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CONTRIBUTIONS TO ESTIMATION IN
A CLASS OF DISCRETE DISTRIBUTIONS

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INTRODUCTION

General Background

Different approaches are possible with regard to the basic structure of the standard discrete distributions like the binomial, Poisson, negative binomial, or logarithmic series. Under plausible assumptions, these distributions may be regarded as descriptive models of populations. For instance, under the usual genetic theory, the distribution of the number $x$ of boys in families of a fixed number of children, say $n$, follows the binomial law. Or again, the number of mistakes per printed page can ordinarily be assumed to have a Poisson distribution. The same is true of the distribution of accidents met with, over a period of time, by a particular individual. However, different persons may have different accident proneness as measured by the average number of accidents to the individual. If this average has, say, a Pearson's type III distribution, the distribution of the number of accidents pooled over the individuals can be shown to follow the negative binomial law. A limiting case of this is the logarithmic series distribution found useful in ecology.

Again, these distributions may arise as a result of the sampling scheme adopted. In sampling with replacement $n$ items from a lot of manufactured items, the number of defectives follows the binomial law. If the proportion of defectives in the lot is small and the number of items sampled is moderately large, one can use the Poisson approximation for the distribution of the number of defectives. On the other hand, if one uses what is known as the inverse binomial sampling procedure (that is, one goes on sampling with replacement until he gets a
fixed number such as k of defectives), the number of items sampled would then follow the negative binomial distribution.

The discrete distributions described above occur sometimes in truncated or censored forms. For instance, in human genetics, to estimate the proportion of albino children produced by couples capable of producing albinos, sampling has necessarily to be restricted to families having at least one albino child. This is because there is no way of distinguishing families incapable of producing albinos from those that are capable but by way of chance have not produced any. The number of albino children x, if sampling is restricted to families of n children, can thus take the values 1, 2, ..., n; the value 0 being excluded. Thus, x follows what is known as the "truncated" binomial distribution--truncated on the left at 1 to be specific. We may similarly think of distributions with both extremes truncated.

Again, some types of counters can record the exact number of radio-active particles (emitted by a radio-active substance in fixed intervals of time) if the number does not exceed a certain limit such as d, otherwise it merely records that the number has exceeded d. Data obtained from such counters are samples from what are called "censored" distributions, censored on the right at d. In such cases, individual counts of all observations not exceeding d are available; only the total count of those exceeding d.

**Review of Previous Work in Estimation in Discrete Distributions**

The problems of estimation of parameters in various discrete distributions and their truncated or censored forms have been considered
by various authors. Generally, the estimate for the case of complete
distributions is neat and easy to compute, but complications come in
for truncated and censored distributions.

For the truncated binomial distribution, Fisher (1936) and
Haldane (1932, 1938) gave the maximum likelihood procedure for estimating the parameter of the distribution. While studying the albinism in
man by sampling from families with variable number of children (but
having at least one albino), Haldane (1938) solved by an elaborate iterative process the complicated maximum likelihood equation based on data obtained simultaneously from different truncated binomial populations. Finney (1949) utilized the method of scores for solving the likelihood equation and provided some tables to facilitate the heavy computation. Because of computational difficulties in getting the maximum likelihood estimate, Moore (1944) suggested an alternative simple estimate which is a ratio of two suitably chosen linear functions of frequencies, whereas Rider (1955) equated the first two sample moments to the corresponding population moments and obtained another simple estimate. The sampling properties of these two simple estimates, however, have not been studied.

Estimation problems in relation to the Poisson distribution have been investigated by many authors. For truncated Poisson distributions, the case of truncation on the left has been considered by David and Johnson (1948) who provided the maximum likelihood estimate and a small numerical table for computational facility. Plackett (1953) gave a simple unbiased and highly efficient estimate which is a ratio of two linear functions of frequencies. Rider (1953) used first two moments and obtained another simple estimate, but did not study its sampling properties.
Truncation on the right has been discussed by Tippett (1932), Bliss (1948) and Moore (1952). Tippett derived the maximum likelihood solution, Bliss developed an approximation to it and Moore suggested a ratio of two linear functions of frequencies as a simple alternative estimate. For estimation in case of doubly truncated Poisson distributions, Cohen (1954) provided maximum likelihood equations, though rather unwieldy to solve; whereas Moore utilized a ratio of suitably constructed linear functions of frequencies to estimate the Poisson-parameter. They also discussed the problems of estimation for samples from censored Poisson distributions. Moore (1952) gave the simple ratio-estimate and Cohen (1954) derived maximum likelihood equations for both singly and doubly censored distributions.

For a negative binomial distribution involving two parameters, Fisher (1941) discussed the efficiency of moment-estimates and derived maximum likelihood equations for simultaneous estimation. He also gave a simple rule as to when one should proceed for getting maximum likelihood estimates. Haldane (1941) reduced the likelihood equations to a simpler form for computational facility. Sampford (1955) gave methods to obtain moment-estimates and likelihood estimates for a truncated negative binomial distribution with "zero" truncated, whereas equating the first three sample moments to the corresponding moments of the truncated negative binomial distribution, Rider obtained simple estimates for the two parameters.

Results for estimation of the parameter of a logarithmic series distribution are rather complicated, even when the distribution is complete. The estimation problems do not seem to have been thoroughly
investigated. Fisher, Corbet and Williams (1943) found it useful in ecology and derived the maximum likelihood estimate of its parameter.

We thus see that previous work on estimation in these discrete distributions can be broadly classified under two heads: (1) estimation by the method of maximum likelihood, and (2) other methods of estimation, the need for the other methods arising from the fact that frequently the method of maximum likelihood leads to complicated equations for estimation.

