Effects of Inhomogeneities in Plasma Density on the Beam-Plasma Instability and Landau Damping

A. T. LIN AND J. E. ROWE

Electron Physics Laboratory, Department of Electrical and Computer Engineering, The University of Michigan, Ann Arbor, Michigan 48104

(Received 21 May 1971; final manuscript received 12 August 1971)

A numerical computer simulation using a charge sheet model is employed to investigate the beam-plasma instability and Landau damping in a one-dimensional, finite-length, inhomogeneous plasma. The results confirm the theoretical predictions that the inhomogeneities in plasma density distribution reduce the spatial growth rate of the beam-plasma instability and increase the Landau damping rate.

I. INTRODUCTION

In most plasmas generated in the laboratory there exists a significant axial density gradient close to the ends of the discharge tube in addition to radial density variations. It is interesting to investigate how such spatial inhomogeneities in plasma density modify the beam-plasma instability and Landau damping. Davis[1] used a one species charge sheet model to study the beam-plasma interaction at the electron plasma frequency in an effort to explain the narrow beam velocity spread in his laboratory experiments. He found that the nonuniform plasma density distribution simulated by a linearized plasma model was the mechanism which reduced the beam-plasma interaction strength. Jackson and Raether[2] selected a special form of plasma density distribution and were able to obtain the oscillation frequency and Landau damping rate as a function of the degree of inhomogeneity. They predicted that an increase in the plasma density inhomogeneity would enhance the Landau damping rate.

The first part of the present investigation is devoted to the development of a linear theory of the beam-plasma instability and Landau damping phenomena in a finite-length nonuniform plasma system. Then, a two-component charge sheet model is used to study the effects of inhomogeneities in plasma density distribution on the beam-plasma interaction and Landau damping by assigning appropriate initial positions to pairs of ion and electron sheets. The comparison between the predictions of linear theory and the numerical computer simulation results is described in detail.

II. THEORY

In the framework of the hydrodynamic approximations the basic system of linearized differential equations for an electron beam nonuniform cold-plasma system in one dimension is written as

$$\frac{\partial v_b}{\partial t}+V_{0b}\frac{\partial v_b}{\partial z}=-\frac{qE}{m}, \qquad (1)$$

$$\frac{\partial n_b}{\partial t}+N_{0b}\frac{\partial v_b}{\partial z}+V_0\frac{\partial n_b}{\partial z}=0, \qquad (2)$$

$$\frac{\partial v_p}{\partial t}=-\frac{qE}{m}, \qquad (3)$$

$$\frac{\partial n_p}{\partial t}+v_p\frac{\partial N_{0p}(z)}{\partial z}+N_{0p}(z)\frac{\partial v_p}{\partial z}=0, \qquad (4)$$

and

$$\frac{\partial E}{\partial z}=-\frac{q}{\epsilon_0}(n_b+n_p), \qquad (5)$$

where the heavier ions are taken as a nonuniform stationary neutralizing background. In the above expressions, v denotes the velocity, n is the density, E is the electric field, q is the magnitude of the electron charge, and ϵ_0 is the dielectric constant of free space. Inserting a time variation of the form $\exp[j\omega t]$ into Eqs. (1)–(5) and neglecting any spatially uniform electric field in the solution yield the following differential equation for the electric field:

$$\left(j\omega+V_{0b}\frac{\partial}{\partial z}\right)^2\left[E\left(1-\frac{\omega_p^2(z)}{\omega^2}\right)\right]+\omega_b^2E=0, \qquad (6)$$

where $\omega_p^2(z)=[q^2N_{0p}(z)]/\epsilon_0 m$ is the spatially varying plasma frequency. Let

$$E=\frac{y(z)}{(1-\omega_p^2(z)/\omega^2)}\exp\left(-\frac{j\omega z}{V_{0b}}\right). \qquad (7)$$

The substitution of Eq. (7) into Eq. (6) yields

$$\frac{\partial^2 y(z)}{\partial z^2}+\frac{\omega_b^2}{V_{0b}^2}\frac{y(z)}{(1-\omega_p^2(z)/\omega^2)}=0. \qquad (8)$$

Consider a linear density profile with

$$\omega_p^2(z)=\omega_{p0}^2(1+\nu^2 z), \qquad (9)$$

where ν is a measure of the inhomogeneity of the plasma density distribution. At $z=0$ the electron beam is modulated at the frequency ω_{p0} and it is assumed that ω_{p0} denotes the dominant mode of the interaction (i.e., $\omega=\omega_{p0}$). The solutions of Eq. (8) can be written in terms of modified Bessel functions of order one and the corresponding physically admissible amplitude of

