

where the parameters [analogs of the  $\alpha_i$  in Eq. (3)] entering in  $V_{n+3}$  are combinations of the  $a$ 's,  $b$ 's, and  $c$ 's.

To be precise, the suggested change of variables for  $V_{n+3}$  consists of letting

$$x_k = \prod_{l=1}^k u_l, \quad k = 1, 2, \dots, n, \quad (9)$$

where the  $u$ 's are the usual integration variables<sup>4-6, 9-11</sup> successively associated with the internal lines of a multiperipheral graph. This prescription

can be easily checked with the aid of Chan's explicit formulas<sup>4-6</sup> for  $n = 3$  and  $n = 4$ , or with the form written down by Bardakci and Ruegg<sup>11</sup>.

In a following paper, we shall study the structure of the generalized Veneziano amplitudes from the point of view of group representations.

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<sup>1</sup> G. Veneziano, *Nuovo Cimento* **57A**, 190 (1968).

<sup>2</sup> K. Bardakci and H. Ruegg, *Phys. Letters* **28B**, 242 (1968).

<sup>3</sup> M. A. Virasoro, *Phys. Letters* **22**, 37 (1969).

<sup>4</sup> H.-M. Chan, *Phys. Letters* **28B**, 425 (1969).

<sup>5</sup> H.-M. Chan and S. T. Tsou, CERN Preprint TH 969, 1968 (unpublished).

<sup>6</sup> H.-M. Chan and S. T. Tsou, *Phys. Letters* **28B**, 485 (1969).

<sup>7</sup> J. L. Hopkinson and E. Plahte, *Phys. Letters* **28B**, 489 (1969).

<sup>8</sup> A. Bialas and S. Pokorski, *Nucl. Phys.* **B10**, 399 (1969).

<sup>9</sup> C. L. Goebel and B. Sakita, *Phys. Rev. Letters* **22**, 259 (1969).

<sup>10</sup> Z. Koba and H. B. Nielson, *Nucl. Phys.* **B12**, 517 (1969).

<sup>11</sup> K. Bardakci and H. Ruegg, *Phys. Rev.* **181**, 1884 (1969).

<sup>12</sup> See, e.g., H.-M. Chan, 'The Generalized Veneziano Model' review paper presented at the Royal Society Meeting on Duality Reggeons and Resonances in Elementary Particle Processes, 1969, CERN Preprint TH 1057, 1969.

<sup>13</sup> I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions*, (Academic, New York, 1966), Vol. V.

<sup>14</sup> M. A. Virasoro, *Phys. Rev.* **177**, 2309 (1969).

<sup>15</sup> S. Mandelstam, *Phys. Rev.* **183**, 1374 (1969); *Phys. Rev. D* **1**, 1720 (1970).

## Structure of the Generalized Veneziano Amplitudes.\* II. Possible Group Theoretic Content

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An attempt is made to generalize a result of Vilenkin that the beta function (being adopted as Veneziano formula for four-particle processes) appears as the kernel (in the integral) of the irreducible representation (in the Mellin-transformed space) of the group of  $2 \times 2$  unimodular triangular matrices. It is shown that with a modified multiplier in the Gel'fand-Naimark prescription for the representation of the group of  $(n+1) \times (n+1)$  unimodular triangular matrices, the (tree-graph) generalized Veneziano amplitude for  $(n+3)$ -particle processes is recovered by a limiting procedure.

### 1. INTRODUCTION

In a previous paper,<sup>1</sup> we discussed the functional structure of the  $(n+3)$ -point (tree-graph) Veneziano amplitudes viewed as the boundary values of a new class of generalized hypergeometric functions having the property that they are the Radon transforms of products of *linear* forms in an  $(n+1)$ -dimensional space.

In this paper, we shall investigate possible group theoretic content of the generalized Veneziano amplitudes.

Many special functions have acquired a new level of respectability as well as a deeper *raison d'être* when it can be shown that a given function occurs naturally in the representation theory of certain groups, even though this connection is in general not one to one. While it is by now more or less a standard textbook exercise<sup>2-5</sup> in going from the representations of known groups to known functions,<sup>6</sup> the converse route (namely, given the function, finding the group) is obviously much more hazardous. Nevertheless, we venture to

point out that, in the mathematical literature, certain results exist for the connection between the representations of the group  $SL(2, \mathbb{R})$  and the Gauss hypergeometric function  ${}_2F_1(a, b; c; x)$  and, in particular, the special case of triangular matrices of  $SL(2, \mathbb{R})$  yields the beta function as the kernel.<sup>7</sup>

The apparent interest in the Veneziano model<sup>8</sup> perhaps justifies an attempt to generalize the Vilenkin result for the beta function.