For maximum likelihood estimation, the authors have derived the estimating equations in individual cases and have suggested the use of Fisher's iterative procedure based on "efficient scores" for the solution of the equations when these turn out to be complicated. Some numerical tables are provided here and there to help in the process of solution. The identity of the maximum likelihood and the moments method has been noticed in a few cases.

Other estimates suggested in individual cases to avoid the computational difficulties of maximum likelihood estimation are of two types. One is derived by equating the first two sample moments with the corresponding theoretical moments. The second is obtained by taking the ratio of two suitably constructed linear functions of frequencies such that the ratio of the expectation of the numerator to that of the denominator is the required parameter. These two types of estimates are easy to compute in the cases suggested, but one has to remember that they are, in general, biased and inefficient though consistent. We note that the sampling properties of these estimates have not been investigated by the previous authors.
Present Contributions to Estimation in Discrete Distributions

It is first shown that the binomial, Poisson, negative binomial and logarithmic series distributions can be regarded as special cases of a general class of discrete distributions which we refer to as generalized power series distribution (gpsd). It is then possible to examine the previous work on estimation in the case of the above discrete distributions from a general point of view. The approach in this thesis is to derive results for this general class of distributions and then apply them to the special cases of binomial, Poisson, etc.

To begin with, we present a few results which bring out some interesting properties of a gpsd. We discover an explicit functional relationship between the variance and mean of a gpsd and based on this fundamental relation, we present some characterization theorems. To mention one, we establish that the equality of variance and mean is necessary and sufficient for a gpsd to be Poisson.

Next, we investigate certain estimation problems connected with a gpsd. We show that the maximum likelihood method and the method of moments give the same estimate when the gpsd involves a single parameter. A computational method for evaluating the maximum likelihood estimate is developed which requires only a table of values of the mean of the gpsd for various values of the parameter at sufficiently close intervals. It is shown how the standard error of the estimate can be approximately evaluated by using this table. The formulae for the amount of bias in the likelihood estimate are obtained to the order of 1/N where N is the sample size.
Large sample methods based on maximum likelihood are then derived for testing the homogeneity of several distributions and providing the estimate for the common parameter in case the distributions are homogeneous. The likelihood equation and a method for solving it are derived for the problem of estimation in censored forms of a gpdf. Methods based on the maximum likelihood principle are given for the treatment of doubtful observations. The problem of estimation when the gpdf involves two parameters has been considered.

For the gpdf, in addition to the maximum likelihood estimate, two other simple estimates are provided. One is called the "two-moments estimate" and is derived by equating the first two sample moments to the corresponding population moments. The other estimate is called the "ratio estimate" as it is obtained by taking the ratio of two suitably constructed linear functions of frequencies such that the ratio of the expected value of the numerator to that of the denominator is equal to parameter. Expressions are derived for the bias and variance of these two estimates correct to terms of order $1/N$ where $N$ is the sample size.

Lastly, the results obtained by general approach are applied to specific distributions - namely, the binomial, Poisson, negative binomial, and logarithmic series. In each case, exhaustive numerical tables are given to facilitate computation of the maximum likelihood estimate. The bias and efficiency of the "ratio" and "two-moments" estimates are numerically evaluated for different values of the parameter and recommendations are given for the suitability of the different methods of estimation. Illustrative examples have been worked out in detail to illustrate the methods suggested.
CHAPTER I

1.0 A CLASS OF DISCRETE DISTRIBUTIONS AND CERTAIN CHARACTERIZATION THEOREMS

1.1 Introduction

Let \( g(\theta) \) be a positive function admitting a power series expansion with non-negative coefficients for non-negative values of \( \theta \) smaller than the radius of convergence of the power series:

\[
g(\theta) = \sum_{z=0}^{\infty} a_z \theta^z. \tag{1.1.1}
\]

Noack (1950) defined a random variable \( Z \) taking non-negative integral values \( z \) with positive probabilities

\[
\text{Prob}\{Z = z\} = \frac{a_z \theta^z}{g(\theta)} \quad (z = 0, 1, 2, \ldots \infty). \tag{1.1.2}
\]

He called the discrete probability distribution given by (1.1.2) a power series distribution (psd) and derived some of its properties relating its moments, cumulants, etc.

To be more general, we note that the set of values of an integral-valued random variable \( Z \) need not be the entire set of non-negative integers \((0, 1, 2, \ldots \infty)\). For, let \( T \) be an arbitrary non-null subset of non-negative integers* and define the generating function

\[
f(\theta) = \sum_{x \in T} a_x \theta^x. \tag{1.1.3}
\]

with \( a_x \geq 0; \theta \geq 0 \) so that \( f(\theta) > 0 \), is finite and differentiable.

* In fact, one can take \( T \) to be a countable subset of real numbers; for purposes of this dissertation, however, \( T \) is chosen to be a subset of non-negative integers.
Then we can define a random variable $X$ taking non-negative integral values in $T$ with probabilities

$$P_X = \text{Prob} \left\{ X = x \right\} = \frac{a_x e^x}{f(\Theta)} \quad x \in T \tag{1.1.4}$$

and call this distribution analogously a generalized power series distribution (gpsd). It may be noted that gpsd reduces to a psd when $T$ is the entire set of non-negative integers. We add here that we call the set of admissible values of the parameter $\Theta$ of gpsd as the parameter space $\Theta$ of the gpsd. Also we refer to set $T$ of values of random variable $X$ defined by the gpsd, as the range $T$ of the gpsd.

