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Compressible Supersonic Flow in Jets under the Kármán-Tsien Pressure-Volume Relation

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The two-dimensional supersonic irrotational flow of a gas in a jet is studied by use of the Kármán-Tsien pressure-volume law. There are two limitations to such a study: (1) since the fluid flow is not continued from the subsonic range, arbitrary boundary conditions must be prescribed; (2) use of the Kármán-Tsien pressure-volume relation implies a restriction on the permissible range of pressure, density, and velocity. On the other hand, use of the Kármán-Tsien law furnishes several advantages: (1) the velocity potential and stream function satisfy the wave equation in the hodograph plane and hence these functions can be easily determined; (2) the mappings between the physical and hodograph planes may be completely characterized and studied in detail. This gain in information should be valuable in the qualitative understanding of phenomena as well as in obtaining first approximations to quantitative solutions. In the case of jets, with free stream lines as boundaries, it is shown that two functions *possessing certain desired properties* completely determine the Kármán-Tsien flow. Further, the phenomenon of the periodic recurrence of the free stream jet boundary is explained *by a folding property of the map of the flow in the hodograph plane.*

1. INTRODUCTION

OUR problem will be to develop a simplified theory for the two-dimensional, irrotational, supersonic flow of a compressible gas in a jet. Owing to the assumption of the validity of the Kármán-Tsien pressure-volume relation, the relation between the maps of the flow in the physical, hodograph, and stream function-velocity potential planes can be explicitly exhibited.

In the physical (x, y) plane, the velocity potential, ϕ, and the stream function, ψ, satisfy a nonlinear second-order partial differential equation. However, in the hodograph (ω, θ) plane (where ω is the angle between the characteristics in the physical plane and θ is the angle between the velocity vector and the x-axis of the physical plane), these functions satisfy a linear second-order partial differential equation. For the adiabatic pressure-volume relation, this equation has variable coefficients. In spite of the fact that this equation can not be explicitly integrated, Frankl[1] has discussed various existence theorems. If the adiabatic pressure-volume curve is approximated by use of the tangent line to this curve at an appropriately chosen point (this is the Kármán-Tsien pressure-volume relation), the above mentioned linear differential equation becomes the wave equation and explicit solutions may be found. When boundary value data are given along the line $\omega=\omega_1$ (this line can be taken arbitrarily close to the sonic line $\omega=\omega_*$), the solution of the wave equation depends upon the region of integration in the (ω, θ) plane. By examining the maps of the flow in the (ω, θ), (ϕ, ψ), and (x, y) planes, the proper integration regions in the (ω, θ) plane are determined. Thus, the velocity potential and stream function can be determined in the (ω, θ) plane and finally these functions can be mapped into the physical (x, y) plane.

Our results are the following: (1) an analytic method, which depends upon a knowledge of two arbitrary functions, for treating fairly uniform supersonic flows in jets with free stream lines is developed; (2) the phenomenon of the periodic recurrence of the free stream jet boundary is explained by a folding property of the map of the flow in the hodograph plane.

Before concluding this introductory section, we shall list some of the formulas[2] used in our previous general study of the Kármán-Tsien pressure-volume relation. If we replace the usual adiabatic gas law by a tangent line drawn at the point $(1/\rho_1, p_1)$ in the pressure-density diagram, one finds that

$$p_1-p=a_1^2\rho_1^2[(1/\rho)-(1/\rho_1)], \tag{1.1}$$

where ρ is density, p is the pressure, a is the local sound speed. From (1.1) and the Bernoulli relation, it follows that

$$a^2\rho^2=k^2, \quad w^2-a^2=l^2, \tag{1.2}$$

where w is the magnitude of the velocity and k, l are constants. Further, it can be shown that

$$(w/w_1)^2=1+M_1^{-2}[(2/\gamma)+(1/\gamma^2)], \tag{1.3}$$

where M_1 is the Mach number of the flow for $w=w_1$, and $\gamma=1.4$. Finally, it may be shown that the angle, ω, between the two families of characteristics in the physical plane is related to the magnitude of the velocity by

$$w=l\sec\tfrac{1}{2}\omega. \tag{1.4}$$

2. THE PHYSICAL (x, y) PLANE

We shall consider a gas jet which emerges from straight walls and is bounded by two free stream lines. As in the subsonic case,[3] we shall assume that along each free stream line, the magnitude of the velocity is constant. In Fig. 1, AB and $A'B'$ represent the two straight walls from which the jet emerges. Along each of these lines, the stream function ψ is constant. Further, AC and $A'C'$ represent the two free stream lines bounding

[1] F. Frankl, Bull. Acad. Sci. U.R.S.S., Ser. Math. (Izvestia Akad. Nauk S.S.S.R.), 9, 121–143 (1945).

[2] N. Coburn, Quart. Appl. Math. 3, No. 2 (July, 1945).