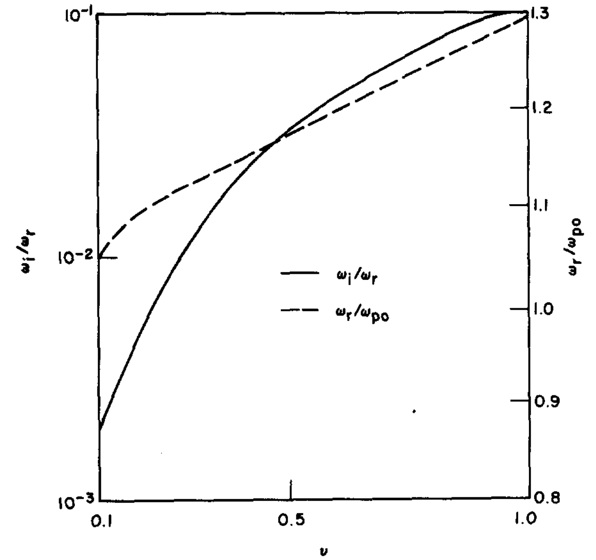

FIG. 1. Landau damping rate and oscillation frequency of the lowest-order mode versus the inhomogeneity parameter for $L/\lambda_D=25$.

the beam velocity is

$$|v_b(z)| = \frac{qA_1}{m\omega_b|\nu|} I_0\left(\frac{2\omega_b}{V_{0b}|\nu|}z^{1/2}\right), \tag{10}$$

where A_1 is an integration constant. It is evident from Eq. (10) and the behavior of the modified Bessel function I_0 that an increase in the inhomogeneity parameter ν will reduce the spatial growth rate of the beam-plasma interaction.

It is believed that for a sufficiently warm plasma in which the resonant interactions between the waves excited by the electron beam and the plasma electrons are important, Landau damping will be the dominant dissipative effect for the beam-plasma instability. In the following the influence of an inhomogeneity in plasma density distribution on Landau damping is investigated. The linearized Vlasov and Maxwell current equations for an inhomogeneous plasma are

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}-\frac{q}{m}E\frac{\partial f_0}{\partial v}-\frac{q}{m}E_0\frac{\partial f}{\partial v}=0 \tag{11}$$

and

$$\frac{\partial E}{\partial t}=-\frac{q}{\epsilon_0}\int_{-\infty}^{\infty} vf\,dv, \tag{12}$$

where $f(z, v, t)$ is the perturbation in the electron distribution function, $f_0(z, v)$ is Maxwellian in velocity space,

$$f_0(z,v)=N_0(z)g(v)=N_0(z)[(2\pi)^{1/2}V_T]^{-1} \times\exp(-v^2/2V_T^2), \tag{13}$$

and $E_0(z)$ arises from the density gradient. The ions are treated as a fixed nonuniform neutralizing background. Weissglas[3] has shown that in Eq. (11), the fourth term can be neglected in comparison with the first term provided that the condition $\lambda_D^2/\lambda L\ll 1$ is satisfied. In other words, if the square of the Debye length is much smaller than the product of the wavelength and the plasma dimension, the variation of the distribution function resulting from the static electric field arising out of the density gradient is much smaller than the local time variation of the distribution function. Following Weissglas the odd and even functions (in velocity) are defined:

$$F_0(z,v,t)=f(z,v,t)-f(z,-v,t), \tag{14}$$

$$F_e(z,v,t)=f(z,v,t)+f(z,-v,t). \tag{15}$$

The plasma is situated in an idealized "square-well" magnetic field such that $f(0, v, t)=f(0, -v, t)$ and $f(L, v, t)=f(L, -v, t)$ which leads to the following boundary conditions on the odd part of the distribution function: $F_0(0, v, t)=F_0(L, v, t)=0$. The plasma neutrality imposes the following boundary conditions on the electric field: $E(0, t)=E(L, t)=0$. In terms of the odd function F_0 and if $\exp[j\omega t]$ is assumed to be the time variation of the perturbations, Eqs. (11) and (12) become

$$F_0+\frac{v^2}{\omega^2}\frac{\partial^2 F_0}{\partial z^2}+\frac{2jq}{m\omega}E\frac{\partial f_0}{\partial v}=0 \tag{16}$$

and

$$E=-\frac{jq}{\epsilon_0\omega}\int_0^{\infty} vF_0\,dv. \tag{17}$$

Now expand F_0 and E in sine series and assume the following plasma density variation:

$$N_0(z)=N_{0p}[1+\nu\sin^2(\pi z/L)]; \tag{18}$$

then, the elimination of F_0 gives the following recurrence

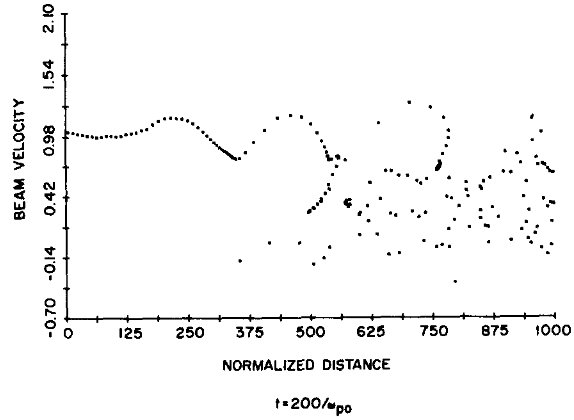

FIG. 2. Electron beam velocity versus distance for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.9$.

