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Instability of a Current-Carrying Fluid Jet Issuing from a Nozzle

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The velocity across a jet becomes nonuniform due to contraction or expansion of fluid in the presence of electric current within the nozzle from which it issues. The results obtained elsewhere on the instability of a jet of uniform velocity due to electric current and surface tension are corrected. It is further shown that velocity nonuniformity reduces this instability. The available data on the instability of a mercury jet issuing from a contraction are for small current density and, hence, low velocity nonuniformity. However, for reasons yet unknown, the data do not agree with the (corrected) theory for the case of uniform velocity.

INTRODUCTION

Neglecting displacement current, the equation for the intensity of magnetic field becomes

$$\partial \mathbf{H}/\partial t = \text{curl}\,(\mathbf{U}\times\mathbf{H}) - (1/\mu\sigma)\,\text{curl curl}\,\mathbf{H}, \quad (1)$$

where μ and σ are assumed constant. The equation of motion for inviscid incompressible fluid is

$$\rho(\partial\mathbf{U}/\partial t - \mathbf{U}\times\text{curl}\,\mathbf{U}) = -\text{grad}\,(p + \tfrac{1}{2}\rho\mathbf{U}^2) + \mathbf{J}\times\mathbf{B}. \quad (2)$$

Consider the steady flow of a current carrying inviscid, incompressible fluid from an axisymmetric nozzle of nonconducting material (see Fig. 1). We assume that electric current density and fluid velocity are constant at the inlet of the nozzle of diameter R_1. The radius of the outlet is R, and J_0, the current density, is uniform here. The directions of the velocity and current coincide for the wall and central streamlines, and, applying the equation of motion separately to these streamlines, we find that[1]

$$U_0^2 - U_c^2 = \mu J_0^2 R^2(1 - R^2/R_1^2)/2\rho, \quad (3)$$

where U_0 and U_c are the exit velocities at the wall and the center of the nozzle. The velocity varies parabolically from the center to the wall for small contractions of the nozzle, and we will assume that for any amount of contraction, the velocity profile for steady unperturbed flow is

$$U_z(r) = U_0[1 - \alpha(1 - r^2/R^2)], \quad (4)$$

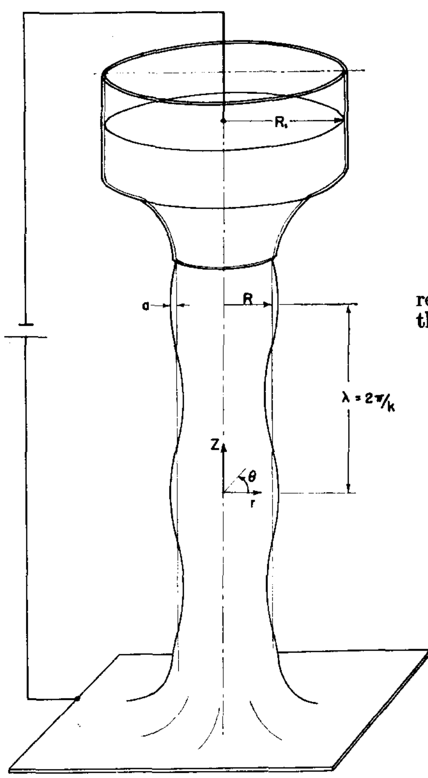

Fig. 1. Schematic representation of the problem.

where

$$\alpha = 1 - (1-\gamma)^{\frac{1}{2}}; \qquad \gamma = (\mu J_0^2 R^2/2\rho U_0^2)(1 - R^2/R_1^2). \quad (5)$$

For a contracting nozzle γ varies from 0 to 1 since U_c ranges from U_0 to zero. For an expanding nozzle, α ranges from 0 to $-\infty$. The jet is subjected to a

[1] M. S. Uberoi, Phys. Fluids 5, 401 (1962).

small axisymmetric perturbation which deforms its surface into a shape given by

$$r_w = R + ae^{ikz+\omega t}; \qquad a/R \ll 1. \tag{6}$$

The governing equations will be solved neglecting squares of small perturbations.

Instability occurs when the quantity

$$\omega = \omega_r + i\omega_i \tag{7}$$

has a positive real part.