Writing the mean $\mu = E(X)$, the crude moments $m_r = E(X^r)$, the central moments $\mu_r = E(X-\mu)^r$, the moment generating function (mgf) $M(t) = E(e^{tX})$ and the cumulants $K_r = \left[ \frac{d^r}{dt^r} \log M(t) \right]_{t=0}$ (in case they exist); we obtain, for a gpsd, the following relations derived on the same lines as shown by Noack (1950) for a psd:

$$\mu = \Theta f'(\Theta)/f(\Theta) \tag{1.1.5}$$

$$m_{r+1} = \Theta m'_r + \mu m_r \tag{1.1.6}$$

$$\mu_{r+1} = \Theta \mu'_r + r \mu_2 \mu_{r-1} \tag{1.1.7}$$

$$M(t) = f(\Theta e^t)/f(\Theta) \tag{1.1.8}$$

$$m_r = \sum_{i=1}^{r} \frac{(r-1)}{i-1} m_{r-1} K_i \tag{1.1.9}$$

$$K_{r+1} = \Theta \sum_{i=1}^{r} \frac{(r-1)}{i-1} m_{r-1} K'_i - \sum_{i=2}^{r} \frac{(r-1)}{i-2} m_{r+1-1} K_i \tag{1.1.10}$$

where primes denote differentiation with respect to $\Theta$. 
We note further that for a gpsd

\[ \mu = \Theta \frac{d}{d\Theta} [\log f(\Theta)] \]  
\[ \mu_2 = \Theta \frac{d \mu}{d\Theta} \]  
\[ K_{r+1} = \Theta \frac{dK_r}{d\Theta} \]  
(1.1.11) (1.1.12) (1.1.13)

Also, writing the factorial moment of order \( r \) as

\[ \mu(r) = E[(X)(X-1)\ldots(X-r+1)] \]

we can derive for a gpsd

\[ \mu(r) = \frac{\theta^r}{f(\theta)} \frac{d^r}{d\theta^r} [f(\theta)] \]  
(1.1.14)

\[ \mu(r+1) = (\mu-r)\mu(r) + \Theta \frac{d}{d\Theta} \left[ \mu(r) \right] \]  
(1.1.15)

The Binomial, Poisson, Negative Binomial and the Logarithmic Series distributions can be obtained as special cases of the gpsd by taking

\[ f(\Theta) = (1+\Theta)^n, \text{ n positive integer for Binomial} \]
\[ f(\Theta) = e^\Theta, \text{ for Poisson} \]
\[ f(\Theta) = (1-\Theta)^{-k}, \text{ k positive for Negative Binomial} \]
\[ f(\Theta) = -\log(1-\Theta), \text{ for Logarithmic Series.} \]

It is interesting to note that the Poisson and Negative Binomial distributions are special cases of a psd also; however, the Binomial and Logarithmic Series are not.

The relations (1.1.5) to (1.1.15) are generalizations of corresponding results obtained by various authors (Romanovsky, Frisch, Haldane) separately for Binomial, Poisson, Negative Binomial distributions.
1.2 Functional Dependence of Variance and Mean of a gbsd

In this section, we present a few results which bring out some interesting properties of a gbsd. We discover an explicit functional relationship between the variance and the mean of a gbsd and based on this relation, present some characterization theorems.

**Theorem 1:** For a gbsd, Variance = Mean + $\phi^2 \frac{d^2}{d\theta^2} [\log f(\theta)]$.

**Proof:** For a gbsd, we have from (1.1.12) and (1.1.11),

\[
\text{Variance } \mu_2(\theta) = \phi \frac{d}{d\theta} [\mu(\theta)] \text{ and }
\]

\[
\text{Mean } \mu(\theta) = \phi \frac{d}{d\theta} [\log f(\theta)].
\]

Consider

\[
\mu_2(\theta) = \phi \frac{d}{d\theta} [\mu(\theta)]
\]

\[
= \phi \frac{d}{d\theta} \left\{ \phi \frac{d}{d\theta} [\log f(\theta)] \right\}
\]

\[
= \phi \frac{d}{d\theta} [\log f(\theta)] + \phi^2 \frac{d^2}{d\theta^2} [\log f(\theta)]
\]

i.e.,

\[
\mu_2(\theta) = \mu(\theta) + \phi^2 \frac{d^2}{d\theta^2} [\log f(\theta)]. \quad (1.2.1)
\]

Hence, the statement of the theorem.

Consider now,

**Lemma 1:** If the parameter space $\Theta$ of a gbsd contains zero, then the range $T$ of the gbsd contains zero and the corresponding random variable takes the value zero with positive probabilities for all $\Theta$ in the parameter space; and conversely.
Proof:

\[ 0 \in \Theta \]

\[ \therefore f(0) > 0. \]

But \( f(\theta) = \sum_{x \in T} a_x \theta^x \)

\[ \therefore 0 \in \Theta \text{ and } a_o > 0 \]

Converse follows by retracing the steps above.

Hence, Lemma 1.

Lemma 2: The logarithm of the generating function of a gpdf is a monotone non-decreasing function of \( \theta \).

Proof: Two cases arise: (1) \( 0 \in \Theta \), i.e., Parameter space of the gpdf does not contain 0, and

(2) \( 0 \in \Theta \), i.e., Parameter space of the gpdf contains 0.

Case 1: Here \( \theta > 0 \), \( \Theta \in \Theta \)

\[ \therefore \mu(\theta) = \sum_{x \in T} \frac{a_x \theta^x}{f(\theta)} > 0 \]

But from (1.1.11), \( \mu(\theta) = \theta \frac{d}{d\theta} \left[ \log f(\theta) \right] \)

\[ \therefore \frac{d}{d\theta} \left[ \log f(\theta) \right] > 0 \]

Case 2: By Lemma 1, we have in this case,

\[ f(\theta) = a_o + \sum_{x \in T - \{0\}} a_x \theta^x, \quad a_o > 0 \]

where \( T - \{0\} \) denotes the set \( T \) without 0.

