A High Speed Product Integrator

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A method of employing the components of a high speed analog computer to evaluate product integrals is presented. Kernel functions of two variables are obtained with time as the variable of integration and the position of ganged potentiometers as the parametric variable. High speed multipliers, function generators, adders, and integrators permit the evaluation of a one-hundred point product integral curve every 1.67 seconds. This curve is displayed on a cathode-ray tube screen having a long persistence P-7 phosphor. Examples of sine and cosine transforms evaluated with the product integrator are given. The observed errors range from 1 to 10 percent, depending upon the problem.

IN the fields of engineering and physics, one is frequently confronted with the problem of evaluating product integrals of the form

$$F(y)=\int_0^{T \text{ or } t} f(t)K(y,t)dt, \qquad (1)$$

where $f(t)$ and the kernel $K(y, t)$ are known functions, and y is a variable parameter. Well-known equations of this form are the Fourier integral equation and the convolution or superposition integral.

The evaluation of product integrals by machine methods was begun at the Massachusetts Institute of Technology in 1928.[1] This work culiminated in 1940 in the completion of the cinema integraph.[2,3] More recently, a more general machine capable of solving a wide range of integral equations has been proposed.[4]

This paper describes how the components of an electronic differential analyzer can be applied to this problem. The product integrator described is capable of evaluating a one-hundred point curve for $F(y)$ every 5/3 seconds. This curve is displayed on the face of a cathode-ray tube having a long persistence P-7 screen.

I. PRODUCT INTEGRATION

A block diagram of the set-up used to evaluate product integrals is shown in Fig. 1. $F(y)$ is evaluated for a series of discrete values of y by a step by step procedure. This is accomplished by setting $y=0$ and evaluating

$$F(0)=\int_0^T f(t)K(0,t)dt, \qquad (2)$$

then changing to $y=y_1$ and evaluating

$$F(y_1)=\int_0^T f(t)K(y_1,t)dt, \qquad (3)$$

and so forth. If a sufficiently large number of values of y are chosen over the range from 0 to y_{max}, an accurate determination of $F(y)$ over this range is obtained. An approximation to the function $F(y)$ can be obtained by varying the parametric variable y continuously. If y varies very slowly relative to the variable of integration t, the difference between $F(y)$ and this approximation will be small.

Three of the computing elements indicated: the integrator, the multiplier, and the function generator for generation of the function of one variable $f(t)$ are units commonly found in any analog computer. The particular units used by the author have been described previously elsewhere.[5] The important features of these units which facilitated their use in the product integrator are: (1) wide band widths and correspondingly short rise times, (2) repetitive operation at 60 cycles per second. All of the computing elements are switched on for a period of ten thousand microseconds and then off for sixty-seven hundred microseconds. Balanced diodes are used as the necessary electronic switches.[6] During the off portion of the computing cycle the electronic switches automatically restore the integrator to zero initial condition.

II. KERNEL GENERATION

The kernel $K(y, t)$ is a function of the two variables y and t. The development of a generator of such a function would be a very difficult problem if it were

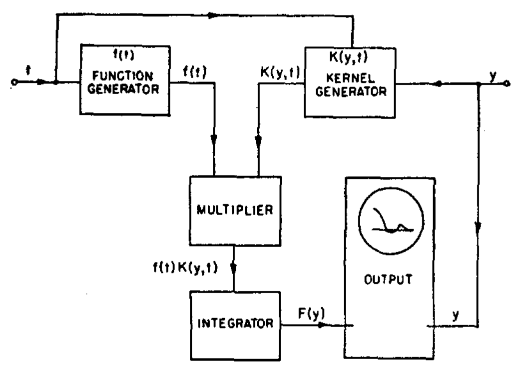

FIG. 1. Product integrator block diagram.

[1] K. E. Gould, J. Math. Phys. 7, 305 (1928).
[2] T. S. Gray, J. Franklin Inst. 221, 77 (1931).
[3] H. L. Hazen and G. S. Brown, J. Franklin Inst. 230, 19 (1940).
[4] H. Wallman, J. Franklin Inst. 250, 45–61.
[5] A. B. Macnee, Proc. Inst. Radio Engr. 37, 1315 (1949).
[6] K. R. Wendt, RCA Rev. 9, 85 (1948).