MAGNETIC FIELD INTENSITY

The magnetic field intensity has θ component only. We may write H_θ and the velocity as

$$H_\theta = \tfrac{1}{2}J_0 r + h(r, z, t), \tag{8}$$

$$\mathbf{U} = \mathbf{i}_z U_z(r) + \mathbf{u}(r, z, t), \tag{9}$$

where $\frac{1}{2}J_0 r$ and $U_z(r)$ are the steady state values, and h and $\mathbf{u}$ the perturbation. In terms of these Eq. (1) becomes, neglecting product of perturbations,

$$\frac{\partial h}{\partial t} = -U_z(r)\frac{\partial h}{\partial z} - \tfrac{1}{2}J_0\left[\frac{\partial}{\partial z}(ru_z) + \frac{\partial}{\partial r}(ru_r)\right] + \frac{1}{\mu\sigma}\left(\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right)h. \tag{10}$$

Making use of the continuity equation for incompressible flow, we have

$$\left[\frac{\partial}{\partial t} + U_z(r)\frac{\partial}{\partial z}\right]h = \frac{d}{dt}h = \frac{1}{\mu\sigma}\left(\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right)h. \tag{11}$$

In most cases of hydromagnetics of incompressible fluid the term on the left-hand side of the above equation may be neglected. That is, the "skin effect" is unimportant which is certainly the case for the experiments to be discussed later. Thus,

$$\left(\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r + \frac{\partial^2}{\partial z^2}\right)h = 0. \tag{12}$$

In this approximation, and because of the special form of the unperturbed magnetic field intensity, the velocity has no effect on h, which is determined by the instantaneous form of the jet boundary. The solution of Eq. (12), which is periodic in the z direction and satisfies the boundary condition that $r = 0$, $h = 0$, and, at $r = R$,

$$\frac{J_r}{J_0} = \frac{dr_w}{dz}, \quad \text{or} \quad -\frac{1}{J_0}\frac{\partial h}{\partial z} = ikae^{ikz+\omega t},$$

is

$$h = -aJ_0[I_1(kr)/I_1(kR)]e^{ikz+\omega t}. \tag{13}$$

The total magnetic field intensity

$$H_\theta = \tfrac{1}{2}J_0 r - aJ_0[I_1(kr)/I_1(kR)]e^{ikz+\omega t}. \tag{14}$$

VELOCITY FIELD

Taking the curl of Eq. (2) we have

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{U} - \operatorname{curl}(\mathbf{U}\times\operatorname{curl}\mathbf{U}) = \frac{\mu}{\rho}\operatorname{curl}(\mathbf{J}\times\mathbf{B})$$

or, neglecting products of perturbation velocity,

$$\frac{\partial}{\partial t}\operatorname{curl}\mathbf{u} + U_z(r)\frac{\partial}{\partial z}\operatorname{curl}\mathbf{u} = \frac{\mu}{\rho}\operatorname{curl}\mathbf{J}\times\mathbf{B}. \tag{15}$$

Now

$$[\operatorname{curl}(\mathbf{J}\times\mathbf{B})]_\theta = -\frac{1}{\mu}\frac{\partial}{\partial z}\left(\frac{H_\theta^2}{r}\right). \tag{16}$$

Substituting for H_θ from Eq. (14) and neglecting the term of order a^2 compared to that of a,

$$[\operatorname{curl}(\mathbf{J}\times\mathbf{B})]_\theta = (i/\mu)J_0^2 k[I_1(kr)/I_1(kR)]e^{ikz+\omega t}. \tag{17}$$

Let ψ be the perturbation velocity stream function such that

$$u_z = \frac{1}{r}\frac{\partial}{\partial r}r\psi \quad \text{and} \quad u_r = -\frac{\partial\psi}{\partial z}. \tag{18}$$

Further, let

$$\psi = f(x)e^{ikz+\omega t}, \tag{19}$$

where $x = kr$. Later we will use X to denote kR.