Direct computation gives, therefore:

\[ \frac{d}{d\theta} \left[ \log f(\theta) \right] = \frac{\frac{d}{d\theta} [f(\theta)]}{f(\theta)} = \frac{\sum_{x \in T - \{0\}} xa_x \theta^{x-1}}{f(\theta)}. \]
clearly, for $\theta > 0$, $\frac{d}{d\theta} [\log f(\theta)] > 0$,

for $\theta = 0$, $\frac{d}{d\theta} [\log f(\theta)] = 0$ if $1 \neq T$ and $a_1 = 0$, or $1 \neq T$

$> 0$ otherwise.

Thus, we have always $\frac{d}{d\theta} [\log f(\theta)] > 0$.

Hence, Lemma 2.

Theorem 2: The necessary and sufficient condition for the variance of a gpsd to equal its mean for every $\theta$ of its parameter space is that the generating function be of the form

$$f(\theta) = e^{k\theta + c}$$

where $k > 0$ and $c$ are arbitrary constants.

Proof: Sufficiency: obvious

Necessity: now, have for a gpsd

$$\mu_2(\theta) = \mu(\theta), \quad \theta \in \Theta$$

By (1.2.1) of Theorem 1,

$$\phi^2 \frac{d^2}{d\phi^2} [\log f(\theta)] = 0.$$  \hspace{1cm} (1.2.2)

Now, two cases arise:

Case 1: $0 \notin \Theta$, i.e., Parameter space of the gpsd does not contain 0. In this case, (1.2.2) reduces to

$$\frac{d^2}{d\phi^2} [\log f(\theta)] = 0$$

$\therefore \frac{d}{d\phi} [\log f(\theta)] = k$, where $k$ is some positive constant by Case 1 of Lemma 2.

$\therefore \log f(\theta) = k\phi + c$, where $c$ is arbitrary constant.
Hence, for all \( \theta \in \Theta \),
\[
f(\theta) = e^{k\theta + c}
\]
where \( k > 0 \) and \( c \) are arbitrary constants.

**Case 2:** \( 0 \in \Theta \), i.e., Parameter space of the gpsd contains 0.

For positive values of \( \theta \), Case 1 applies and we get for all \( \theta \in \Theta - \{ 0 \} \),
\[
f(\theta) = e^{k\theta + c}
\]
where \( k > 0 \) and \( c \) are arbitrary constants.

To verify that this form holds for \( \theta = 0 \), we have by Lemma 1,
\[
0 \in T \text{ and } a_0 > 0, \text{ so that,}
\]
\[
f(0) = a_0 = e^{\log a_0} = e^{k \cdot 0 + c},
\]
where \( c = \log a_0 \).

Hence, the statement of the theorem.

**Theorem 3:** The equality of mean and variance is necessary and sufficient for a gpsd to become Poisson. (Characterization of Poisson distribution.)

**Proof:** We first prove the following lemma.

**Lemma:** A positive constant multiple of the generating function of a gpsd does not affect it; i.e., gives rise to the same original gpsd.

Let the generating function be as given in (1.1.3) and the corresponding gpsd as given in (1.1.4).
Consider the new generating function
\[ h(\Theta) = k f(\Theta) \text{ where } k \text{ is some positive constant} \]

i.e., \[ h(\Theta) = \sum_{x \in T} k a_x \Theta^x \]

and the gpsi corresponding to \( h(\Theta) \) becomes
\[ \text{Prob} \{ X = x \} = \frac{k a_x \Theta^x}{h(\Theta)} = \frac{a_x \Theta^x}{f(\Theta)} \quad x \in T \]

which is the original gpsi given in (1.1.4).

Hence the Lemma.

Now, Theorem 3 follows immediately by applying the above lemma to Theorem 2.

Hence, the characterization of Poisson distribution as stated by Theorem 3.

**Theorem 4:** The necessary and sufficient condition for the variance of a gpsi to exceed its mean for every non-zero \( \Theta \) of its parameter space \( \Theta \) is that the generating function be of the form
\[ f(\Theta) = e^{P(\Theta) + R\Theta + Q} \]

where \( Q \) and \( R \) are arbitrary constants and \( P(\Theta) \), along with its derivative, is a positive monotone increasing function of \( \Theta \).

**Proof:** Necessity: Let \( \mu_2(\Theta) - \mu(\Theta) = \Theta^2 p(\Theta) \),

where \( p(\Theta) \) is a positive function of \( \Theta \).

Then by Theorem 1, we have
\[ \Theta^2 \frac{d^2}{d\Theta^2} \left[ \log f(\Theta) \right] = \Theta^2 p(\Theta). \]
\[ \frac{d^2}{d\Theta^2} [\log f(\Theta)] = p(\Theta). \]  
(1.2.3)

Integrating (1.2.3),

\[ \frac{d}{d\Theta} [\log f(\Theta)] = p'(\Theta) + R, \]  
(1.2.4)

where R is an arbitrary constant and

\[ R + p'(\Theta) = \int p(\Theta) d\Theta, \] a positive monotone increasing function of \( \Theta \).

Integrating (1.2.4), we have

\[ \log f(\Theta) = P(\Theta) + R\Theta + Q, \]  
(1.2.5)

where Q is an arbitrary constant and

\[ Q + P(\Theta) = \int p'(\Theta) d\Theta, \] a positive monotone increasing function of \( \Theta \).

Hence, from (1.2.5), we have the required form for the generating function, namely,

\[ f(\Theta) = e^{P(\Theta)} + R\Theta + Q, \]

where symbols carry the meaning as stated.

**Sufficiency:** Sufficiency follows from above by retracing the steps.

**Theorem 5:** The necessary and sufficient condition for the variance of a gbsd to be less than its mean for every non-zero \( \Theta \) of its parameter space \( \Theta \) is that the generating function be of the form

\[ f(\Theta) = e^{A(\Theta)} + B\Theta + c, \]

where B and C are arbitrary constants and \( A(\Theta) \) is such that its derivative is a monotone decreasing function of \( \Theta \).
Proof : On the same lines as that of Theorem 4.