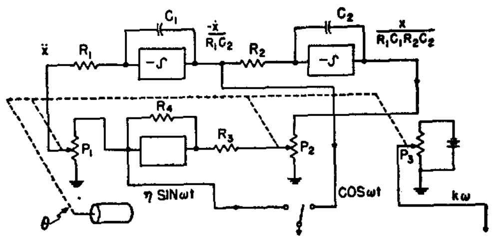

FIG. 2. Set-up for the generation of the kernel functions cosωt and sinωt.

not for one important characteristic of the kernel function, namely, that the parametric variable y need only vary slowly relative to the variable of integration t.

The output unit is an ordinary oscilloscope employing dc amplification and a cathode-ray tube having a long persistence screen. The intensity grid is gated on for two 50-microsecond periods during each computing cycle; once just before the beginning of the on period, when the output of the integrator is at zero, and again just before the end of the on period. Thus, for each computing cycle two points appear on the output screen, one giving the zero level and one giving the value of the integral.

Since the kernel $K(y, t)$ is a function of t alone over each integration from 0 to T, one way of obtaining many useful kernels is to use an electronic differential analyzer to generate the necessary functions of time and to introduce some way of slowly varying the required parameter in these functions. Since the parameter variation is slow, it can be produced by mechanical means such as a motor-driven potentiometer. For a fixed value of the parametric variable, the kernel is a function of t alone. If this function of time can be represented as the solution of an ordinary differential

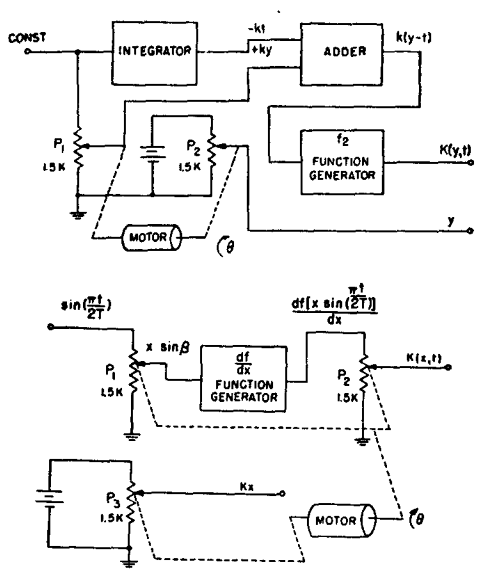

FIG. 3. Block diagrams of set-ups for the generation of typical kernel functions. (a) $K(y, t)=f(y\text{-}t)$, (b) $K(y, t)=y\cdot f[y\sin(kt)]$.

equation, the analyzer can be made to solve this equation directly.

A. Fourier Transforms

As an example of this technique of kernel generation, consider the Fourier transformation

$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt. \tag{4}$$

This integral of a complex quantity is conveniently broken down into two real product integrals

$$g(\omega)=g_1(\omega)+jg_2(\omega), \tag{5}$$

where

$$g_1(\omega)=\frac{1}{\pi}\int_0^{\infty} f_1(t)\cos\omega t dt \tag{6}$$

and

$$g_2(\omega)=\frac{1}{\pi}\int_0^{\infty} f_2(t)\sin\omega t dt. \tag{7}$$

The functions $f_1(t)$ and $f_2(t)$ are the even and odd parts respectively of the original time function $f(t)$. To evaluate the product integrals of Eqs. (6) and (7) it is necessary to generate the two kernel functions cosωt and sinωt. This is done by solving the differential equation

$$d^2x/dt^2=-\omega^2 x \tag{8}$$

on an electronic differential analyzer. The set-up for solving this equation is shown in Fig. 2. The potentiometers P_1 and P_2 are introduced to permit variation of the parametric variable ω. These potentiometers are capable of continuous rotation. They are geared together and driven by a synchronous motor, through a reduction gear box at a speed of 0.6 rps. The differential analyzer used by the author solves equations 60 times a second; therefore in one rotation of the potentiometers one-hundred kernel functions are obtained. The differential equation actually solved by the set-up of Fig. 2 is

