Characteristic Manifolds in Three-Dimensional Unsteady Magnetohydrodynamics

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(Received July 11, 1958; revised manuscript received February 20, 1959)

The technique developed in the general theory of discontinuities is applied to the basic equations of unsteady magnetohydrodynamics in order to find the conditions to be satisfied by the discontinuities in the derivatives of the significant flow and magnetic field parameters. In the formulation of the basic equations use is made of the "magnetohydrodynamic approximation." This amounts to the assumption that the magnetic energy is very large compared with the electric energy, or physically, that the displacement current is negligible. The fluid itself is considered to be infinitely conductive, inviscid, and compressible. With the aid of the relations satisfied by the jumps in the derivatives of the parameters the various characteristic manifolds are found. Finally, it is shown that these manifolds are hypersurfaces along which small disturbances and weak shocks are propagated.

1. INTRODUCTION

THE theory of discontinuities in conventional hydrodynamics has been treated by many authors culminating in the work of J. Hadamard[1] and the texts by Courant and Hilbert[2] and Courant and Friedrichs.[3] However, Hadamard's theory is based on the Lagrangian form of the equations of motion and the classical method treated in the texts mentioned above are rather cumbersome and long, especially when the number of parameters involved is large. Following a method developed by R. K. Luneberg[4] for electromagnetic theory one may obtain a very general theory of discontinuities[5] in terms of the Eulerian form of the equations of motion. This method, as adapted by N. Coburn in conventional flows of compressible fluids, may be employed very effectively in the study of discontinuities in magnetohydrodynamics.

It is the purpose of this paper to illustrate the application of this theory to three-dimensional unsteady flows of a compressible, inviscid, magnetohydrodynamic medium, resulting in a simple method of obtaining the various "characteristic manifolds" of the flow. Furthermore, from the resulting relations to be satisfied by the discontinuities in the derivatives of the flow parameters, it will be shown that small disturbances and weak shocks are propagated along these manifolds.

[1] J. Hadamard, *Propagation des ondes* (Chelsea Publishing Company, New York, 1949).
[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953) Vol. II.
[3] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Interscience Publishers, Inc., New York, 1948).
[4] R. K. Luneberg, "Asymptotic development of steady state electromagnetic fields," New York University Mathematics Research Group, Research Rept. No. EM 14, July, 1949.
[5] N. Coburn, Math. Mag. **27**, 245 (1954).

In the next section the fundamental equations of magnetohydrodynamics are formulated and the method of the general theory of discontinuities is applied to obtain the relations for the jumps in the derivatives of the flow variables. The details concerning the general theory of discontinuities are given in reference 5.

2. MAGNETOHYDRODYNAMIC EQUATIONS

For the case of the continuum hydrodynamics of an electrically conducting fluid immersed in an electromagnetic field the equations describing the motion are obtained by combining the equations of conventional macroscopic hydrodynamics with the equations of electromagnetism. For a general inviscid, compressible, infinitely conducting fluid in the absence of external nonelectromagnetic body forces these are (in Gaussian units)[6]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\mu}(\rho v^\mu) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho v_\lambda) + \frac{\partial}{\partial x^\mu}(\rho v^\mu v_\lambda) + \frac{\partial}{\partial x^\mu}\left(p + \frac{\mu_e H^2}{8\pi}\right)\delta_\lambda{}^\mu - \frac{\partial}{\partial x^\mu}\left(\frac{\mu_e}{4\pi} H^\mu H_\lambda\right) = 0, \tag{2}$$

$$\frac{\partial H}{\partial t} + v^\mu \frac{\partial H_\lambda}{\partial x^\mu} - H^\mu \frac{\partial v_\lambda}{\partial x^\mu} + H_\lambda \frac{\partial v^\mu}{\partial x^\mu} = 0, \tag{3}$$

where, x^λ, ($\lambda = 1, 2, 3$) denote coordinate axes of a Cartesian coordinate system, ρ denotes density, v_λ, fluid velocity vector, H_λ, magnetic field vector, p, pressure, and μ_e, magnetic permeability (assumed constant).