**Theorem 6:** The mean \( \mu(\theta) \) of a gpsd is a non-negative monotone non-decreasing function of \( \theta \).

Proof : Consider the relation (1.1.12), which states

\[
\mu_2(\theta) = \theta \frac{d}{d\theta}[\mu(\theta)]. \quad (1.2.6)
\]

We know that \( \mu_2(\theta) \geq 0 \). Also \( \theta \geq 0 \).

\[\therefore\] from (1.2.6) follows that

\[
\frac{d}{d\theta}[\mu(\theta)] \geq 0; \text{ also } \mu(\theta) \geq 0.
\]

\[\therefore\] \( \mu(\theta) \) is a non-negative monotone non-decreasing function of \( \theta \).

**Theorem 7:** The graph of the mean of a gpsd with parameter space containing zero is convex or concave or linear in accordance with the variance of the gpsd being greater than or less than or equal to the mean and conversely.

Proof : Suppose \( \mu_2(\theta) > \mu(\theta) \)

\[\therefore \theta \frac{d}{d\theta}[\mu(\theta)] > \mu(\theta)\]

\[\therefore \frac{d}{d\theta}[\mu(\theta)] > \frac{\mu(\theta)}{\theta} \text{ when } \theta \neq 0.
\]

Also, as the gpsd is taken with parameter space containing zero, we can speak of \( \mu(0) \) which is clearly = 0.

Hence, follows the convexity of the graph of the mean when the variance exceeds the mean.

On similar lines the rest of the statement can be very easily established.
Theorem 8: The mean of a gpdf with parameter space containing zero is respectively a linear or convex or concave function of $\theta$ if and only if the generating function is respectively of the form of Theorem 2, Theorem 4, or Theorem 5.

Proof : The proof follows immediately from Theorem 2, Theorem 4, Theorem 5, and Theorem 7.
CHAPTER II

2.0 LIKELIHOOD ESTIMATION AND ALLIED PROBLEMS IN A CLASS OF DISCRETE DISTRIBUTIONS

We show first in this chapter that for gpsd (1.1.4), the maximum likelihood method and the method of moments give the same estimate of the gpsd parameter. The likelihood equation and a method for solving it are derived for the problem of estimation in truncated and censored forms of the gpsd (1.1.4). We mention here that we call gpsd (1.1.4) a complete gpsd as opposed to its truncated and censored forms.

Large sample methods based on maximum likelihood are then derived for testing the homogeneity of several gpsd's and providing the estimate for the common parameter in case the gpsd's are homogeneous. A treatment for doubtful observations is given. Lastly, the problem of estimation when the general distribution involves two parameters is discussed.

2.1 Estimation by Likelihood for a Complete gpsd

2.1.1 Let $x_i$ (i=1,2,...,N) be a random sample of size N from the gpsd (1.1.4). Then the logarithm of the likelihood function $L$ is

$$\log L = \text{constant} + \sum_{i=1}^{N} x_i \log \theta - N \log f(\theta)$$

so that the "efficient score" for $\theta$ is

$$\psi(\theta) = \frac{d}{d\theta} [\log L]$$

$$= \sum_{i=1}^{N} x_i / \theta - N f'(\theta) / f(\theta)$$

$$= \frac{N}{\theta} (\bar{x} - \mu), \quad (2.1.1)$$

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\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i/N \text{ is the sample mean and by (1.1.5)} \]

\[ \mu = \frac{\Theta f'(\Theta)}{f(\Theta)}, \text{ the mean of the gpsd (1.1.4)}. \]

The likelihood equation \( \psi(\hat{\Theta}) = 0 \) for estimating \( \Theta \) thus reduces to

\[ \bar{x} = \mu(\hat{\Theta}) (\equiv \hat{\mu}, \text{ say}) \]  \hspace{1cm} (2.1.2)

which means equating the sample mean to the population mean. The method of maximum likelihood and the method of moments thus lead to the same estimate in the case of a gpsd.

Denoting this estimate by \( \hat{\Theta} \), the asymptotic variance is given by \( 1/I(\Theta) \) where

\[ I(\Theta) = -E\left( \frac{d\psi}{d\Theta} \right) \]

\[ = -E\left[ - \frac{N}{\Theta} \frac{d\mu}{d\Theta} - \frac{N}{\Theta^2} (\bar{x} - \mu) \right] \]

\[ = \frac{N}{\Theta} \frac{d\mu}{d\Theta} \]  \hspace{1cm} (2.1.3)

also,

\[ = \frac{N}{\Theta^2} \cdot \mu_2(\Theta), \text{ because of (1.1.12)}. \]  \hspace{1cm} (2.1.4)

Thus,

\[ \text{Var}(\hat{\Theta}) = \frac{\Theta}{N} \left( \frac{d\mu}{d\Theta} \right) \]  \hspace{1cm} (2.1.5)

also,

\[ = \frac{\Theta^2}{N} / \mu_2(\Theta). \]  \hspace{1cm} (2.1.6)