$$\frac{d^2x}{dt^2}+\left[\frac{\theta}{2\pi}\right]^2\omega_0^2x=0, \quad 0\leq\theta\leq 2\pi \tag{9}$$

where

$$\omega_0^2=\frac{1}{R_1C_1R_2C_2}\frac{R_4}{R_3} \tag{10}$$

and θ= potentiometer shaft position in radians. Choosing initial conditions $x_0=0$, $\dot{x}_0=1$, voltages proportional to cosωt and sinωt become available at the points shown, with the parametric variable

$$\omega=\omega_0(\theta/2\pi). \tag{11}$$

As has been indicated, the differential analyzer components of Fig. 2 are turned on 60 times a second for a period of 1/100 sec. If the potentiometer shafts were held at a fixed value over this 1/100-sec period and then stepped to the next value during the off time of the analyzer, the sine and cosine kernels would be generated exactly. In practice, these potentiometer shafts are continuously moving even when the differential analyzer components are turned on. There is, therefore, a small variation of the parametric variable ω during each of the individual integrations from 0 to T seconds. If this variation in θ is considered, one finds that the set-up of Fig. 2 solves the differential equation

$$\frac{d^2x}{dt^2}+\left(\frac{\theta}{2\pi}\right)^2\left[1+\frac{\pi}{100\,\theta}\,\frac{t}{T}\right]^2\omega_0{}^2x=0, \qquad (12)$$

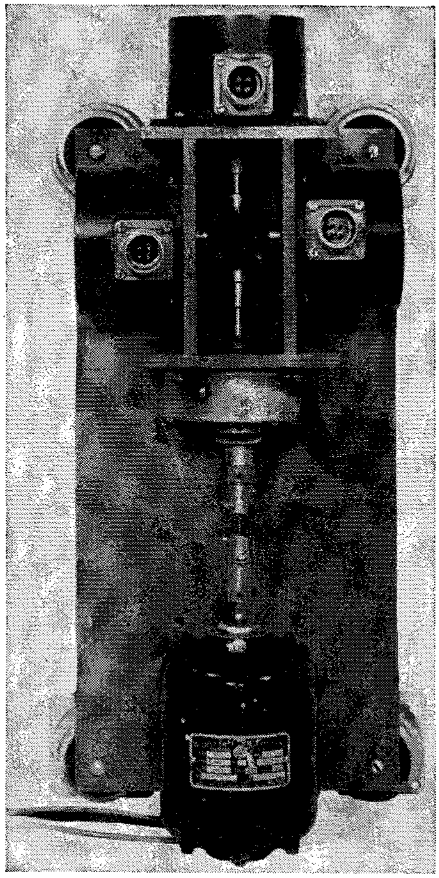

FIG. 4. Potentiometers and motor drive used to generate the parametric variable.

where now θ is constant over each integration cycle. This equation has the approximate solution, for $T=1/100$ sec,

$$x=C_1\cos\left[\frac{\theta}{2\pi}\omega_0 t+\omega_0\frac{t^2}{4}\right]+C_2\sin\left[\frac{\theta}{2\pi}\omega_0 t+\omega_0\frac{t^2}{4}\right]. \qquad (13)$$

By making C_1 or C_2 zero, either the sine or cosine kernels can be generated.

Blocking diagrams of the set-ups necessary to generate other useful kernel functions are given in Fig. 3. These

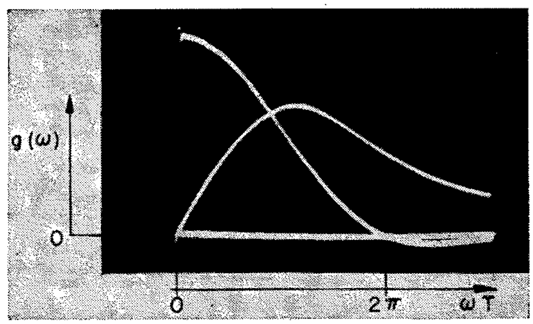

FIG. 5. Cosine and sine transforms of $f_1(t)$ as observed at the output of product integrator.

set-ups for the generation of $f(y\text{-}t)$ and $y\cdot f[y\sin(kt)]$ have been used successfully by the author.