[3] S. A. Chaplygin, "On gas jets," Sci. Ann. Imp. Univ. Moscow, Physico-Math. Division, Pub. No. 21 (Moscow, 1904), translated from the Russian by M. Slud (Brown University Notes, Providence, Rhode Island, 1944).

the jet. Along each of these free stream lines, the magnitude of the velocity, w, and the stream function, ψ, are constant. From symmetry considerations, we deduce that the velocity vector has the same magnitude on both free stream lines. Further, from symmetry, we see that the y-axis (line ED) is a stream line along which ψ is constant.

The following values will be assigned to the stream constants:

$$B'A'C',\ \psi=M/2;\ ED,\ \psi=0;\ BAC,\ \psi=-M/2. \quad (2.1)$$

If ρ is the density and (u, v) are the x and y components, respectively, of the velocity vector, then

$$M=-\int_{A'}^{A} d\psi=\int_{A'}^{A} \rho v dx-\rho u dy. \quad (2.2)$$

Evidently, M is the mass of gas entering the jet per unit time.

We shall specify the inclination of the walls by use of the negative angle $\bar{\theta}$ (see Fig. 1). A and A' are situated at the jet entrance, where the velocity vector has the magnitude, w_2. The points B and B' are two symmetrically situated points on the walls where the velocity vector has the magnitude, w_1. For given values of w_2 and θ, the quantity w_1 will be fixed by a relation to be determined (see Eq. 3.9). Further, the curves $B'E$ and EB will represent the locus of points along which $w=w_1$. Evidently, the curves $B'E$ and EB are symmetric. The points C and C', which are symmetrically situated on the two free stream lines, and the point D will be fixed by specifying their hodograph coordinates.

Evidently, we may write Eq. (1.2) in the form

$$l^2=w_1^2-a_1^2=w_2^2-a_2^2, \quad (2.3)$$

where a_1, a_2 are the local sound speeds corresponding to w_1, w_2, respectively. By use of the angle ω, one can simultaneously treat the cases: (1) $w_1<w_2$; (2) $w_1>w_2$. From Eq. (1.4), it follows that these two cases can be characterized by: (1) $0\leqslant\omega_1<\omega_2<\pi$; (2) $-\pi<\omega_1<\omega_2\leqslant 0$. In mapping from the (ω, θ) plane to the (ϕ, ψ) plane, we shall assume only that $\omega_1<\omega_2$. Hence, this phase of our work will be applicable to either of the above cases. However, it will be shown that in order for the (ϕ, ψ) plane to map into the proper region of the (x, y) plane, the angle ω must be negative ($w_2<w_1$). Finally, we note that by use of Eq. (1.3) an upper bound for the ratio w/w_2 can be given in terms of the Mach number at A or A'.

3. THE HODOGRAPH (ω, θ) PLANE

We introduce the variable θ which denotes the angle between the velocity vector and the x-axis at any point of the physical (x, y) plane. The plane determined by ω and θ will be called the hodograph plane. From Fig. 1, the hodograph map can be determined. We see that

$$\begin{aligned} B'A'&:\ \theta=\bar{\theta};\\ ED&:\ \theta=-\pi/2;\\ AB&:\ \theta=-\pi-\bar{\theta}. \end{aligned} \quad (3.1)$$

Further, from the discussion in the preceding section, we find that

$$A, A':\ \omega=\omega_2;\quad B'EB:\ \omega=\omega_1. \quad (3.2)$$

In order properly to specify the coordinates of D, C, C', it will be necessary to discuss some of the properties of the stream function and the velocity potential in the (ω, θ) plane. We shall prove:

Theorem 1. If the mapping functions connecting the (ϕ, ψ) and the (ω, θ) planes are continuous, with second partial derivatives and nonvanishing jacobian, then ψ and ϕ satisfy the wave equation.

From our previous work,[2] we find

$$\omega=F\left(\frac{\phi}{2l}+\frac{\psi}{2k}\right)+G\left(\frac{\phi}{2l}-\frac{\psi}{2k}\right), \quad (3.3)$$

$$2\theta=F\left(\frac{\phi}{2l}+\frac{\psi}{2k}\right)-G\left(\frac{\phi}{2l}-\frac{\psi}{2k}\right), \quad (3.4)$$

where F and G are arbitrary functions of their respective arguments which possess second partial derivatives (at least sectionally), l is the constant defined by Eq. (2.3) and k is the constant, $\rho a=\rho_1 a_1=\rho_2 a_2$. If the jacobian of Eqs. (3.3), (3.4) does not vanish, we may solve these equations for ϕ, ψ. We find

$$\frac{\phi}{l}=f\left(\frac{\omega}{2}+\theta\right)+g\left(\frac{\omega}{2}-\theta\right), \quad (3.5)$$

$$\frac{\psi}{k}=f\left(\frac{\omega}{2}+\theta\right)-g\left(\frac{\omega}{2}-\theta\right), \quad (3.6)$$

where f and g are the inverse functions to F and G, respectively. From Eq. (3.6), we see that the stream function (as well as the velocity potential) is a solution of the wave equation

$$(\partial^2\psi/\partial\theta^2)-4(\partial^2\psi/\partial\omega^2)=0. \quad (3.7)$$