2.1.2 If Equation (2.1.2) does not readily give an algebraic solution, one may use an iterative process of solution (which converges; see Rao, 1952) by starting with an approximation \( \Theta_0 \). An improved approximation \( \Theta_1 \) is then obtained from

\[ \Theta_1 = \Theta_0 + \psi(\Theta_0)/I(\Theta_0) \]

\[ = \Theta_0 + [\bar{x} - \mu(\Theta_0)]/\left( \frac{d\mu}{d\Theta} \right) \Theta_0, \]  \hspace{1cm} (2.1.7)
or from the equivalent formula

\[ \theta_1 = \theta_0 + \theta_0 [X - \mu(\theta_0)] / \mu_2(\theta_0) \]

\[ = \theta_0 [1 + \frac{X - \mu(\theta_0)}{\mu_2(\theta_0)}] \]  \hspace{1cm} (2.1.8)

and the process is repeated till one gets a sufficiently accurate solution. To carry out this process by Formula (2.1.8), a table of numerical values of \( \mu(\theta) \) and \( \mu_2(\theta) \) of the gpdf under consideration for sufficiently close values of \( \theta \) could be very useful. Formula (2.1.7) would require a table of numerical values of \( \mu(\theta) \) and \( \frac{d\mu}{d\theta} \). However, it may be observed that a table for only \( \mu(\theta) \) of the gpdf under consideration for sufficiently close values of \( \theta \) would do. Because \( \frac{d\mu}{d\theta} \) can be approximated by the finite difference ratio \( \frac{\Delta \mu}{\Delta \theta} \) and this approximation is expected to be good if the tabular interval is small. Illustrative examples (4.2.5; 5.2.11; 7.2.6) given later substantiate this observation.

2.1.3 To find the amount of bias in the maximum likelihood estimate \( \hat{\theta} \), following Haldane and Smith (1956), we have the amount of bias in \( \hat{\theta} \) to order \( \frac{1}{N} \) as

\[ b(\hat{\theta}) = -\frac{1}{2} \cdot \frac{1}{N} \left( \frac{B_1}{A_1^2} \right) \]  \hspace{1cm} (2.1.9)

where

\[ A_1 = \sum_{x \in T} \left( \frac{dP_x}{d\theta} \right)^2 / P_x \]  \hspace{1cm} (2.1.10)

and

\[ B_1 = \sum_{x \in T} \left( \frac{dP_x}{d\theta} \right) \left( \frac{d^2P_x}{d\theta^2} \right) / P_x \]  \hspace{1cm} (2.1.11)
in which, as usual,

\[ P_x = a_x \frac{\Theta^x}{f(\Theta)} \quad x \in T \]

\[ \mu = \Theta \frac{f'(\Theta)}{f(\Theta)} \quad \text{and} \quad \mu_2 = \Theta \frac{d\mu}{d\Theta}, \]

\[ \frac{dP_x}{d\Theta} = \frac{P_x}{\Theta} (x - \mu) \]

and

\[ \frac{d^2 P_x}{d\Theta^2} = \frac{P_x}{\Theta^2} \left[ (x - \mu)^2 - \mu_2 - (x - \mu) \right] \]

so that

\[ A_1 = \frac{\mu_2}{\Theta^2} \]

and

\[ B_1 = \frac{\mu_3 - \mu_2}{\Theta^3}. \]

Therefore, from (2.1.9), we have the amount of bias in \( \hat{\Theta} \), to order \( \frac{1}{N} \),

\[ b(\hat{\Theta}) = -\frac{1}{N} \cdot \frac{\Theta}{2} \frac{\mu_3 - \mu_2}{\mu_2^2}. \quad (2.1.12) \]

2.1.4 Next, to estimate a differentiable function of \( \Theta \), such as \( \omega(\Theta) \), the method of maximum likelihood leads to the estimate

\[ \hat{\omega} = \omega(\hat{\Theta}) \]

with variance

\[ \text{Var} (\hat{\omega}) = \frac{\Theta (\frac{d\omega}{d\Theta})^2}{N \frac{d\mu}{d\Theta}} = \frac{\Theta^2}{N \mu_2} (\frac{d\omega}{d\Theta})^2 \quad (2.1.13) \]

where \( \hat{\Theta} \) is the maximum likelihood estimate of \( \Theta \).
To find the amount of bias in \( \hat{\omega} \), following Haldane and Smith (1956), we have that the amount of bias to order \( \frac{1}{N} \) in \( \hat{\omega} \) is given by:

\[
b(\hat{\omega}) = - \frac{1}{2} \frac{1}{N} \frac{B_2}{A_2^2}
\]

(2.1.14)

where

\[
A_2 = \sum_{x \in T} \frac{dP_x}{P_x} = \frac{A_1}{2} \left( \frac{d\omega}{d\theta} \right)_2
\]

(2.1.15)

and

\[
B_2 = \sum_{x \in T} \frac{dP_x}{P_x} \left( \frac{d^2P_x}{d\omega^2} \right)_x
\]

\[
= \frac{1}{(d\omega/d\theta)_3} \left[ B_1 - \frac{d^2\omega}{d\omega d\theta} \right] A_1
\]

(2.1.16)

where \( A_1 \) and \( B_1 \) are defined by (2.1.10) and (2.1.11), respectively.

Then from (2.1.14),

\[
b(\hat{\omega}) = - \frac{1}{N} \frac{\Phi}{2} \frac{d\omega}{d\theta} \cdot \frac{\mu_3 - \mu_2(1 + \frac{d^2\omega}{d\omega d\theta} \cdot \Phi)}{\mu_2}
\]

(2.1.17)

2.2 Estimation by Likelihood for a Truncated gpd

Let \( T^* \) be a non-null subset of the range \( T \) of gpd (1.1.4), and consider the distribution (1.1.4) truncated to the subset \( T^* \). In this case, it can be easily verified that the (truncated) random variable \( X^* \) has the probability distribution:

\[
P^*_x = \text{Prob} \{ X^* = x \} = \frac{a_x \theta^x}{f^*(\theta)} , \quad x \in T^*
\]

(2.2.1)
where
\[ f^*(\theta) = \sum_{x \in T^*} a_x \theta^x. \quad (2.2.2) \]