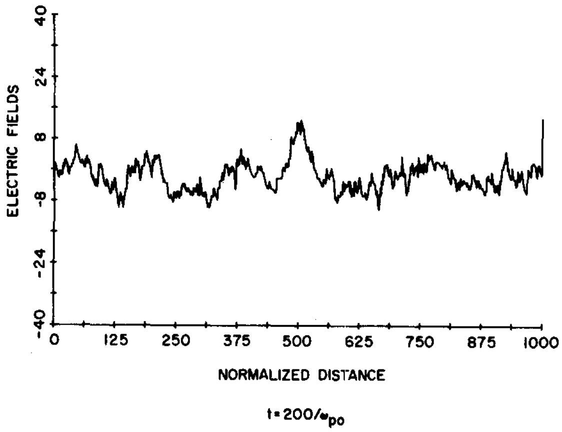

FIG. 3. Electric field versus distance for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.9$.

relation for the electric field:

$$\epsilon_k E_k - [\nu(\epsilon_k - 1)/2(2+\nu)](E_{k+2}+E_{k-2}) = 0, \quad (19)$$

where

$$\epsilon_k = 1 + \frac{\omega_{p0}{}^2 L}{\omega\pi k}\left(1+\frac{\nu}{2}\right)\int_{-\infty}^{\infty}\frac{(\partial g/\partial v)\,dv}{1-(v\pi k/\omega L)}. \quad (20)$$

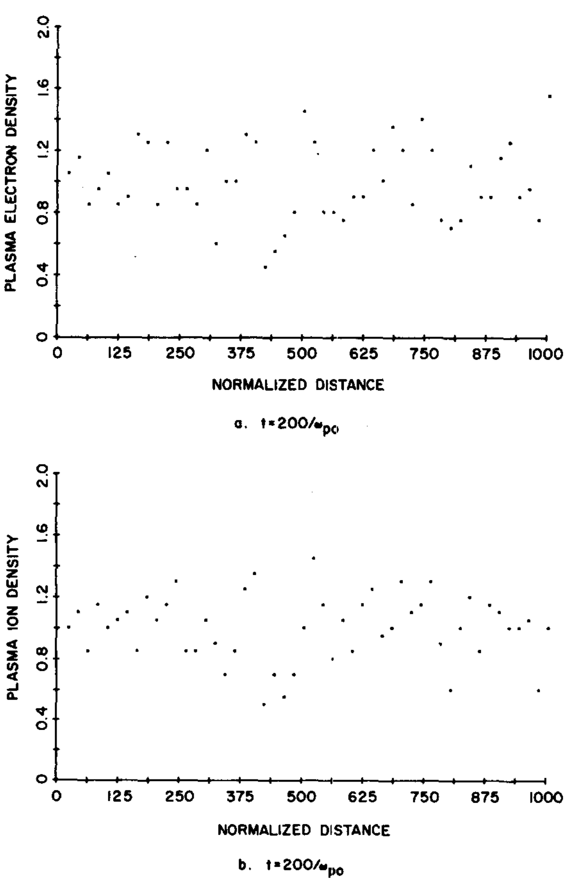

FIG. 4. Plasma electron and ion density distributions for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.9$.