Repeated indices indicate summation and $\delta_\lambda{}^\mu$ is

[6] G. H. Cole, "Some aspects of magnetohydrodynamics," Advances in Physics (Taylor and Francis, Ltd., London, 1956), Vol. 5, p. 452.

the usual Kronecker delta. Together with the system of Eqs. (1) to (3) must be adjoined the relation

$$\frac{\partial}{\partial x^\mu} H^\mu = 0 \tag{4}$$

which expresses the condition that the divergence of the magnetic field is zero.

The metric of the space is expressed by the tensor $g_{\lambda\mu}$ satisfying the relation

$$g_{\lambda\mu} = \cos(x^\lambda, x^\mu). \tag{5}$$

Since a general Cartesian coordinate system is used, contravariant components of a vector or tensor are to be distinguished from covariant ones. The relation between them is given by

$$v_\lambda = g_{\lambda\mu} v^\mu, \quad \text{etc.} \tag{6}$$

Consider a four-dimensional Euclidean space-time manifold E_4. The time variable will be denoted by t, and the space variable by x^λ, $\lambda = 1, 2, 3$. The space variables are assumed to be Cartesian and define ∞^1 Euclidean three spaces E_3 in E_4. The coordinate lines, t = variable, will be assumed to be Euclidean and thus they are lines in E_4.

The hypersurface (or lower dimensional manifold) in space-time along which the discontinuities occur will be denoted by

$$\phi(t, x^\lambda)_j = c_j, \tag{7}$$

where the c_j are constants and $j = 0$, or $j = 0, 1$, or $j = 0, 1, 2$, or $j = 0, 1, 2, 3$. If $j = 0$, then this system consists of only one equation, and in this case the discontinuity manifold defines a hypersurface. Similarly, if the system consists of two equations ($j = 0, 1$) the discontinuity manifold defines a surface, etc. The vector fields for the various types of j, $[(\partial\phi)/(\partial t)]_j$, $[(\partial\phi)/(\partial x^\mu)]_j$ ($\mu = 1, 2, 3$), determine vectors normal to the discontinuity manifold.

3. RELATIONS FOR THE JUMPS IN THE DERIVATIVES OF THE FLOW AND MAGNETIC FIELD VARIABLES

In order to determine the fundamental relations for the study of the discontinuities in the derivatives of the parameters the general covariant differentiation operator, ∇_j, $j = 0, 1, 2, 3$ is introduced where

$$\nabla_0 = \frac{\partial}{\partial t}, \qquad \nabla_j = \frac{\partial}{\partial x^j}, \qquad j = 1, 2, 3. \tag{8}$$

By differentiation of Eqs. (1) to (3) the following relations are obtained:

$$\nabla_j\left(\frac{\partial \rho}{\partial t}\right) + \nabla_j \nabla_\mu(\rho v^\mu) = 0, \tag{9}$$

$$\nabla_j \frac{\partial}{\partial t}(\rho v_\lambda) + \nabla_j \nabla_\mu \left\{ (\rho v^\mu v_\lambda) + \left(p + \frac{\mu_e H^2}{8\pi}\right)\delta_\lambda{}^\mu - \frac{\mu_e}{4\pi} H_\lambda H^\mu \right\} = 0, \tag{10}$$

$$\nabla_j\left(\frac{\partial H_\lambda}{\partial t}\right) + \nabla_j \nabla_\mu(v^\mu H_\lambda - v_\lambda H^\mu) = 0. \tag{11}$$

Since the space E_4 is Euclidean the interchange of the order of convariant differentiation is permissible. Thus, $\nabla_j \nabla_k v_\lambda = \nabla_k \nabla_j v_\lambda$.

In order to obtain the equations for the various characteristic manifolds the jumps in the derivatives of the flow parameters will be investigated. Hence, it is assumed in the future work that v_λ, p, ρ, H_λ are continuous but that the derivatives of these quantities may be discontinuous. Also the discontinuities of s will be assumed negligible.