2.2.1 It is easy to see that the truncated gbsd (2.2.1) is in turn a gbsd in its own right with the generating function given by (2.2.2), and consequently, all the properties of (1.1.4) are valid for (2.2.1). To be explicit, distinguishing the characteristics of this truncated distribution (2.2.1) from the complete gbsd (1.1.4) by means of an asterisk(*), it immediately follows that relations analogous to those in (1.1.5) - (1.1.15) will hold for the mean \( \mu^* \), crude moments \( m_r^* \), central moments \( \mu_r^* \), mgf \( M^*(t) \), etc., such as

\[
\mu^* = \theta \frac{d}{d\theta} [f^*(\theta)]
\]
\[
\mu_2^* = \theta \frac{d\mu^*}{d\theta}, \text{ etc.}
\]

2.2.2 Similarly, the maximum likelihood estimate \( \hat{\theta} \) (which in this case, is also equivalent to the moments estimate) for \( \theta \) is to be computed from the likelihood equation:
\[
\bar{x}^* = \mu^*(\hat{\theta}) \quad (2.2.3)
\]
where \( \bar{x}^* \) is the mean of a random sample of size \( N \) from the truncated gbsd (2.2.1).

The asymptotic variance of \( \hat{\theta} \) is similarly given by
\[
\text{Var} \ (\hat{\theta}) = \frac{\theta}{N \frac{d\mu^*}{d\theta}} = \frac{\theta^2}{N \mu_2^*(\theta)}. \]

The iterative process of solving (2.2.3) can again be put down in the form
\[
\theta_1 = \theta_0 + [\bar{x}^* - \mu^*(\theta_0)]/\left(\frac{d\mu^*}{d\theta}\right) \theta_0,
\]
or

$$\hat{\Theta}_1 = \Theta_0 [1 + \frac{x^* - \mu^*(\Theta_0)}{\mu^*_2(\Theta_0)}] .$$

The formulae for the amount of bias to order $\frac{1}{N}$ in $\hat{\Theta}$ [and $\hat{\omega} = \omega(\hat{\Theta})$] can be written down similarly.

2.3 Estimation by Likelihood for Censored gspd

Let $T^*$, $T_j$ ($j = 1, 2, \ldots, k$) be $k+1$ mutually exclusive and exhaustive subsets of the range $T$ of the gspd (1.1.4). Suppose that in a random sample of size $N$ from the gspd (1.1.4), we have a record of the number $n_j$ of observations in the subset $T_j$ ($j = 1, 2, \ldots, k$) and of the $n^*$ observations $x_i$ ($i = 1, 2, \ldots, n^*$) in the subset $T^*$, so that

$$N = n^* + \sum_{j=1}^{k} n_j$$

and we write

$$\bar{x}^* = \sum_{i=1}^{n^*} x_i / n^* .$$

(2.3.1)

2.3.1 The logarithm of the likelihood function may be written as

$$\log L = \text{constant} + \sum_{j=1}^{k} n_j \log f_j + \sum_{i=1}^{n^*} x_i \log \Theta - N \log f$$

where

$$f_j = f_j(\Theta) = \sum_{x \in T_j} a_x \Theta^x \text{ and } f = f(\Theta) = \sum_{x \in T} a_x \Theta^x .$$

(2.3.2)
The "efficient score" for $\Theta$ is

$$
\psi(\Theta) = \frac{d}{d\Theta}[\log L]
$$

$$
= \frac{k}{\theta} \sum_{j=1}^{n^*} n_j f_j \overline{x}_j / f_j + \sum_{i=1}^{n^*} x_i / \theta - N f^1 / f
$$

$$
= \frac{1}{\theta} [n^* \overline{x} - (N \mu - \sum_{j=1}^{k} n_j v_j)]
$$

(2.3.3)

where

$$
v_j = v_j(\Theta) = \frac{\sum_{x \in T_j} x \theta^x / f_j}{\sum_{x \in T_j} \theta^x / f_j}
$$

(2.3.4)

is the mean of the $j$-th class $T_j$ and $\mu$ is the mean of the gpsd (1.1.4).

Thus, the likelihood equation for estimating $\Theta$ is

$$
n^* \overline{x} = N \mu - \sum_{j=1}^{k} n_j v_j
$$

(2.3.5)

where

$$
\mu = \mu(\Theta) \text{ and } v_j = v_j(\Theta)
$$

The asymptotic variance of the estimate $\hat{\Theta}$ derived from (2.3.5) is $1/I(\Theta)$ where

$$
I(\Theta) = - E(\frac{d^2 \psi}{d\Theta^2}) = \frac{N}{\theta} \left[ \mu^1 - \sum_{j=1}^{k} p_j v_j^1 \right]
$$

(2.3.6)

where $p_j = f_j / f$ is the probability for the $j$-th class $T_j$ and primes denote the differentiation with respect to $\Theta$.

It may be noted that when the subsets $T_j \ (j = 1, 2, \ldots k)$ are all empty, we get our previous formula (2.1.2) for the estimate from sample for a complete gpsd.