We shall now determine the hodograph coordinates of D, C, C'. The slopes of the characteristic lines of

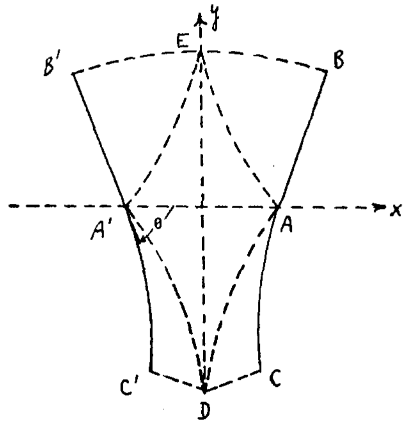

FIG. 1. Flow through a jet in the physical plane.

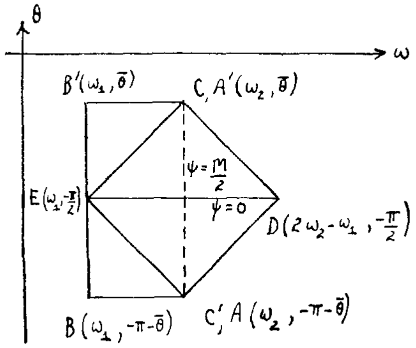

FIG. 2. The hodograph map of the flow.

Eq. (3.7) are

$$d\theta/d\omega = \pm\tfrac{1}{2}. \tag{3.8}$$

First, we require that the point A' lie on the characteristic through E. Hence, the variables $\bar{\theta}$, ω_1, ω_2 satisfy the equation

$$\bar{\theta} + \pi/2 = (\omega_2 - \omega_1)/2. \tag{3.9}$$

The point D will be fixed as the intersection of the characteristic of negative slope passing through A' and the line $\theta = -\pi/2$. The resulting coordinates of D are indicated on Fig. 2. If $2\omega_2 - \omega_1 > \pi$ for positive ω (or $2\omega_2 - \omega_1 > 0$ for negative ω), then the point D must be replaced by two points D and D'. These latter points would be the intersections of characteristics through A and A', respectively, with the line $\omega = \pi$ for positive ω (or $\omega = 0$ for negative ω). For simplicity, we assume $2\omega_2 - \omega_1 < \pi$ for positive ω (or $2\omega_2 - \omega_1 < 0$ for negative ω). C and C' will be fixed by assuming that these points coincide with A' and A, respectively.

In the following sections, we shall consider only three regions of the (ω, θ) plane: $EB'A'$ (henceforth denoted as region I); $EA'D$ (henceforth denoted as region II); $A'C'D$ (henceforth denoted as region III). The regions EBA, EAD, ACD may be treated by symmetry considerations. Further, the extension of our method to other regions will be apparent.

Next, we consider the boundary conditions on ψ for the integration of Eq. (3.7). In addition to the boundary conditions (2.1), data along $B'E$ must be given. We shall assume boundary conditions along $B'E$ which are equivalent to Cauchy data

$$B'E\colon\ \psi = \tfrac{1}{2}M + kh(\theta), \quad \phi = lp(\theta), \quad -\pi/2 \le \theta \le \bar{\theta}, \tag{3.10}$$

where k and l are the constants previously introduced and $h(\theta)$ and $p(\theta)$ possess second derivatives. In order for the stream function to be continuous along $B'E$, it is necessary that

$$h(\bar{\theta}) = 0, \quad h(-\pi/2) = -M/2k. \tag{3.11}$$

We shall now show that:

Theorem 2. A solution for ψ exists in region I with the following properties: (1) the solution depends upon a single function f of the variables $\tfrac{1}{2}(\omega - \omega_2) + \theta - \bar{\theta}$ and $\tfrac{1}{2}(\omega - \omega_2) - \theta + \bar{\theta}$; (2) the range of θ in $f(\theta)$ is $\omega_1 - \omega_2 \le \theta \le 0$.

Replacing ψ by $\psi - M/2$, ω by $\omega - \omega_2$, θ by $\theta - \bar{\theta}$ in Eqs. (3.5), (3.6), we obtain

$$\psi = \frac{M}{2} + k\left[f\left(\frac{\omega - \omega_2}{2} + \theta - \bar{\theta}\right) - g\left(\frac{\omega - \omega_2}{2} - \theta + \bar{\theta}\right)\right], \tag{3.12}$$

$$\phi = l\left[f\left(\frac{\omega - \omega_2}{2} + \theta - \bar{\theta}\right) + g\left(\frac{\omega - \omega_2}{2} - \theta + \bar{\theta}\right)\right]. \tag{3.13}$$

In order to determine the range of the arguments of f and g, we write

$$\tfrac{1}{2}(\omega - \omega_2) + \theta - \bar{\theta} = \theta_0, \quad \tfrac{1}{2}(\omega - \omega_2) - \theta + \bar{\theta} = \theta_1. \tag{3.14}$$