Consequently from the formulation of the gas law as $p = p(\rho, s)$ it is seen readily that

$$\nabla_j p = a^2 \nabla_j \rho + \left(\frac{\partial p}{\partial s}\right)_\rho \nabla_j s, \tag{12}$$

where $a^2 = (\partial p/\partial \rho)_s$ = local velocity of sound.

Let $[\nabla_j v_\lambda]$ denote the jump in the value of the derivative of v_λ across the discontinuity manifold, $[\nabla_j v^\mu v_\lambda]$ the jump in the derivative of $v^\mu v_\lambda$, etc. In view of our assumption the discontinuity in $\nabla_j p$ may be expressed as

$$[\nabla_j p] = a^2 [\nabla_j \rho]. \tag{13}$$

Then by means of the integral forms of Eqs. (9), (10), and (11) the jumps in the derivatives of the parameters satisfy the following equations[5]:

$$\frac{\partial\phi}{\partial t}[\nabla_j \rho] + \frac{\partial\phi}{\partial x^\mu}[\nabla_j \rho v^\mu] = 0, \tag{14}$$

$$\frac{\partial\phi}{\partial t}[\nabla_j \rho v_\lambda] + \frac{\partial\phi}{\partial x^\mu}\left\{[\nabla_j \rho v^\mu v_\lambda] - \left[\nabla_j \frac{\mu_e}{4\pi} H^\mu H_\lambda\right]\right\} + \frac{\partial\phi}{\partial x^\lambda}\left(a^2[\nabla_j \rho] + \left[\nabla_j \frac{\mu_e}{8\pi} H^\mu H_\lambda\right]\right) = 0, \tag{15}$$

$$\frac{\partial\phi}{\partial t}[\nabla_j H_\lambda] + \frac{\partial\phi}{\partial x^\mu}\{[\nabla_j v^\mu H_\lambda] - [\nabla_j H^\mu v_\lambda]\} = 0. \tag{16}$$

In the preceding equations $(\partial\phi/\partial t)$, $(\partial\phi/\partial x^\mu)$ represents the components of any vector normal to the discontinuity manifold. The equations (14) through (16) are the fundamental equations for the study of discontinuities in the first derivatives of the flow parameters.

The auxiliary equation (4) becomes

$$\frac{\partial\phi}{\partial x^\mu}[\nabla_j H^\mu] = 0. \tag{17}$$

4. VARIOUS "DISCONTINUITY MANIFOLDS"

In this section the (hyper) surfaces associated with discontinuities in one or more of the first derivatives of the flow parameters will be investigated.

By means of the usual property of derivatives the following expansions in the jumps may be made:

$$[\nabla_j \rho v_\lambda] = v_\lambda[\nabla_j \rho] + \rho[\nabla_j v_\lambda],$$

$$[\nabla_j \rho v^\mu v_\lambda] = v_\lambda[\nabla_j \rho v^\mu] + \rho v^\mu[\nabla_j v_\lambda], \qquad \text{etc.}$$

Introducing these expansions into Eqs. (14) through (16) and making use of Eq. (17) the jump relations may be written as

$$L[\nabla_j \rho] + \rho\phi_\mu[\nabla_j v^\mu] = 0, \tag{18}$$

$$L\rho[\nabla_j v_\lambda] + \phi_\lambda\left(a^2[\nabla_j \rho] + \frac{\mu_e}{4\pi} H^\mu[\nabla_j H_\mu]\right) - \frac{\mu_e}{4\pi} H^\mu\phi_\mu[\nabla_j H_\lambda] = 0, \tag{19}$$

$$L[\nabla_j H_\lambda] + H_\lambda\phi_\mu[\nabla_j v^\mu] - H^\mu\phi_\mu[\nabla_j v_\lambda] = 0, \tag{20}$$

where

$$L \equiv \frac{\partial\phi}{\partial t} + v^\alpha \frac{\partial\phi}{\partial x^\alpha},$$

$$\phi_\mu \equiv \frac{\partial\phi}{\partial x^\mu}.$$

(a) Manifolds along which $\nabla_j v_\lambda$ may be discontinuous but $\nabla_j\rho$, ∇H_λ are continuous: Equations (18), (19), and (20) yield

$$\phi_\mu[\nabla_j v^\mu] = 0; \quad L[\nabla_j v_\lambda] = 0; \quad H^\mu\phi_\mu[\nabla_j v_\lambda] = 0.$$