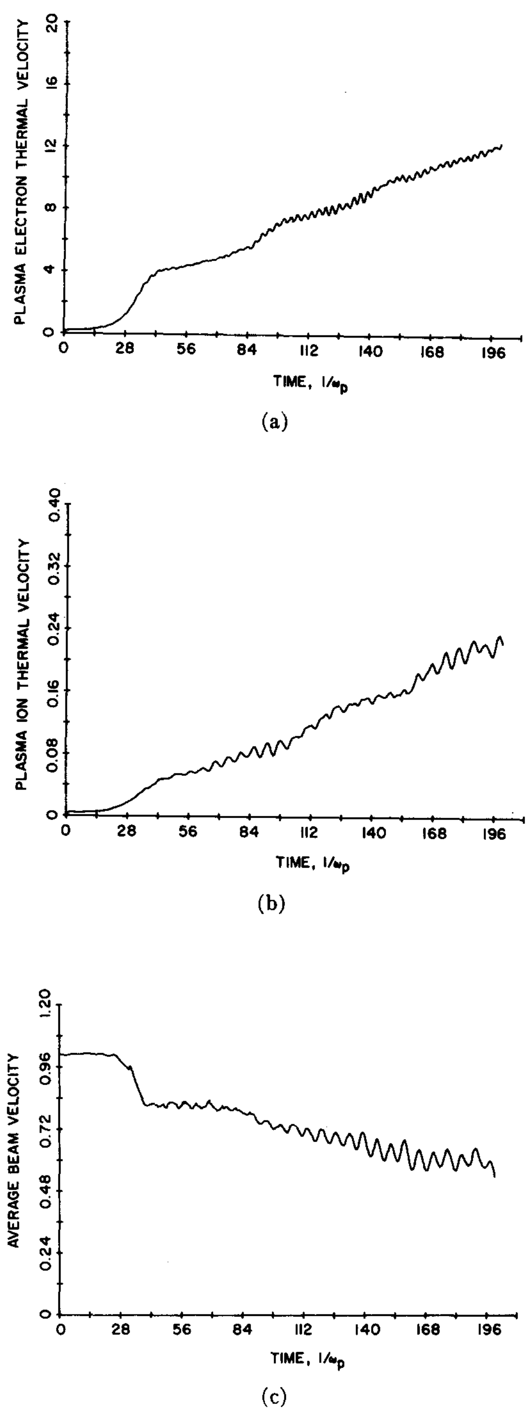

FIG. 5. Time evolution of plasma electron and ion thermal velocity and average beam velocity for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.9$. (a) Plasma electron thermal velocity versus time; (b) plasma ion thermal velocity versus time; (c) average beam velocity versus time.

Under the condition that the damping rate is small in comparison with the frequency ω, the perturbation method can be used to investigate the effects of plasma density inhomogeneities on Landau damping. In the

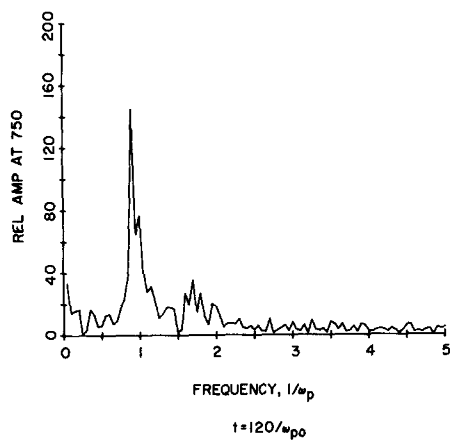

FIG. 6. Frequency spectrum of the electric field at $x=750$ $(0.02\ V_{01}/\omega_{p0})$.

limit of long wavelength the real part of Eq. (19) gives

$$\left[1-\frac{\omega_r^2}{\omega_{p0}^2(1+\frac{1}{2}\nu)}+\frac{3\pi^2k^2}{L^2\omega_{p0}^2}V_T^2\right]E_k$$

$$-\frac{\nu}{2(2+\nu)}(E_{k+2}+E_{k-2})=0, \quad (21)$$

and the imaginary part of Eq. (19) gives the damping rate as

$$\frac{\omega_i}{\omega_r}=\sum (E_k^r)^2\,\tfrac{1}{2}(\tfrac{1}{2}\pi)^{1/2}\left(\frac{L}{\pi k}\right)^3\frac{\omega_r^3}{V_T^3}\exp\left[-\left(\frac{L}{\pi k}\right)^2\frac{\omega_r^2}{2V_T^2}\right], \quad (22)$$

where $\omega=\omega_r+i\omega_i$. Equation (21) can be identified as the recurrence relation among the coefficients of the Mathieu equation[4] which in standard form is written as

$$\frac{d^2E}{d\xi^2}+(b-2s\cos 2\xi)E=0, \quad (23)$$

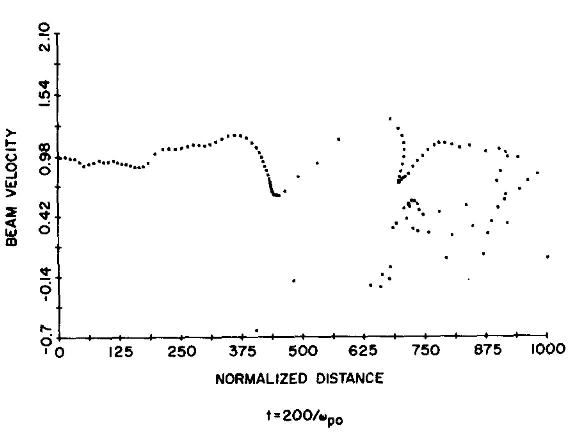

FIG. 7. Electron beam velocity versus distance for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.5$.

where

$$b=(L^2\omega_{p0}^2/3\pi^2V_T^2)\{[\omega_r^2/\omega_{p0}^2(1+\tfrac{1}{2}\nu)]-1\} \quad (24)$$

and

$$s=-(L^2\omega_{p0}^2/3\pi^2V_T^2)[\nu/2(2+\nu)]. \quad (25)$$

For each Mathieu function $S_{e_r}(\xi, s)$, b is a function of s, and therefore the resonance frequency ω_r for a given value of the inhomogeneity parameter ν can be determined from Eqs. (24), (25), and the function $b(s)$.