III. RESULTS

The high speed product integrator described here has been used to evaluate integrals of the Fourier, Schlomilch, and convolution types.

A. Fourier Integral

The equipment of Fig. 1 used in conjunction with that of Fig. 2 is capable of evaluating sine and cosine integrals. Figure 4 is a photograph of the motor-driven potentiometers used by the product integrator.

Functions of time which lend themselves to analytic integration were evaluated in order to check the accuracy of the product integrator. The first of these is

$$f_1(t)=\frac{\pi}{T}[1+\cos(\pi/T)t] \quad \text{for } 0\leq t\leq T$$
$$f_1(t)=0 \quad \text{for } t\geq T, \text{ and } t\leq 0. \qquad (14)$$

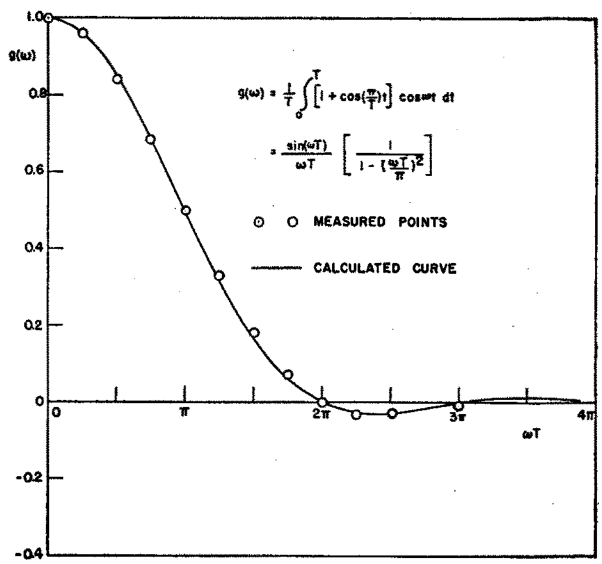

FIG. 6. Calculated and measured cosine transform of $f_1(t)$.

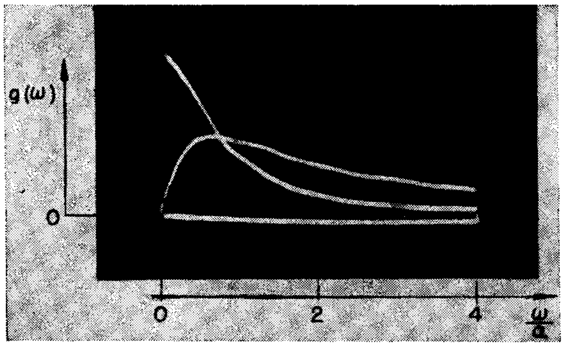

FIG. 7. Cosine and sine transforms of $f_2(t)$ as observed at the output of the product integrator.

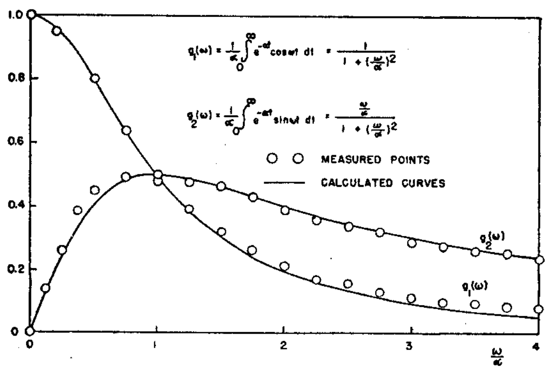

FIG. 8. Calculated and measured cosine and sine transforms of $f_2(t)$.

Figure 5 shows the cosine and sine transforms of this function as they are observed at the product integrator output. This is a double exposure photograph; normally only one of these curves is displayed. The calculated cosine transform for this time function is shown in Fig. 6. The points plotted in this figure were taken from Fig. 5 graphically. The negative for this figure was viewed on a microfilm viewer, and the points measured with a pair of dividers and a scale.