Substituting Eqs. (17), (18), and (19) into (15) and using U_z from Eq. (4) we have

$$\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \left(1 - \frac{1}{x^2}\right)f = -i\frac{a\mu J_0^2R^2}{\beta\rho U_0}\frac{I_1(x)}{I_1(X)}\frac{1}{1 + i(\alpha/\beta)x^2}, \tag{20}$$

where

$$\beta = (X^2/kU_0)(\omega + ikU_0) - i\alpha X^2. \tag{21}$$

Then Eq. (20) may be simplified by letting

$$f(x) = -i\frac{a\mu J_0^2R^2}{\beta\rho U_0}\frac{I_1(x)}{I_1(X)}g(x). \tag{22}$$

After substituting Eq. (22) into Eq. (20), g does not appear in the equation, and the first integral of the equation gives

$$\frac{dg}{dx} = \frac{1}{xI_1^2(x)}\int_0^x \frac{yI_1^2(y)\,dy}{1 + i(\alpha/\beta)y^2}. \tag{23}$$

The boundary condition

$$(u_r)_{r=R} = \partial r_w/\partial t + U_0\,\partial r_w/\partial z, \tag{24}$$

gives the relation

$$g(X) = -(\beta\rho U_0/k\mu J_0^2R^2)(\omega + ikU_0). \tag{25}$$

PRESSURE FIELD AND DISPERSION RELATION

The pressure in the jet is the sum of steady state and perturbation pressures

$$p = p_0(r) + ap_1(r)e^{ikz+\omega t}. \tag{26}$$

By substituting the above equation, together with Eqs. (14), (18), (19), and (22), in the z component of Eq. (2), we obtain the expression for the perturbed pressure just inside the jet when we let $x = X$,

$$p_1(R) = \tfrac{1}{2}\mu J_0^2 R + \rho(\omega + ikU_0)\frac{\mu J_0^2 R^2}{\beta\rho U_0} \cdot \left[g(X)\frac{I_0(X)}{I_1(X)} + \left(\frac{dg}{dx}\right)_{x=X}\right]. \tag{27}$$

This pressure is determined by the surface tension T of the fluid and the surface geometry of the jet[2]

$$p_1(R) = (T/R^2)(X^2 - 1). \tag{28}$$

Equating Eqs. (27) and (28), and substituting from Eqs. (21), (23), and (25), the dispersion relation is obtained,

$$(\omega + ikU_0)^2 = -\frac{T}{\rho R^3}X(X^2 - 1)\frac{I_1(X)}{I_0(X)} + \frac{\mu J_0^2}{\rho}\frac{I_1(X)}{I_0(X)}\left\{\frac{X}{2} + \frac{\omega + ikU_0}{\alpha kU_0}\frac{1}{XI_1^2(X)} \cdot \int_0^X \frac{xI_1^2(x)\,dx}{(\omega + ikU_0)/\alpha kU_0 - i[1 - (x/X)^2]}\right\}. \tag{29}$$

STABILITY IN ABSENCE OF SURFACE TENSION

We neglect the effect of surface tension and substitute Eq. (7) into Eq. (29), which leads to two equations,

$$\Sigma_r^2 - \Sigma_i^2 = 1 + \frac{X}{2}\left[2\frac{I_1(X)}{I_0(X)} - \frac{I_0(X)}{I_1(X)}\right] - \frac{1}{XI_0(X)I_1(X)} \cdot \int_0^X \frac{(X^2 - x^2)(X^2 - x^2 - (\gamma^{\frac{1}{2}}/\alpha)\Sigma_i X)xI_1^2(x)}{[X^2 - x^2 - (\gamma^{\frac{1}{2}}/\alpha)\Sigma_i X]^2 + (\gamma/\alpha^2)\Sigma_r^2 X^2}\,dx, \tag{30}$$

and

$$\Sigma_i = \frac{\gamma^{\frac{1}{2}}}{2\alpha I_0(X)I_1(X)} \cdot \int_0^X \frac{(X^2 - x^2)xI_1^2(x)\,dx}{[X^2 - x^2 - (\gamma^{\frac{1}{2}}/\alpha)\Sigma_i X]^2 + (\gamma/\alpha^2)\Sigma_r^2 X^2}, \tag{31}$$

where

$$\Sigma_r^2 = \rho\omega_r^2/\mu J_0^2, \tag{32}$$

[2] H. Lamb, *Hydrodynamics* (Cambridge University Press, New York, 1952), p. 473.