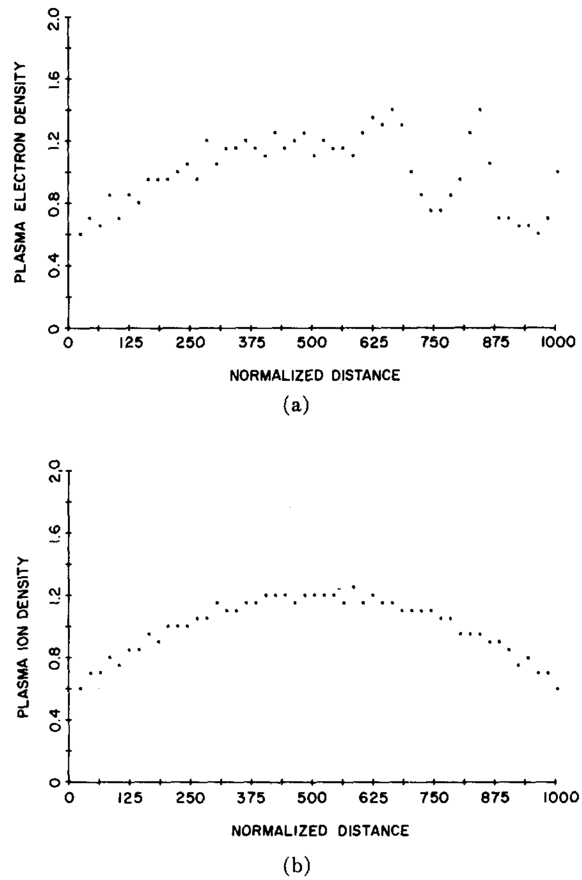

FIG. 8. Plasma electron and ion density distributions for an electron beam-plasma system with $(N_{02})_{\min}/(N_{02})_{\max}=0.5$. (a) Plasma electron density distribution at $t=40/\omega_{p0}$; (b) plasma electron density distribution at $t=40/\omega_{p0}$.

The corresponding Landau damping rates can be calculated through Eq. (22). The resulting values of ω_i/ω_r and ω_r/ω_{p0} as a function of ν for the lowest-order mode with $L/\lambda_D=25$ are shown in Fig. 1. It is observed that the Landau damping rate is enhanced by the particular form of inhomogeneity in density profile investigated here.

III. RESULTS OF NUMERICAL EXPERIMENTS

Two numerical experiments have been carried out in order to study the effects of plasma inhomogeneities