It is easily shown that the characteristic lines (3.14) will intersect region I if and only if

$$\omega_1 - \omega_2 \le \theta_0 \le \theta_0, \quad \tfrac{1}{2}(\omega_1 - \omega_2) \le \theta_1 \le 0. \tag{3.15}$$

Thus, the arguments of f and g have the common range $\tfrac{1}{2}(\omega_1 - \omega_2)$ to 0. In this common range, it follows from Eq. (3.12) and the boundary condition, $\psi = \tfrac{1}{2}M$ when $\theta = \bar{\theta}$, that $f(\theta) = g(\theta)$ for $\tfrac{1}{2}(\omega_1 - \omega_2) \le \theta \le 0$. Thus, we may consider f and g as single function, f, defined over the interval $\omega_1 - \omega_2$ to 0.

The boundary conditions (3.10) furnish relations for determining $f(\theta)$. Thus, from Eqs. (3.10), (3.12), (3.13), we find

$$2f\left(\frac{\omega_1 - \omega_2}{2} + \theta\right) = h(\theta + \bar{\theta}) + p(\theta + \bar{\theta}), \tag{3.16}$$

$$2f\left(\frac{\omega_1 - \omega_2}{2} - \theta\right) = -h(\theta + \bar{\theta}) + p(\theta + \bar{\theta}). \tag{3.17}$$

Equation (3.16) determines $f(\theta)$ for

$$\omega_1 - \omega_2 \le \theta \le \tfrac{1}{2}(\omega_1 - \omega_2);$$

Eq. (3.17) determines $f(\theta)$ for $\tfrac{1}{2}(\omega_1 - \omega_2) \le \theta \le 0$. Further, the solutions for ψ and ϕ in region I may be written in terms of this function, f, as

$$\text{I:}\quad \psi = \frac{M}{2} + k\left[f\left(\frac{\omega - \omega_2}{2} + \theta - \bar{\theta}\right) - f\left(\frac{\omega - \omega_2}{2} - \theta + \bar{\theta}\right)\right], \tag{3.18}$$

$$\text{I:}\quad \phi = l\left[f\left(\frac{\omega - \omega_2}{2} + \theta - \bar{\theta}\right) + f\left(\frac{\omega - \omega_2}{2} - \theta + \bar{\theta}\right)\right]. \tag{3.19}$$

It should be noted that the function $f(\theta)$ as determined by Eq. (3.16) for the range $\omega_1 - \omega_2 \le \theta \le \tfrac{1}{2}(\omega_1 - \omega_2)$

may differ completely in analytic form from the function $f(\theta)$ as determined by Eq. (3.17) for the range $\frac{1}{2}(\omega_1-\omega_2)\leq\theta\leq 0$. However, by use of Eqs. (3.16), (3.17), it can be easily shown that:

Theorem 3. The necessary and sufficient conditions that $f(\theta)$ be represented by the same function in the above two intervals are that $h(\theta)$ have odd symmetry about $\theta=\bar{\theta}$ and $p(\theta)$ have even symmetry about $\theta=\bar{\theta}$.

We shall continue our solution for ψ into regions II ($EA'D$) and III ($A'C'D$) by use of the well-known methods[4] for integrating the wave equation. The solution for ϕ in these regions can then be determined by use of Eq. (3.5) and continuity considerations. We find

$$\text{II:}\quad \psi=k\left[f\left(\frac{\omega-\omega_2}{2}+\theta-\bar{\theta}\right)-f\left(\frac{\omega-\omega_2-2\pi}{2}-\theta-\bar{\theta}\right)\right], \tag{3.20}$$

$$\text{II:}\quad \phi=\frac{Ml}{2k}+l\left[f\left(\frac{\omega-\omega_2}{2}+\theta-\bar{\theta}\right)+f\left(\frac{\omega-\omega_2-2\pi}{2}-\theta-\bar{\theta}\right)\right], \tag{3.21}$$

$$\text{III:}\quad \psi=\frac{M}{2}+k\left[f\left(\frac{-\omega+\omega_2-2\pi}{2}-\theta+\bar{\theta}\right)-f\left(\frac{\omega-\omega_2-2\pi}{2}-\theta-\bar{\theta}\right)\right], \tag{3.22}$$

$$\text{III:}\quad \phi=\frac{Ml}{k}+l\left[f\left(\frac{-\omega+\omega_2-2\pi}{2}-\theta+\bar{\theta}\right)+f\left(\frac{\omega-\omega_2-2\pi}{2}-\theta-\bar{\theta}\right)\right]. \tag{3.23}$$

It is easily shown that the functions in the right-hand side of the above equations are defined by Eqs. (3.16), (3.17) in regions II and III.

With the aid of Eqs. (3.20) through (3.23), we shall prove three results relating the (ω, θ) and (ϕ, ψ) plane maps of the flow.

Theorem 4. The necessary and sufficient condition for the continuity across EA' of the mapping functions ψ and ϕ is that $h(-\pi/2)=-M/2k$; the necessary and sufficient condition that the first partial derivatives of ψ and ϕ be continuous across EA' is that $p'(-\pi/2)=0$.

