator \( R(-b) \) which is a pseudoresolvent and satisfies
\[
\lim_{b \to \infty} (-b) R(-b) = I \quad (48)
\]
in the norm operator topology.

**Proof:** Estimates (44) through (47) imply that the \( n \)th term in (42) is bounded by \( c\psi b^{1-\alpha} \). Therefore, the series converges for \( b > c\psi b^{1/4} \) to an operator \( R(-b) \).

Clearly,
\[
\lim_{b \to \infty} \prod_{i=1}^\infty \| R(-b;M_i) L_i R(-b;M_i) \| = 0. \quad (49)
\]
This implies (48). It is not hard to prove that \( R(-b) \) is a pseudoresolvent.

**Proof of Theorem 1:** We follow the proof of Theorem 2.2 in Ref. 1. Equality (48) implies that \( R(-b) \) is a non-negative operator. Then \(-b - (R(-b))^{-1}\) defines \( M \) whose domain is independent of \( b \) because of the pseudoresolvent property of \( R(-b) \). The self-adjointness follows from the next lemma.

**Lemma 4:** If \( T : \mathcal{X} \to \mathcal{X} \) is an operator with dense domain on the Hilbert space \( \mathcal{X} \), and if \( T^{-1} \) exists and has a dense domain, then \((T^*)^{-1} = (T^{-1})^*\).

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3. While this paper was in preparation, a set of notes by J. Glimm and A. Jaffe, “Boson Quantum Fields” came out, in which they treat the self-adjointness of the local Lorentz generator without using second-order estimates.