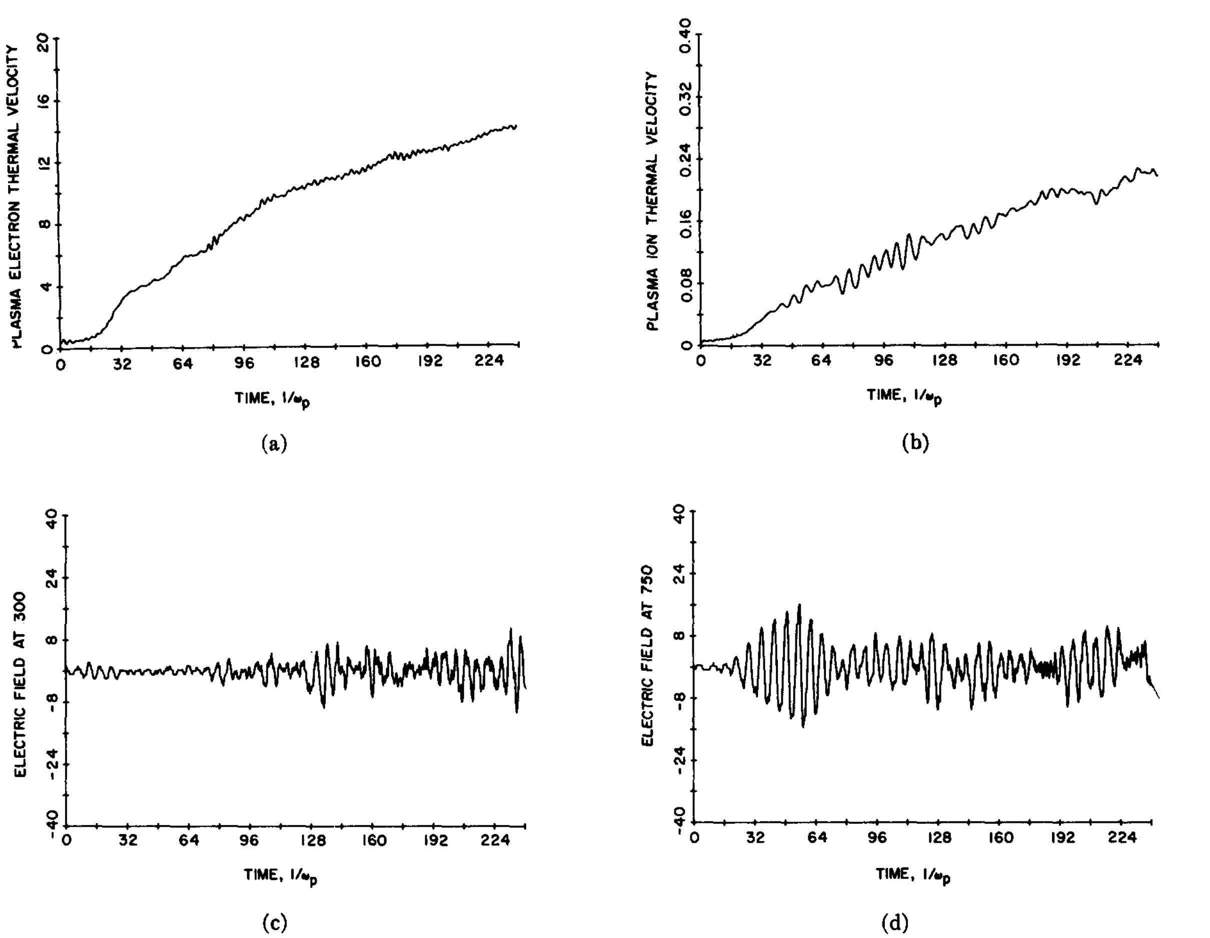

FIG. 9. Time evolution of plasma and ion thermal velocity, electric field at $z=300$ ($0.02\ V_{01}/\omega_{p0}$) and $z=750$ ($0.02\ V_{01}/\omega_{p0}$) for an electron beam-plasma system $(N_{02})_{min}/(N_{02})_{max}=0.5$. (a) Plasma electron thermal velocity versus time; (b) plasma ion thermal velocity versus time; (c) electric field at $z=300$ ($0.02\ V_{01}/\omega_{p0}$) versus time; (d) electric field at $z=750$ ($0.02\ V_{01}/\omega_{p0}$) versus time.

on the electron beam-plasma instability and the plasma heating process. In this investigation a charge sheet model[5] for electrons and ions is used to follow the time-dependent, nonlinear, and inhomogeneous evolution of the beam-plasma instability in a finite-length system. The model contains 1000 sheets of plasma electrons, 1000 sheets of plasma ions, and on the average 100 sheets of beam electrons. Electron beam sheets are injected continuously at the left of the system, pass through the plasma region, and are collected at the right. The system is situated in an idealized "square-well" mirror magnetic field. The mirror ratio is assumed to have such a value that all beam sheets are lost at the ends, whereas plasma sheets are reflected back into the system. This is equivalent to the assumption that all beam-sheet velocities lie in the loss-cone region and all plasma sheet velocities lie outside of it. The electron beam density is chosen to be approximately 0.025 that of the plasma density and the beam is 3% velocity modulated at the electron plasma frequency at the entrance plane. The normalized beam dc velocity is taken to be 50 which gives approximately three nominal wavelengths ($2\pi V_0/\omega_{p0}$) of interaction length in the system.

Initially, the plasma electrons and ions are cold. A parabolic plasma density distribution with its maximum at the center of the interaction region and minima at each end is generated for both plasma electrons and ions by initially assigning appropriate intersheet spacings. The ion to electron mass ratio is taken to be 100 to 1 in the numerical experiments rather than the realistic value, because in the latter case a prohibitive amount of computer time would be required to observe ion participation in the beam-plasma interaction. Figures 2–10 show some of the resulting numerical results. The ratios of maximum to minimum plasma density are 0.9 and 0.5 for runs 1 and 2, respectively. In the calculations, distance is normalized to the average plasma intersheet spacing which is equal to ($0.02V_0/\omega_{p0}$), time to $1/\omega_{p0}$, and acceleration to $0.02\ V_0\omega_{p0}$.