To show the necessity of the condition $h(-\pi/2)=-M/2k$, we compute the values of Eqs. (3.18), (3.19), (3.20), (3.21) along EA'. We find

$$\psi(\text{II})-\psi(\text{I})=-\frac{M}{2}+k\left[f\left(\frac{\omega_1-\omega_2+\pi}{2}+\bar{\theta}\right)-f\left(\frac{\omega_1-\omega_2-\pi}{2}-\bar{\theta}\right)\right], \tag{3.24}$$

$$\phi(\text{II})-\phi(\text{I})=\frac{Ml}{2k}-l\left[f\left(\frac{\omega_1-\omega_2+\pi}{2}+\bar{\theta}\right)-f\left(\frac{\omega_1-\omega_2-\pi}{2}-\bar{\theta}\right)\right]. \tag{3.25}$$

By use of Eqs. (3.9), (3.16), (3.17), we obtain

$$f\left(\frac{\omega_1-\omega_2+\pi}{2}+\bar{\theta}\right)-f\left(\frac{\omega_1-\omega_2-\pi}{2}-\bar{\theta}\right)=f(0)-f(\omega_1-\omega_2)=-h\left(\frac{-\pi}{2}\right). \tag{3.26}$$

Comparing the last three equations, we see the necessity of our condition. The sufficiency of this condition follows from the fact that a continuous solution has been constructed with the aid of this condition.

In order to verify the second part of our result, we compute the derivatives of ψ and ϕ in regions I and II. Denoting the jump of the derivative across EA' by brackets, and using Eqs. (3.9), (3.16), (3.17), we find

$$f'(\omega_1-\omega_2)-f'(0)=p'\left(-\frac{\pi}{2}\right)=\frac{1}{k}\left[\frac{\partial\psi}{\partial\theta}\right]=-\frac{2}{k}\left[\frac{\partial\psi}{\partial\omega}\right]=-\frac{1}{l}\left[\frac{\partial\phi}{\partial\theta}\right]=\frac{2}{l}\left[\frac{\partial\phi}{\partial\omega}\right]. \tag{3.27}$$

The primes on f and p denote differentiation of these functions with respect to their respective arguments. Equation (3.27) verifies our second result.

In the same manner, we can prove:

Theorem 5. The necessary and sufficient condition for the continuity of ψ and ϕ across $A'D$ (the boundary of II and III) is that $h(-\pi/2)=-M/2k$; the necessary and sufficient condition for the continuity across $A'D$ of the first partial derivatives of ψ and ϕ is that $h'(-\pi/2)=0$.

Finally, we examine the Jacobian of the transformation from the (ω, θ) to the (ϕ, ψ) plane. We shall prove:

Theorem 6. The necessary and sufficient conditions that the jacobian, $J(\phi, \psi/\omega, \theta)$, shall not vanish in the closed regions I, II, III are $h'(\theta)+p'(\theta)\neq 0$, $h'(\theta)-p'(\theta)\neq 0$.

[4] J. D. Tamarkin and W. Feller, *Partial Differential Equations* (Brown University Notes, Providence, Rhode Island, 1941), pp. 24–28.

By use of the expressions for ψ and ϕ in region I, we find

$$J\left(\frac{\phi,\psi}{\omega,\theta}\right)=2klf'\left(\frac{\omega-\omega_2}{2}+\theta-\bar{\theta}\right)f'\left(\frac{\omega-\omega_2}{2}-\theta+\bar{\theta}\right). \quad (3.28)$$

Similar expressions are valid for regions II and III. Thus, the nonvanishing of the jacobian is equivalent to the nonvanishing of $f'(\theta)$. From Eqs. (3.16), (3.17), it follows that this latter condition is satisfied if and only if $h'(\theta)\pm p'(\theta)\neq 0$.

4. SOME MAPPING RELATIONS

The functions relating the (ϕ, ψ) and the (x, y) plane can be analyzed by use of Eq. (1.4). If u and v are the

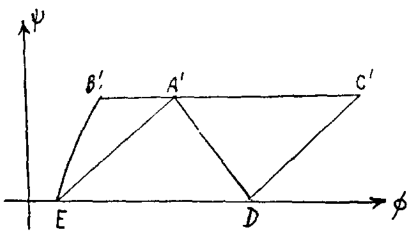

FIG. 3. The $\phi-\psi$-plane map of the flow.

x and y components of the velocity, respectively, and ρ is the density, then[2]

$$u=\frac{\partial\phi}{\partial x}=\frac{1}{\rho}\frac{\partial\psi}{\partial y}=l\sec\tfrac{1}{2}\omega\cos\theta, \quad (4.1)$$

$$v=\frac{\partial\phi}{\partial y}=-\frac{1}{\rho}\frac{\partial\psi}{\partial x}=l\sec\tfrac{1}{2}\omega\sin\theta. \quad (4.2)$$