It is seen that these equations are satisfied if and only if $\phi(x^\lambda, t)$ is the solution of $(\partial\phi/\partial t) + v^\mu(\partial\phi/\partial x^\mu) = 0$. These manifolds consist of families of lines of particle motion. This follows from the fact that the equation implies that $(d\phi)/(dt) = 0$ as the motion of the fluid is followed. It is to be noted that associated with it the component of the magnetic field vector normal to the discontinuity manifolds is zero.

(b) Manifolds along which $\nabla_j\rho$ may be discontinuous but $\nabla_j v_\lambda$, $\nabla_j H_\lambda$ are continuous. Equations (18) and (19) yield

$$L[\nabla_j \rho] = 0; \qquad \phi_\lambda[\nabla_j \rho] = 0.$$

These equations are satisfied if and only if $\phi(x^\lambda, t)$ is the solution of the system

$$\frac{\partial\phi}{\partial t} + v^\mu \frac{\partial\phi}{\partial x^\mu} = 0; \qquad \frac{\partial\phi}{\partial x^\mu} = 0.$$

This system possesses the solutions $\phi(x^\lambda, t)_j = c_j$, $j = 0, 1, 2, 3$. Hence there exists only a finite number of points at which $\nabla_j\rho$ may be discontinuous, but $\nabla_j v_\lambda$, $\nabla_j H_\lambda$ are continuous.

(c) Manifolds along which $\nabla_j H_\lambda$ may be discontinuous but $\nabla_j v_\lambda$, $\nabla_j\rho$ are continuous: Equations (19) and (20) yield

$$\phi_\lambda H^\mu[\nabla_j H_\mu] - H^\mu\phi_\mu[\nabla_j H_\lambda] = 0; \qquad L[\nabla_j H_\lambda] = 0.$$

Hence the discontinuity manifolds consists of lines of particle motion.

(d) Manifolds along which $\nabla_j\rho$, $\nabla_j H_\lambda$ may be discontinuous, but $\nabla_j v_\lambda$ is continuous: Equations (18), (19), and (20) yield

$$L[\nabla_j \rho] = 0; \qquad a^2\phi_\lambda[\nabla_j \rho] + \frac{\mu_e}{4\pi}\phi_\lambda H^\mu[\nabla_j H_\mu] - \frac{\mu_e}{4\pi} H^\mu\phi_\mu[\nabla_j H_\lambda] = 0; \qquad L[\nabla_j H_\lambda] = 0.$$

Multiplying the second equation by H^λ and sum over λ gives

$$H^\lambda\phi_\lambda = 0.$$

Hence the discontinuity manifolds are lines of particle motion with the condition that $H^\lambda\phi_\lambda = 0$.

(e) Manifolds along which $\nabla_j v_\lambda$, $\nabla_j H_\lambda$ may be discontinuous, but $\nabla_j\rho$ is continuous: Equations (18), (19), and (20) yield

$$\phi_\mu[\nabla_j v^\mu] = 0; \qquad L\rho[\nabla_j v_\lambda] + \frac{\mu_e}{4\pi}\phi_\lambda H^\mu[\nabla_j H_\mu] - \frac{\mu_e}{4\pi} H^\mu\phi_\mu[\nabla_j H_\lambda] = 0;$$

$$L[\nabla_j H_\lambda] - H^\mu\phi_\mu[\nabla_j v_\lambda] = 0.$$

Multiplying the second equation by H^λ and sum over λ gives

$$LH^\lambda[\nabla_j v_\lambda] = 0.$$

This yields either $L = 0$ or $H^\lambda[\nabla_j v_\lambda] = 0$.

(i) $L = 0 \Rightarrow$ the discontinuity manifolds consist of lines of particle motion.