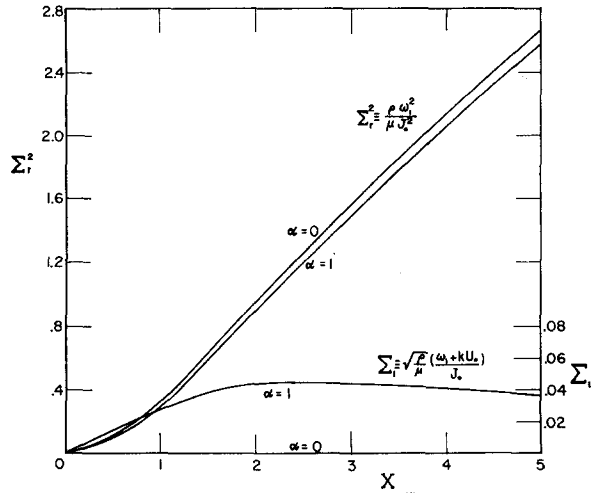

FIG. 2. Influence of initial velocity nonuniformity (α) in absence of surface tension

and

$$\Sigma_i^2 = \rho(\omega_i + kU_0)^2/\mu J_0^2. \tag{33}$$

For an expanding nozzle α is negative, Σ_r is unchanged, and the sign of Σ_i changes. The results are shown in Fig. 2. In the case of uniform initial flow $\alpha = \Sigma_i = 0$, and Σ_r is given in terms of the known functions. The results for $\alpha = 1$ were obtained by an iterative process starting with the solution for $\alpha = 0$. The process converged rapidly. As $X \to \infty$, $\Sigma_i \to 0$ and Σ_r^2 approaches $1 + \frac{1}{2}X$ for all values of α.

STABILITY IN PRESENCE OF SURFACE TENSION

Substituting Eq. (7) in Eq. (29) we get

$$\Omega_r^2 - \Omega_i^2 = -X(X^2 - 1)\frac{I_1(X)}{I_0(X)} + 4N\left\{1 + \frac{X}{2}\left[2\frac{I_1(X)}{I_0(X)} - \frac{I_0(X)}{I_1(X)}\right]\right\} - \frac{4N}{XI_0(X)I_1(X)} \cdot \int_0^X \frac{(X^2 - x^2)\left[X^2 - x^2 - \frac{1}{2\alpha}\left(\frac{\gamma}{N}\right)^{\frac{1}{2}}\Omega_i X\right]xI_1^2(x)}{\left[X^2 - x^2 - \frac{1}{2\alpha}\left(\frac{\gamma}{N}\right)^{\frac{1}{2}}\Omega_i X\right]^2 + \frac{\gamma}{4\alpha^2 N}\Omega_r^2 X^2}\,dx, \tag{34}$$

$$\Omega_i = \frac{\gamma^{\frac{1}{2}}N^{\frac{1}{2}}}{\alpha I_0(X)I_1(X)} \cdot \int_0^X \frac{(X^2 - x^2)xI_1^2(x)\,dx}{\left[X^2 - x^2 - \frac{1}{2\alpha}\left(\frac{\gamma}{N}\right)^{\frac{1}{2}}\Omega_i X\right]^2 + \frac{\gamma}{4\alpha^2 N}\Omega_r^2 X^2}, \tag{35}$$

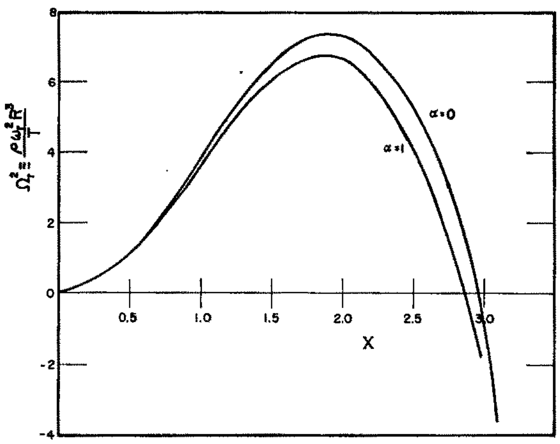

FIG. 3. Influence of initial velocity nonuniformity (α) in presence of surface tension

where

$$\Omega_r^2 = \rho\omega_r^2 R^3/T, \tag{36}$$

$$\Omega_i^2 = \rho(\omega_i + kU_0)^2 R^3/T, \tag{37}$$

$$N = \mu J_0^2 R^3/4T = \mu I^2/4\pi^2 RT, \tag{38}$$

and I is the total current.