From the known relations $a=w\sin\omega/2$, $\rho a=k$, where w is the magnitude of the velocity vector, one can determine ρ as a function of ω and the above relations may be written as

$$\frac{\partial y}{\partial\psi}=\frac{1}{k}\sin\tfrac{1}{2}\omega\cos\theta,\quad \frac{\partial x}{\partial\psi}=-\frac{1}{k}\sin\tfrac{1}{2}\omega\sin\theta, \quad (4.3)$$

$$\frac{\partial y}{\partial\phi}=\frac{1}{l}\cos\tfrac{1}{2}\omega\sin\theta,\quad \frac{\partial x}{\partial\phi}=\frac{1}{l}\cos\tfrac{1}{2}\omega\cos\theta. \quad (4.4)$$

In the next few paragraphs, we shall be concerned with the (ϕ, ψ) plane map of the flow. Evidently, the line ED and the curve $B'A'C'$(see Fig. 1) map into the lines $\psi=0$ and $\psi=M/2$ in the (ϕ, ψ) plane. We shall show:

Theorem 7. If we move from B' to A', A' to C', E to D (see Fig. 1) then ϕ is increasing. That is, the maps of the boundaries $B'A'C'$ and ED in the (x, y) and (ϕ, ψ) planes are in one-to-one correspondence.

To verify this result, we note that in moving from B' to A' along the wall, x increases and y decreases. From (4.4) and the fact that $\theta<0$ along $B'A'$, it follows that $\partial x/\partial\phi>0$, $\partial y/\partial\phi<0$. Hence, ϕ must increase as one moves from B' to A'. The other results may be shown in similar manner.

The following result is an immediate consequence of the assumptions and the form of the equations for ϕ and ψ in the various regions (or the formulas 3.3 through 3.6).

Theorem 8. If the boundaries $B'A'C'$ and ED of the (ω, θ) and (ϕ, ψ) planes are in one-to-one correspondence and if the jacobian, $J(\phi, \psi/\omega, \theta)$, does not vanish over the regions I, II, III, then: (1) the characteristics of the (ω, θ) plane transform into two families of straight lines (which will be called characteristics) in the (ϕ, ψ) plane; (2) to each characteristic of positive (negative) slope in (ω, θ) plane, there corresponds a unique characteristic of positive (negative slope) in (ϕ, ψ) plane and conversely.

If the above assumptions are valid, then by use of our results, one may immediately construct the (ϕ, ψ) plane map of the flow (see Fig. 3). In order to obtain conditions under which the boundaries $B'A'C'$ and ED of the (ω, θ) and (ϕ, ψ) planes are in one-to-one correspondence, we shall prove:

Theorem 9. The necessary and sufficient conditions that the boundaries $B'A'C'$ and ED of the (ω, θ) and (ϕ, ψ) planes be in one-to-one correspondence are $h'(\theta)+p'(\theta)>0$, $h'(\theta)-p'(\theta)>0$.

Consider the line $B'A'$. In moving from B' to A', we have shown in theorem 7 that ϕ is increasing. From Eq. (3.19), we find that along $B'A'$

$$\phi=2lf\left(\frac{\omega-\omega_2}{2}\right),\quad \omega_1\leq\omega\leq\omega_2. \quad (4.5)$$

Equation (4.5) implies that the range of θ in $f(\theta)$ is $\frac{1}{2}(\omega_1-\omega_2)\leq\theta\leq 0$. Hence, the $f(\theta)$ as defined by Eq. (3.17) must be used. Replacing θ in Eq. (3.17) by $\frac{1}{2}(\omega_1-\omega_2)$, we obtain

$$\phi=2lf\left(\frac{\omega-\omega_2}{2}\right)=2l\left[-h\left(\bar{\theta}+\frac{\omega_1+\omega}{2}\right)+p\left(\bar{\theta}+\frac{\omega_1-\omega}{2}\right)\right]. \quad (4.6)$$

Since in moving from B' to A', both ϕ and ω are increasing (ω is increasing, since $-\pi<\omega_1<\omega_2<0$, or $0<\omega_1<\omega_2<\pi$), then $d\phi/d\omega>0$. By differentiation of Eq. (4.6), we find that this last statement is equivalent to

$$h'(\theta)-p'(\theta)>0,\quad -\pi/2\leq\theta\leq\bar{\theta}. \quad (4.7)$$

A similar argument for the boundaries $A'C'$ and ED will verify the necessity of the remaining condition.

Further, by reversing the order of the above steps, one may verify the sufficiency of the above conditions.