(ii) $H^\lambda[\nabla_j v_\lambda] = 0$. In this case Eq. (20) yields $H^\lambda[\nabla_j H_\lambda] = 0$. Hence, Eq. (19) gives

$$L\rho[\nabla_j v_\lambda] - \frac{\mu_e}{4\pi} H^\mu\phi_\mu[\nabla_j H_\lambda] = 0.$$

Also from Eq. (20)

$$[\nabla_j H_\lambda] = \frac{H^\mu\phi_\mu[\nabla_j v_\lambda]}{L}.$$

Therefore,

$$L\rho[\nabla_i v_\lambda] - \frac{\mu_e}{4\pi} H^\alpha H^\beta \phi_\alpha \phi_\beta \frac{[\nabla_i v_\lambda]}{L} = 0$$

or

$$\left(L^2 - \frac{\mu_e}{4\pi\rho} H^\alpha H^\beta \phi_\alpha \phi_\beta\right)[\nabla_i v_\lambda] = 0.$$

Hence in this case the discontinuity manifolds are described by the partial differential equation:

$$L^2 - \frac{\mu_e}{4\pi\rho} H^\alpha H^\beta \phi_\alpha \phi_\beta = 0.$$

(f) Manifolds along which $\nabla_i v_\lambda$, $\nabla_i \rho$ may be discontinuous, but $\nabla_i H_\lambda$ is continuous: Equations (18) and (19) yield

$$L[\nabla_i \rho] + \rho\phi_\mu[\nabla_i v^\mu] = 0,$$

$$L\rho[\nabla_i v_\lambda] + a^2\phi_\lambda[\nabla_i \rho] = 0.$$

Multiplying the last equation by ϕ_λ and sum over λ gives

$$L\rho\phi_\lambda[\nabla_i v^\lambda] + a^2 g^{\lambda\mu}\phi_\lambda\phi_\mu[\nabla_i \rho] = 0.$$

Hence, using (18),

$$L^2 - a^2 g^{\lambda\mu}\phi_\lambda\phi_\mu = 0.$$

It is to be noted that this is exactly the relation which describes the characteristic surfaces in the case of conventional hydrodynamics.[5]

(g) Finally the manifolds along which $\nabla_i\rho$, $\nabla_i v_\lambda$, and $\nabla_i H_\lambda$ may all be discontinuous will be determined: It will be seen that these are hypersurfaces and they are usually known as the "characteristic manifolds" of the flow.

Multiplying Eq. (19) by H^λ and summing over λ gives

$$L\rho H^\lambda[\nabla_i v_\lambda] + a^2 H^\lambda\phi_\lambda[\nabla_i \rho] = 0. \tag{21}$$

Multiplying Eq. (20) by H^λ, summing over λ, and using Eqs. (18) and (21) yields

$$\begin{aligned} L^2\rho H^\lambda[\nabla_i H_\lambda] &+ L\rho H^\alpha H_\alpha \phi_\lambda[\nabla_i v^\lambda] \\ &+ a^2 H^\alpha H^\beta \phi_\alpha\phi_\beta[\nabla_i\rho] = 0. \end{aligned} \tag{22}$$

Multiplying Eq. (19) by ϕ_λ, summing over λ, and using Eq. (18) gives

$$(L^2 - a^2 g^{\alpha\beta}\phi_\alpha\phi_\beta)[\nabla_i\rho] = \frac{\mu_e}{4\pi} g^{\alpha\beta}\phi_\alpha\phi_\beta H^\lambda[\nabla_i H_\lambda].$$

Using this equation together with (18), (22), and the fact that $[\nabla_i\rho] \neq 0$, it is seen that the characteristic manifolds are described by the differential equation:

$$\begin{aligned} L^2\left\{L^2 - \left(a^2 + \frac{\mu_e H^2}{4\pi\rho}\right)g^{\alpha\beta}\phi_\alpha\phi_\beta\right\} \\ + \frac{\mu_e a^2}{4\pi\rho} H^\lambda H^\mu \phi_\lambda\phi_\mu g^{\alpha\beta}\phi_\alpha\phi_\beta = 0. \end{aligned} \tag{23}$$

The characteristic relation (23) was also obtained by Friedrichs and Kranzer by using a method of weak shocks.[7] It may be easily shown, e.g., by computing the determinant of (18), (19), and (20), etc., that this condition is also a sufficient one.