The second time function chosen is

$$f_2(t)=\frac{T}{6}e^{-6t/T} \quad \text{for } 0\leq t\leq T,$$
$$f_2(t)=0 \quad \text{for } t\leqq 0 \text{ and } t>T. \tag{15}$$

This is a difficult function for the product integrator to handle because $f_2(t)$ is rather close to zero over most of each integration period. The multiplier is, therefore, operating in a very unfavorable manner most of the time, and one might expect the errors to be high. The observed sine and cosine transforms are shown in Fig. 7. Figure 8 shows the measured curves and the calculated points. The error in this case is greater than that observed in Fig. 6, but is less than 10 percent over most of the output range.

This second example illustrates one of the applications of the cosine and sine integrals. If the function $f(t)$ is considered as the desired impulse response of a linear passive network, then the real and imaginary parts of the transfer impedance of this network are respectively the cosine transform and the negative of the sine transform of this $f(t)$. Thus the curves of Fig. 8 are the real part and the negative imaginary part of the impedance of a parallel RC network.

Another application of cosine transforms is the evaluation of the power density spectrum of experimentally observed autocorrelation functions. The high speed integrator has been used for this purpose by N. H. Knudtzon.[7] A typical autocorrelation function of some filtered

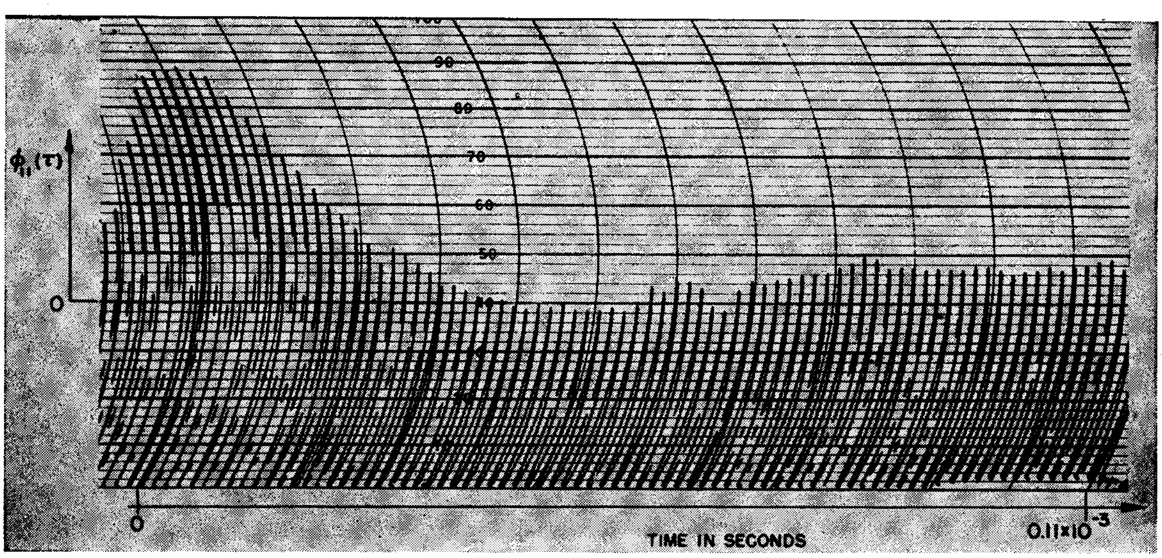

FIG. 9. Autocorrelation function of filtered random noise as measured by an electronic correlator.

[7] N. H. Knudtzon, Technical Report No. 115, Research Laboratory of Electronics, M.I.T. (1949).

random noise is shown in Fig. 9. The data for this curve were obtained with an electronic correlator developed at the Massachusetts Institute of Technology.[8] These data were plotted on an 8×11-in. piece of paper, and photographically reduced to a mask for the arbitrary function generator. Figure 10(a) shows the output of the arbitrary function generator. Figure 10(b) is the power density curve for this correlation function as observed at the output of the product integrator.