From theorem 9, it follows that:

Theorem 10. If $h'(\theta)>0$, $p'(\theta)>0$ *(or* $p'(\theta)<0$*) and* $h'(\theta)>p'(\theta)$, *then the* (ω, θ) *and* (ϕ, ψ) *maps of the boundaries of regions I, II, III are in one-to-one correspondence.*

The proof can be given in three steps. First, the above conditions imply $h'(\theta)\pm p'(\theta)>0$ and hence that $f(\theta)$ is an increasing function of θ [see Eqs. (3.16, 3.17)]. Thus, by theorem 9, the maps of ED, $B'A'C'$, are in one-to-one correspondence. From Eq. (3.10) and our conditions, we see that the maps of EB' are in one-to-one correspondence. Finally, the mapping functions along the characteristic EA', which bounds region I, have the values

$$EA': \quad \begin{aligned} \psi &= \tfrac{1}{2}M + k[f(\omega-\omega_2)-f(0)], \\ \phi &= l[f(\omega-\omega_2)+f(0)]. \end{aligned} \tag{4.8}$$

Since $f(\theta)$ is an increasing function of θ, the maps of EA' are in one-to-one correspondence. In a similar manner, it may be shown that the (ω, θ) and (ϕ, ψ) maps of $A'D$, $C'D$ are in one-to-one correspondence.

Finally, we consider the mapping of the interior points of the (ω, θ) and (ϕ, ψ) planes. We prove:

Theorem 11. If the boundaries of the (ω, θ) *and* (ϕ, ψ) *planes are in one-to-one correspondence (see theorem 10), then the interior points of these two maps of the flow are in one-to-one correspondence.*

Consider a point P in the interior of region I in the (ω, θ) plane. Through P one can pass two characteristics of the (ω, θ) plane. These characteristics will be such that they will intersect either $B'A'$ or EB' in two points. Consider the case where the lines intersect EB' in the points Q and R. Using subscripts to denote the values of the variables at various points, we see that $\theta_Q<\theta_P<\theta_R$. Since Q and R lie on characteristics through P, we find that (ω_P, θ_P) satisfy

$$\begin{aligned} (\theta_P-\theta_R)/(\omega_P-\omega_R) &= -\tfrac{1}{2}, \\ (\theta_P-\theta_Q)/(\omega_P-\omega_Q) &= \tfrac{1}{2}. \end{aligned} \tag{4.9}$$

Substituting Eq. (4.9) into the formula for ψ in region I and noting that $\omega_R=\omega_Q=\omega_1$, we find

$$\psi_P=\frac{M}{2}+k\left[f\left(\frac{\omega_1-\omega_2}{2}+\theta_R-\bar{\theta}\right) -f\left(\frac{\omega_1-\omega_2}{2}+2\theta_R-2\theta_P-\theta_R+\bar{\theta}\right)\right], \tag{4.10}$$

$$\psi_P=\frac{M}{2}+k\left[f\left(\frac{\omega_1-\omega_2}{2}+2\theta_P-2\theta_Q+\theta_Q-\bar{\theta}\right) -f\left(\frac{\omega_1-\omega_2}{2}-\theta_Q+\bar{\theta}\right)\right], \tag{4.11}$$

$$\psi_R=\frac{M}{2}+k\left[f\left(\frac{\omega_1-\omega_2}{2}+\theta_R-\bar{\theta}\right) -f\left(\frac{\omega_1-\omega_2}{2}-\theta_R+\bar{\theta}\right)\right], \tag{4.12}$$

$$\psi_Q=\frac{M}{2}+k\left[f\left(\frac{\omega_1-\omega_2}{2}+\theta_Q-\bar{\theta}\right) -f\left(\frac{\omega_1-\omega_2}{2}-\theta_Q+\bar{\theta}\right)\right]. \tag{4.13}$$

By noting that $f(\theta)$ is an increasing function of θ and $\theta_P-\theta_Q>0$, $\theta_R-\theta_P>0$, we see from Eqs. (4.10) through (4.13) that $\psi_Q<\psi_P<\psi_R$. A similar argument shows that $\phi_P>\phi_Q$, $\phi_P>\phi_R$, and also that P must lie to the left of the intersection of the characteristics RP and EA'. Further, the same argument will show that interior points of regions II and III of the (ω, θ) plane map into interior points of the corresponding regions of the (ϕ, ψ) plane.

Again, since the inverse function to $f(\theta)$ is also an increasing function, it can be shown that a point P in the (ϕ, ψ) plane interior maps into a unique point P in the (ω, θ) plane interior.

Our final results are concerned with the mapping from the (x, y) plane to the (ϕ, ψ) plane. We shall show that it is essential that ω be negative (or $w_2<w_1$). In particular, we shall prove:

Theorem 12. If, (1) no back flow exists (θ *is negative), (2)* $w_2<w_1$ *(*ω *is negative), (3) the* (x, y) *plane map of* EB' *is in one-to-one correspondence with* (ϕ, ψ) *plane map of* EB', *(4)* $h'(\theta)>0$, $p'(\theta)>0$ *(or* $p'(\theta)<0$*)* $h'(\theta)\pm p'(\theta)>0$, *then there exists a one-to-one correspondence between the interior points of the* (x, y) *and* (ϕ, ψ) *planes.*

Through any point P in the interior of the (ϕ, ψ) plane, there pass two lines, the line $\phi=$constant and the line $\psi=$constant. These lines intersect the boundaries, $B'E$ and $B'A'C'$, in the points, R and S, respectively. By theorem 7 and assumption (3), the boundaries of the (x, y) and (ϕ, ψ) planes are in one-to-one correspondence. Hence, the points R and S have unique maps in the (x, y) plane. Evidently, $\psi_P<\psi_S$, $\phi_P>\phi_R$. From assumptions (1) and (2) of the theorem and Eqs. (4.3), (4.4), it is seen that along SP, $\partial x/\partial\psi<0$, and along RP, $\partial y/\partial\phi<0$. Thus, $x_P>x_S$, $y_P>y_R$, and to each interior point of the (ϕ, ψ) plane, there corresponds at least one interior point of the (x, y) plane.