A straightforward application of the Gel'fand-Naimark scheme<sup>9</sup> for the irreducible representations of  $(n+1) \times (n+1)$  unimodular triangular matrices will yield in general a class of functions different from the class of functions  $F^{(n)}$  discussed in Paper I. The differences generally are twofold. (i) In one aspect, the multiplier in the Gel'fand-Naimark scheme, which consists of the products of  $n$  principal minors<sup>10</sup>  $\Delta_{p, p+1 \dots n+1}$  ( $p = 2, \dots, n+1$ ), is found to be insufficient in generating all the desired tree-graph links in the Veneziano formula. One can remedy this if one modifies the

multiplier by a product of  $\frac{1}{2}(n-1)(n-2)$  uniquely defined cofactors. As an example, the case  $n = 4$  will call for factors of  $\Delta_{23}, \Delta_{34},$  and  $\Delta_{234}$  in addition to those  $\Delta_{2345}, \Delta_{345}, \Delta_{45},$  and  $\Delta_5$  inherent in the Gel'fand-Naimark scheme. (ii) On the other hand, the group of  $(n+1) \times (n+1)$  unimodular triangular matrices for  $n > 2$  are actually richer than necessary for generating the Veneziano functions. To specialize, we only need the non-vanishing entries in the elements  $g_{i+1(i=1, \dots, n)}$  besides the diagonal elements. In order to preserve the group property, we cannot arbitrarily set those unwanted elements directly equal to zero since this special subset of triangular matrices is not closed under group multiplication. So, one can only approach the desired result by letting some parameters vanish in the limit sense.

Under these two provisos, the tree-graph generalized Veneziano functions are recovered from representations of  $(n+1) \times (n+1)$  unimodular triangular matrices.

For the sake of readability, the essential steps in the establishment<sup>7</sup> of the connection between the beta function and the representation of the  $2 \times 2$  unimodular triangular matrices are summarized in Sec. 2. This will serve as a prototype upon which our generalization will be based. We hope that by going over this simplest case, our generalization to the higher-rank case will not be obscured by the mounting algebraic complexities. To keep the notation straight, we devoted Sec. 3 to setting up the Gel'fand-Naimark machinery for the representations of  $(n+1) \times (n+1)$  unimodular matrices. A modified multiplier will be defined and the representation operator will be carried into the Mellin-transformed space. A limiting procedure will be stated to get to the generalized Veneziano function.

**2. BETA FUNCTION AND GROUP OF  $2 \times 2$  UNIMODULAR TRIANGULAR MATRICES**

The essential steps in the derivation are as follows:<sup>7</sup>

**A. Irreducible Representation of  $SL(2, R)$**

*Step 1:* The representation is to be realized in the space of (infinitely differentiable, square-integrable) homogeneous functions of degree  $\rho$ . A signature factor  $\epsilon(x)$  is included. On account of homogeneity, we have, for

$$g \equiv \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

$$T_g f(z) = |g_{12}z + g_{22}|^\rho [\epsilon(g_{12}z + g_{22})]^m \times f\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right), \tag{1}$$

where  $\epsilon(x)$  is a short-hand notation for  $x/|x|$ .

Note that the representation is unitary for pure imaginary  $\rho$ .

*Step 2:* (a) Carry the representation into the Mellin-transformed space. It will be convenient to separate out the support properties over the positive and negative half-spaces:

$$\tilde{T}_g \tilde{f}_\pm(s) = \int_{-\infty}^{\infty} dz z_\pm^{s-1} |g_{12}z + g_{22}|^\rho \times [\epsilon(g_{12}z + g_{22})]^m f\left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right), \tag{2}$$

where

$$\tilde{f}_\pm(s) \equiv \int_{-\infty}^{\infty} dz z_\pm^{s-1} f(z), \tag{3}$$

$$z_+^s \equiv \theta(z)z^s, \text{ [}\theta(z) \text{ is the usual step function]}, \tag{4a}$$

$$z_-^s \equiv \theta(-z)(-z)^s. \tag{4b}$$

(b) Re-expressing  $f(x)$  in terms of  $\tilde{f}_\pm(t)$  by making an inverse Mellin transform and interchanging the order of integration which can be justified, we get, by writing in the matrix form,

$$\tilde{T}_g \begin{pmatrix} \tilde{f}_+(s) \\ \tilde{f}_-(s) \end{pmatrix} = \int_{c-i\infty}^{c+i\infty} dt \begin{pmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{pmatrix} \begin{pmatrix} \tilde{f}_+(t) \\ \tilde{f}_-(t) \end{pmatrix}, \tag{5}$$

where the kernel is given by

$$K_{\pm\pm}(s, t; \rho, m; g) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz z_\pm^{s-1} \times \left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right)_\pm^{-t} |g_{12}z + g_{22}|^\rho [\epsilon(g_{12}z + g_{22})]^m. \tag{6}$$

*Step 3: Identification:* For the case of  $2 \times 2$  Hermitian or unitary matrices, the kernel is readily recognized as the Gauss hypergeometric function  ${}_2F_1$ .

**B. The Special Case of Triangular Matrices**

In this case, the kernel is reduced to the beta function. Take

$$g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \tag{7}$$

$$K_{++}(s, t; \rho; g) = (2\pi i)^{-1} \int_0^\infty dz z^{s-t-1} (1+az)^{\rho+t} = (2\pi i)^{-1} a^{t-\rho} \beta(s-t, \rho-s). \tag{8}$$

**3. REPRESENTATIONS OF  $(n+1) \times (n+1)$  UNIMODULAR TRIANGULAR MATRICES**

We now generalize the discussion of Sec. 2 to the case of general  $n$ .

*Step 1:* The representation will be realized