For uniform initial velocity $\alpha = \Omega_i = 0$, and[3]

$$\Omega_r^2 = -X(X^2 - 1)\frac{I_1(X)}{I_0(X)} + 4N\left\{1 + \frac{X}{2}\left[2\frac{I_1(X)}{I_0(X)} - \frac{I_0(X)}{I_1(X)}\right]\right\}. \tag{39}$$

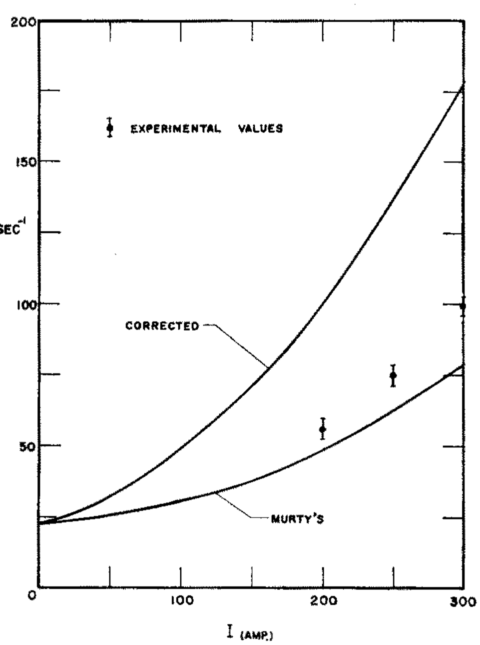

FIG. 4. Variation of growth constant $(\omega_r)_{\max}$ with axial current I for mercury jet of 0.2 cm radius, $T = 487$ dyn cm^{-1}, $\rho = 13.6$ g cm^{-3}.

[3] G. Murty, Arkiv Fysik **18**, 14 (1960).

Due to some oversight, Murty obtained the erroneous result which in our notation is

$$\Omega_r^2 = -X(X^2 - 1)\frac{I_1(X)}{I_0(X)} + 4N\left\{1 + \frac{X}{2}\left[\frac{I_1(X)}{I_0(X)} - \frac{I_0(X)}{I_1(X)}\right]\right\}. \tag{40}$$

The results for $N = 3$ and $\alpha = 1$ are shown in Fig. 3, and were obtained by iterative process starting with the results for $\alpha = 0$. The nonuniformity of the initial velocity profile caused by either converging or diverging nozzle decreased the instability.

DISCUSSION AND COMPARISON WITH EXPERIMENTS

The assumed initial velocity profile agrees with the detailed analysis of flow in a nozzle of small contraction or expansion. Further, the profile does not introduce instability of its own as rotational flows often do. Equation (29) shows that $\omega \equiv 0$ if $J_0 = T = 0$, i.e., the instabilities analyzed here are due to current and surface tension.

Assuming that the observed rate of growth corresponds to maximum instability, the results for $\alpha = 0$ are compared with experiments[4] on a mercury jet in Fig. 4. The discrepancy cannot be explained by the decrease in instability caused by the initial nonuniformity of the velocity, since $\alpha \ll 1$ for these experiments.