The first half of the interaction region corresponds

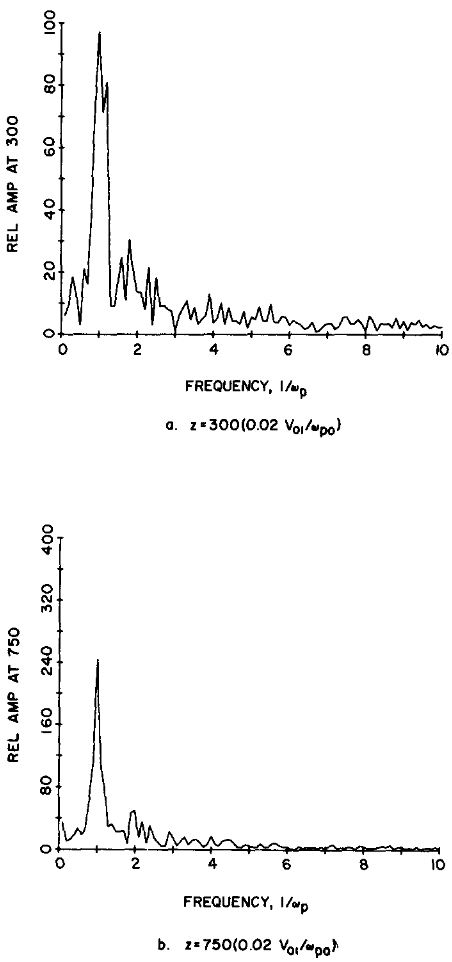

FIG. 10. Frequency spectrum of the electric field at $z=300$ and $z=750$, ($t=200/\omega_{p0}$).

qualitatively to the case considered in the analytic study except that a parabolic density distribution has been generated instead of a linear density variation. In the initial stages of the interaction the electron plasma oscillations excited by the passage of the electron beam through the plasma have been observed and the resulting beam velocity variations (Figs. 2 and 7) and the electric field illustrate that the increase in the inhomogeneities of the plasma density distribution does reduce the spatial growth rate of the dominant mode in the beam-plasma interaction. However, at a later time, the electron plasma oscillations die out. The mechanisms of beam-plasma instability and the resulting phenomena such as harmonic generation, amplitude modulation, and wave–wave interaction are discussed in detail in Ref. 6.

On the other hand, there is no appreciable difference in the time evolution of the plasma electron and ion thermal velocities for the two experiments. This might be attributed to the fact that the mechanism responsible for plasma heating is not Landau damping since the plasma sheet velocity is much less than the phase velocity of the waves in both experiments. The frequency analysis for the electric field at $z=300$ $(0.02V_0/\omega_{p0})$ (Fig. 10), shows that there are satellite frequencies $(\omega_{p0}\pm0.1\omega_{p0})$ generated around the fundamental frequency, whereas at $z=750$ $(0.02V_0/\omega_{p0})$ only harmonics of the electron plasma oscillations have been generated.

To verify the theory on the effects of plasma density inhomogeneities on the Landau damping rate and the oscillation period, a special form of plasma density variation [Eq. (18)] is simulated by assigning the appropriate intersheet spacings to pairs of electron and ion charge sheets. The electron beam is turned off and the initial perturbation is excited by giving each electron sheet the following velocity[7]:

$$v_i=v_{i0}+\epsilon V_T \sin(ik\pi/L), \qquad (26)$$

where v_{i0} is the unperturbed velocity of the ith sheet, $V_T=40$ is the normalized thermal velocity, $\epsilon=0.1$ is a dimensionless perturbation, and $k=5$ is the wavenumber. The Debye length is defined as $\lambda_D=V_T/\omega_{p0}$. In this case λ_D is equal to $(0.8\ V_0/\omega_{p0})$ which contains 40 plasma sheets and corresponds to $L/\lambda_D=25$. The behavior of the voltage across the system is shown in Fig. 11 for $\nu=0.1$ and 0.5. For the case of $\nu=0.1$, the theoretical predictions are $\omega_r/\omega_{p0}=1.05$ and $\omega_i/\omega_r=1.975\times10^{-3}$, where the numerical simulations give $\omega_r/\omega_{p0}\approx1.02$ and $\omega_i/\omega_r<10^{-2}$. For the case of $\nu=0.5$, the theory predicts $\omega_r/\omega_{p0}=1.175$ and $\omega_i/\omega_r=3.42\times10^{-2}$, whereas the numerical simulation yields $\omega_r/\omega_{p0}\approx1.18$ and $\omega_i/\omega_r\approx3\times10^{-2}$. Thus, it appears that the numerical experimental results are in good agreement with the theory.

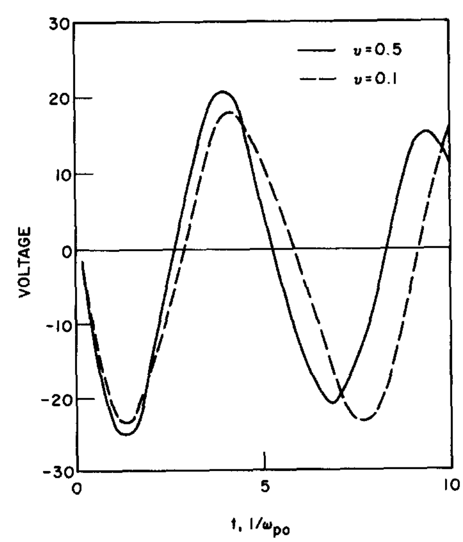

FIG. 11. Time evolution of the voltage across the system for $L/\lambda_D=25$.