B. Errors

Of the four computing elements indicated in Fig. 1 the two most subject to error are the multiplier and the kernel generator. The time integrator can easily be made accurate to within 0.1 percent. The photocell-feedback arbitrary-function generator used was accurate to within 2 percent of the maximum output. This maximum output corresponded to a deflection of two and one-half inches on the face of a 5UP11 cathode-ray tube. Greater accuracy than this could be achieved by suitably compensating the function masks.

The principal error in kernel generation as described in Section II is caused by the continuous rotation of the parametric variable potentiometers. This results in an error which would not be encountered if the parametric variable were varied in a quantized manner during the off time of the integration cycle. This source of error has been investigated in some detail for the cosine and sine transforms.[9] In this case the kernel functions obtained are those given by Eq. (13). In this equation the frequency ω_0 is the maximum value of the parametric variable ω, and the kernel function error is largest when the ω_0 is large. This source of error can always be reduced by reducing the rate of change of the parametric variable. For the cosine and sine transforms this would amount to reducing ω_0 or modifying the set-up of Fig. 2 to give a range of frequencies from ω_1 to ω_2 instead of 0 to ω_0.

The other important source of error in the product integrator is the multiplier. No analog multiplier is capable of perfect zero adjustment. To a first approximation the output of any multiplier having inputs x and y will be

$$F(x, y) = x \cdot y + a_1 x + a_2 y + a_3. \tag{16}$$

When such a multiplier is used in a product integrator, the product integrator output will be

$$F'(y) = F(y) + E_1(y) + E_2, \tag{17}$$

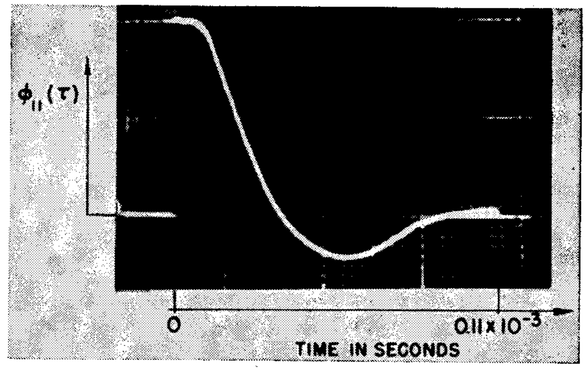

FIG. 10(a). Autocorrelation function of Fig. 9 as generated by cathode-ray tube function generator.

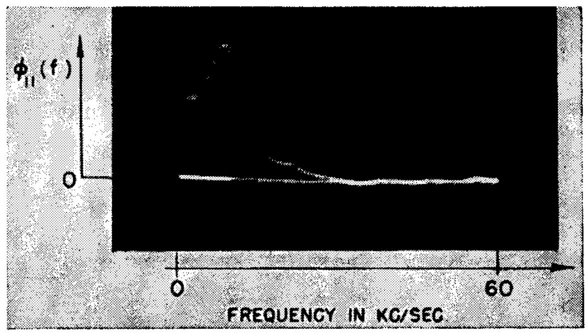

FIG. 10(b). Power density spectrum observed at output of product integrator by taking the cosine transform of the correlation function shown in Fig. 10(a).

where $F(y)$ is the desired output,

$$E_1(y) = a_2 \int_0^T K(y, t) dt, \tag{18}$$

and

$$E_2 = \int_0^T [a_1 f(t) + a_3] dt. \tag{19}$$

The second of these error terms E_2 is independent of the parametric variable y and can, therefore, be subtracted from the output of the integrator shown in Fig. 1. This is easily done experimentally by adding a constant to give zero product integrator output with $K(y, t)$ disconnected from the multiplier input.

The first error $E_1(y)$ being a function of y is not so easily removed. It can be evaluated analytically by Eq. (18), or experimentally using the product multiplier itself to do the work! This error is not a function of $f(t)$; it is minimized by making the desired $F(y)$ as large as possible without overloading the multiplier. This means the scale of the $f(t)$ voltage should always be such as to use all of the available multiplier dynamic range. This last effect is the principal source of error in Figs. 6 and 8.

[8] T. P. Cheatham, Technical Report No. 122, Research Laboratory of Electronics, M.I.T. (1949).
[9] A. B. Macnee, Technical Report No. 136, Research Laboratory of Electronics, M.I.T. (1949).