Depending on the background disturbances, the

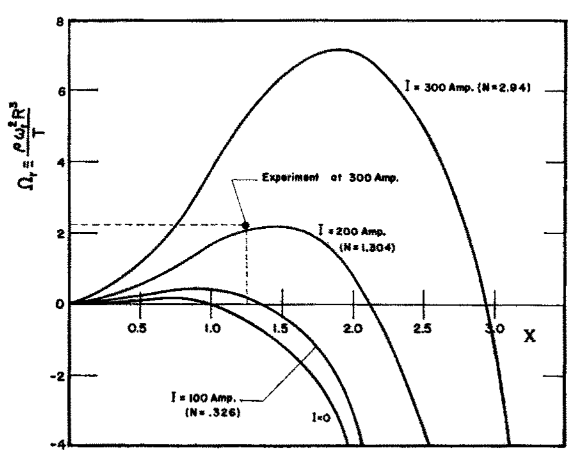

FIG. 5. Comparison of experimental with theoretical results, not considering the effect of velocity distribution across the jet. $R = 0.2$ cm; $T = 487$ dyn cm^{-1}; $\rho = 13.6$ g cm^{-3}.

[4] A. Dattner, B. Lehnert, and S. Lundquist, in *Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 325. The rates of growth are taken from Murty's paper.

observed rates of growth may not correspond to maximum instability. In Fig. 5 we have calculated the rate of growth vs wavenumber corresponding to experimental conditions. The rate of growth and the wavenumber of disturbance are given for only one value of current in Murty's paper, and this is shown in Fig. 5. Both the rate of growth and wavenumber are off by nearly 100%. The discrepancy is unresolved.

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Nonuniform Expansion of a Piston Into an Ionized Medium with a Weak Magnetic Field

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The nonuniform expansion of a rigid, perfectly conducting piston into an infinitely conducting fluid wherein there exists a weak uniform magnetic field is considered. A solution for the state of the plasma and the magnetic field between the piston and the magnetohydrodynamic shock is obtained by a small perturbation method; the dependent variables are expressed in double expansions about a zero order (hydrodynamic) solution. The solution is valid for pistons (either cylindrical or spherical) whose expansion velocity is expressible by a small perturbation on a constant velocity. The development includes second order terms; however, numerical results are restricted to first order. Within the first order perturbation, it is shown that the fluid dynamic analysis is uncoupled from the magnetic field, and the shock shape retains its cylindrical or spherical symmetry. Calculations are carried out for a spherical piston, for several values of γ (specific heat ratio), and a_∞/v_0 where a_∞ is the ambient speed of sound and v_0 is the uniform piston velocity. The results show that the shock velocity is nearly insensitive to changes in the piston velocity.

I. INTRODUCTION

THE expansion of a piston into a fluid medium propagates a shock ahead of it which separates the disturbed fluid from the quiescent region. The classical solution to this problem was obtained by Taylor[1] who considered a spherical piston expanding with constant velocity into a fluid in the absence of a magnetic field. Across the shock, the Rankine–Hugoniot relations were satisfied; behind the shock, the fluid was governed by the fluid equations while, at the piston, the kinematic condition equating the piston velocity to the fluid velocity was imposed. Taylor's solution retained spherical symmetry and remained self-similar for all times because of the constant piston velocity. Rogers[2] extended the Taylor analysis to other geometries.

When the fluid is infinitely conducting and a magnetic field exists, the expanding piston propagates ahead of it a magnetohydrodynamic shock across which properties are governed by modified Rankine–Hugoniot relations as first derived by de Hoffmann and Teller.[3] Behind the shock, the fluid is governed by the magnetohydrodynamic equations and Maxwell's equations, while at the piston, a constraint on the magnetic field, in addition to the kinematic condition of Taylor, is imposed.

The expansion of a piston into an ionized medium wherein a magnetic field exists is relevant to explosions in the ionosphere or to the astrophysical problem of exploding stars.

Kulsrud *et al.*[4] have considered the uniform expansion of an infinitely conducting spherical piston into an infinitely conducting fluid in which initially there exists a constant magnetic field. They have solved this problem by a perturbation procedure about the zero order (hydrodynamic) solution of Taylor[1]; their perturbation parameter ϵ was related to the ratio between the undisturbed Alfvén velocity and the piston velocity, and was assumed small. Their solution is still self-similar, even though

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[2] M. H. Rogers, Quart. J. Mech. Appl. Math. **11**, 411 (1958).
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[4] R. M. Kulsrud, I. B. Bernstein, J. B. Fanucci, M. D. Kruskal, and N. Ness, AFSWC-TDR-62-12, Vol. II, Chap. 2, March 1962 (to be published).