IV. SUMMARY

The two-component charge sheet model of a plasma system is used to investigate the effects of inhomogeneity in the plasma density distribution on beam-plasma interaction and Landau damping in a finite-length one-dimensional system. The theoretical predictions that an increase in the inhomogeneity will reduce the spatial growth rate of the dominant mode in the beam-plasma interaction and enhance the Landau damping rate are demonstrated by the numerical experiments and the results are in good agreement with theory.

ACKNOWLEDGMENT

The authors acknowledge support from National Science Foundation Grant No. GK-15689.

[1] J. A. Davis, Ph.D. dissertation, Massachusetts Institute of Technology (1969).
[2] E. A. Jackson and M. Raether, Phys. Fluids **9**, 1257 (1966).
[3] P. Weissglas, J. Nucl. Energy **C6**, 251 (1964).
[4] *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series No. 55 (U.S. Government Printing Office, 1964), p. 721.
[5] J. M. Dawson, Phys. Fluids **5**, 445 (1962).
[6] A. T. Lin and J. E. Rowe, Phys. Fluids **15**, 166 (1972).
[7] J. M. Dawson and R. Shanny, Phys. Fluids **11**, 1506 (1968).

THE PHYSICS OF FLUIDS VOLUME 15, NUMBER 1 JANUARY 1972

Nonlinear Evolution of the Dory–Guest–Harris Instability

GLENN JOYCE AND GEORG KNORR
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52240
(Received 6 May 1971)

The nonlinear evolution of the Dory–Guest–Harris type instability is investigated by numerical integration of the Vlasov equation in one spatial and two velocity dimensions. Two methods are used: a particle simulation and the Fourier–Hermite transform. The two methods are compared for $f_0(v_\perp)=(2\pi 5!)^{-1}\times(v_\perp/\sqrt{2})^{10}\exp(-v_\perp^2/2)$ for k values which are unstable. A method of predicting the limiting amplitude of the electric field is presented for the case in which one mode dominates the system.

I. INTRODUCTION

The linear behavior of the Dory–Guest–Harris instability is well known.[1,2] Recently, a numerical study of the nonlinear behavior was made by Byers and Grewal[3] using a particle simulation model of the plasma. They found that the electric field saturates after reaching a certain level as is expected and thereafter the field rises and falls with a period of roughly the cyclotron period.

We concentrate on a rather special type of Dory–Guest–Harris instability, that of zero frequency. We model this instability in two ways; by Fourier–Hermite transformations of the Vlasov equation and by particle simulation. The simplifying assumption is made that it suffices to consider only one spatial dimension (x) and two velocity dimensions (v_x, v_y) while the magnetic field is assumed to be constant and directed along the z axis.

Sections II and III present a general outline of the two methods, Sec. IV compares the results, and Sec. V attempts an explanation of the electric-field saturation.

II. THE NONLINEAR VLASOV EQUATION

If we consider a constant magnetic field directed along the z axis and all quantities to be functions only of the spatial dimension x, and the velocity dimensions v_x, v_y, the Vlasov–Poisson system can be written in dimensionless variables as

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}+E_x\frac{\partial f}{\partial v_x}+R\left(v_y\frac{\partial f}{\partial v_x}-v_x\frac{\partial f}{\partial v_y}\right)=0,$$

$$\frac{\partial E_x}{\partial x}=1-\int f\,dv_x\,dv_y, \qquad (1)$$

where t is in units of ω_p, v is in units of the thermal velocity, x is in units of the Debye length, and $R=\omega_c/\omega_p$, the ratio of the cyclotron frequency to the plasma frequency.

The Vlasov equation may be Fourier transformed in x, v_x, v_y by the transformation (compare Ref. 4)

$$F_n(y_1,y_2,t)=\int_{-\infty}^{\infty}dv_x\int_{-\infty}^{\infty}dv_y\int_0^L dx\,\exp(ink_0x)\times\exp[(v_xy_1+v_yy_2)]f(x,v_x,v_y,t) \qquad (2)$$

which yields

$$\frac{\partial F_n}{\partial t}+nk_0\frac{\partial F_n}{\partial y_1}+R\left(y_2\frac{\partial F_n}{\partial y_1}-y_1\frac{\partial F_n}{\partial y_2}\right)-i\sum_{m=-\infty}^{\infty}E_my_1F_{n-m}=0, \qquad (3)$$

$$ik_0mE_m=F_m(0,0,t).$$