It is to be noted that the above equation reduces to the familiar characteristic equation for conventional hydrodynamics in the absence of a magnetic field.

5. PROPAGATION OF SMALL DISTURBANCES OR WEAK SHOCKS

It will be shown in this section that small disturbances in the parameters v_λ, p, ρ, H_λ are propagated along the characteristic manifolds. Suppose v_λ, p, ρ, H_λ, and their space-time derivatives are continuous except that in crossing the discontinuity manifold the parameters themselves and their derivatives may be discontinuous. Let $[\rho v_\lambda]$ denote the jump in the value of ρv_λ across the discontinuity, $[\rho v^\mu v_\lambda]$ the jump in the values of $\rho v^\mu v_\lambda$, etc. Let the subscripts 1 and 2 indicate values on the sides of the discontinuity manifold, then

$$\begin{aligned} [ab] = (ab)_2 - (ab)_1 &= (a_2 - a_1)(b_2 - b_1) \\ &+ a_1 b_2 + a_2 b_1 - 2a_1 b_1. \end{aligned}$$

Upon simplifying the right-hand side the jump in (ab) may be expressed in the form

$$\begin{aligned} [ab] &= [a][b] + [a]b_1 + a_1[b] \\ &= -[a][b] + [a]b_2 + a_2[b]. \end{aligned} \tag{24}$$

Similar to the case of conventional hydrodynamics the jumps in the flow parameters satisfy the relations[5]

$$\frac{\partial\phi}{\partial t}[\rho] + \frac{\partial\phi}{\partial x^\mu}[\rho v^\mu] = 0, \tag{25}$$

$$\begin{aligned} \frac{\partial\phi}{\partial t}[\rho v_\lambda] + \frac{\partial\phi}{\partial x^\mu}\left\{[\rho v^\mu v_\lambda] + \left([p] + \left[\frac{\mu_e H^2}{8\pi}\right]\right)\delta_\lambda{}^\mu \right. \\ \left. - \left[\frac{\mu_e}{4\pi} H^\mu H_\lambda\right]\right\} = 0, \end{aligned} \tag{26}$$

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[7] K. O. Friedrichs and H. Kranzer, "Non-linear wave motion in magnetohydrodynamics," New York University, Inst. Math. Sci. Rept. MH-8, 1958 (unpublished). This was brought to the attention of the author by the referee of this paper.

$$\frac{\partial\phi}{\partial t}[H_\lambda] + \frac{\partial\phi}{\partial x^\mu}([v^\mu H_\lambda] - [H^\lambda v_\mu]) = 0. \quad (27)$$

The auxiliary equation becomes

$$\frac{\partial\phi}{\partial x^\mu}[H^\mu] = 0. \quad (28)$$

This means that the normal component of the magnetic field vector is continuous across a discontinuity manifold. As before, entropy changes are assumed negligible and hence

$$[p] = a^2[\rho]. \quad (29)$$

Expanding the jump relations (25), (26), and (27) by means of Eq. (24), neglecting the quadratic and cubic jumps, and using the relation (28), the equation for the characteristic manifold (23) is obtained. Thus small disturbances and weak shocks are indeed propagated along the characteristic manifolds as they are in the case of conventional hydrodynamics.

6. CONCLUSION

As was shown in the foregoing the general theory of discontinuities is readily adaptable to the study of discontinuity manifolds in unsteady flow of magnetohydrodynamics. The characteristic hypersurfaces of the flow were determined in a very elegant manner which avoids the extensive algebraic manipulations one usually encounters in the classical method of characteristics. Moreover, the method of the general theory of discontinuities may be extended to discuss the shock and contact discontinuities in unsteady magnetohydrodynamics [see (25)–(28)]. This may possibly lead to useful information concerning the effect of the magnetic field on the nature of the shock manifolds.

ACKNOWLEDGMENT

The author is indebted to Professor N. Coburn of the Department of Mathematics, University of Michigan, for many helpful discussions and valuable advice on this work.