Further, it is easily seen that a unique interior point corresponding to P must exist. From Eqs. (4.3), (4.4), we find

$$x_P-x_S=\int_{\psi_S}^{\psi_P} -\frac{1}{k}\sin\frac{\omega}{2}\sin\theta d\psi, \tag{4.14}$$

$$y_P-y_R=\int_{\phi_R}^{\phi_P} \frac{1}{l}\cos\frac{\omega}{2}\sin\theta d\phi. \tag{4.15}$$

By assumption (4) and theorems 10 and 11, to each interior point of the (ϕ, ψ) plane, there corresponds a unique interior point of the (ω, θ) plane. Hence, the above integrals are single-valued and define unique values of (x_P, y_P). A similar argument shows that to each interior point in the (x, y) plane, there corresponds a unique interior point of the (ϕ, ψ) plane.

5. THE FUNCTIONS CONNECTING THE (ω, θ) AND (x, y) PLANES

We shall merely state some results. By use of Eqs. (4.3), (4.4), the expressions (3.18) through (3.23), and the chain rule for differentiation, we can easily obtain the expressions for $\partial x/\partial\omega$, $\partial x/\partial\theta$, $\partial y/\partial\omega$, $\partial y/\partial\theta$ in the regions I, II, III. It can be shown with the aid of some detailed computation that:

Theorem 13. The partial derivatives $\partial x/\partial\omega$, $\partial x/\partial\theta$, $\partial y/\partial\omega$, $\partial y/\partial\theta$ *are such that; (1) the integrability conditions are satisfied; (2) the tangential derivatives of the function* $x(\omega, \theta)$ *along* EA' *in region I is equal to the tangential derivative of this function along* EA' *in region II (similar results are valid for* $y(\omega, \theta)$ *along* EA' *and for both functions along* $A'D$*); (3) thus, these functions define a continuous map from the* (ω, θ) *plane to the* (x, y) *plane.*

6. SOME REMARKS

It should be noted that our methods can be extended to other regions of the (x, y) plane. The fundamental idea can be obtained by use of Fig. 3. Evidently, the extension of ED and the two characteristics through C' furnish region IV. Thus, region IV in the (ω, θ) plane should be the region bounded by ED, EC', and $C'D$. This folding property of the (ω, θ) plane map will furnish a type of periodic recurrence of the jet boundary in the physical (x, y) plane.

Finally, it should be noted that the above mapping idea is applicable to the two-dimensional jet for an arbitrary gas law. Here, the problem is to determine the Riemann function in the hodograph plane.[5]

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A Mechanical Determination of Biaxial Residual Stress in Sheet Materials

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A method is given for determining the residual stress in a sheet material by removing successive uniform layers of material from the surface of a test specimen and measuring the resulting curvature. From the condition of equilibrium of a free specimen, a stress *vs* curvature relation is derived which holds over the depth to which material has been removed. The method applies when the stress is constant in the plane of the specimen and varies through the thickness. An experimental technique is described which is believed to satisfy the essential requirement that the removal of surface layers should not affect the stress in the remaining material, and a practical example is given.

I. INTRODUCTION

Residual stresses, that is, stresses which may exist in a material free of external load, generally result from nonhomogeneous plastic deformation, which may be introduced in many ways including probably all of the fabrication methods.[1] These stresses have an important effect on the physical properties of metals. Fatigue life in bending, for example, depends on the maximum stress developed at the surface including residual stress. The important practical problems of measuring residual stresses have been attacked in many ways. One method, for which a bibliography is given,[1] involves unbalancing the self-equilibrium of the internal stresses by removing or sectioning a part of the material and observing the resulting deformation.[2–4] The present treatment of this method differs from the previous ones in two respects: first, two principal components of stress are considered; second, these components are explicitly expressed as functions of curvature, thereby eliminating the need for calculation by successive approximations.

The method applies to sheet materials and involves removing successive uniform layers from the surface of a sample. In an initially straight sample symmetry makes it necessary to remove material to a depth of only half the thickness. When the following three conditions are satisfied, the accuracy of the method is limited only by the precision of the measurements.

(i) In order to determine the stress from the curvature it is necessary that the sample be linear in pure bending, over the range of curvatures involved, and that the elastic constants be the same throughout the material. If these conditions are not satisfied, the stress cannot be obtained from the curvature. Instead, it is necessary to measure the bending moment required to straighten the strip after every removal of material. A

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