2.4 Estimation with Doubtful Observations

Let $T^*$ be a subset of the range $T$ of the gpsd (1.1.4). Consider the situation where the experimenter has doubts about sample
observations not in $T^*$, and therefore, merely records the number of observations not in $T^*$. The model considered is:

$$\text{Prob} \{ X = x \} = \beta \quad \text{for } x \notin T^*$$

$$= (1-\beta) \frac{a_x \Theta^x}{f^*(\Theta)} \quad \text{for } x \in T^* \quad (2.4.1)$$

where

$$0 < \beta < 1 \text{ and } f^*(\Theta) = \sum_{x \in T^*} a_x \Theta^x. \quad (2.4.2)$$

2.4.1 If, in a sample of size $N$, the number of observations not in $T^*$ is $n_1$ and the records of the other $N-n_1 = n^*$ (say), are $x_i (i = 1, 2, \ldots, n^*)$; the logarithm of the likelihood function is then given by

$$\log L = \text{constant} + n_1 \log \beta + n^* \log (1-\beta)$$

$$+ \sum_{i=1}^{n^*} x_i \log \Theta - n^* \log f^*(\Theta).$$

The "efficient scores" for $\beta$ and $\Theta$ are then

$$\psi_1 = \frac{\partial \log L}{\partial \beta} = \frac{n_1}{\beta} - \frac{n^*}{1-\beta} \quad (2.4.3)$$

$$\psi_2 = \frac{\partial \log L}{\partial \Theta} = \frac{n^*}{\Theta} [x^* - \mu^*(\Theta)] \quad (2.4.4)$$

where

$$x^* = \sum_{i=1}^{n^*} x_i/n^*. \quad (2.4.5)$$

and

$$\mu^*(\Theta) = \sum_{x \in T^*} x a_x \Theta^x / f^*(\Theta) \quad (2.4.6)$$

is the mean of the subset $T^*$. 
The likelihood estimates of $\beta$ and $\Theta$ are thus given by
\[ \hat{\beta} = \frac{n_1}{N} \quad (2.4.7) \]
\[ \mu^*(\Theta) = \bar{x}^* \quad (2.4.8) \]
so that, the estimate of $\Theta$ is derived simply by neglecting the $n_1$ observations not in $T^*$ and treating the sample of $n^*$ as one from gpsi (1.1.4) truncated to $T^*$.

The elements of the "information matrix" are given by:
\[ I_{11} = -E \left( \frac{d\psi_1}{d\beta} \right) = \frac{N}{\beta(1-\beta)} \quad (2.4.9) \]
\[ I_{12} = -E \left( \frac{d\psi_1}{d\Theta} \right) = -E \left( \frac{d\psi_2}{d\beta} \right) = 0 \quad (2.4.10) \]
\[ I_{22} = -E \left( \frac{d\psi_2}{d\Theta} \right) = \frac{N(1-\beta)}{\Theta} \left( \frac{d\mu^*}{d\Theta} \right) \quad (2.4.11) \]

2.4.2 The hypothesis $H_0$ of interest is that the proportion of doubtful observations conforms to the gpsi (1.1.4), that is,
\[ H_0: \beta = 1 - f^*(\Theta)/f(\Theta) \]

If $H_0$ is true, the estimate of $\Theta$ is obtained by consideration of the sample as a censored one as in Section 2.3. The estimate of $\Theta$ under $H_0$ is thus obtained from:
\[ n^*\bar{x}^* = N\mu - n_1\nu_1 \quad (2.4.12) \]
where
\[ \nu_1 = \sum_{x \in T^*} \frac{x \alpha x^X}{[f(\Theta) - f^*(\Theta)]} \quad (2.4.13) \]
is the mean of the subset complementary to $T^*$.

The estimate of $\Theta$ under $H_0$ derived from this equation will be denoted by $\hat{\Theta}_0$. The estimate of $\beta$ under $H_0$ is then given by
\[ \hat{\beta}_0 = 1 - f^*(\hat{\Theta}_0)/f(\hat{\Theta}_0) \quad (2.4.14) \]
Following the method suggested by Rao (1948) for testing $H_0$, we have the criterion

$$
\chi^2_1 = \frac{[\psi_1(\widehat{\theta}_0)]^2}{I_{11}(\widehat{\theta}_0)} + \frac{[\psi_2(\widehat{\theta}_0)]^2}{I_{22}(\widehat{\theta}_0, \widehat{\theta}_0)}
$$

(2.4.15)

which is asymptotically distributed as a Chi-square with one degree of freedom. In the present case, (2.4.15) takes the following form:

$$
\chi^2_1 = \frac{N}{\theta_0(1-\theta_0)} \left( \frac{n_1}{N} - \theta_0 \right)^2 + \frac{n_{x^2}}{N(1-\theta_0)} \left[ x^* - \mu*(\widehat{\theta}_0) \right]^2
$$

(2.4.16)

where

$$
\mu^*_2(\theta) = \theta \cdot \frac{d\mu^*(\theta)}{d\theta}
$$

2.5 Homogeneity and Combined Estimation

In the light of random samples from a number of gpdf's, it may be required to examine if the distributions are homogeneous in respect to the parameter $\theta$ and if so, to make a combined estimate of $\theta$.

Let $x_{j1}$ ($i = 1, 2, \ldots, N_j$) be a random sample from the $j$-th gpdf characterized by the probability law:

$$
a_{x}(j)\theta_j^x f_j(\theta_j) \quad x \in T_j
$$

(2.5.1)

where

$$
f_j(\theta_j) = \sum_{x \in T_j} a_{x}(j)\theta_j^x,
$$

(2.5.2)

$T_j$ is the range of the $j$-th gpdf, and $j = 1, 2, \ldots, k$.

2.5.1 The logarithm of the joint likelihood function is

$$
\log L = \text{constant} + \sum_{j=1}^{k} N_j \bar{x}_j \log \theta_j - \sum_{j=1}^{k} N_j \log f_j(\theta_j),
$$
where
\[ \bar{x}_j = \frac{\sum_{i=1}^{N_j} x_{ji}}{N_j} \]
is the mean of the sample from the \(j\)-th gpdf. The \(j\)-th "efficient score" is then
\[ \psi_j = \frac{\partial}{\partial \theta_j} \log L = \frac{N_j}{\theta_j} [\bar{x}_j - \mu_j(\theta_j)] \]  
(2.5.3)

where \(\mu_j(\theta_j)\) is the mean of the \(j\)-th gpdf.

The elements of the "information matrix" are
\[ I_{jj} = \frac{N_j}{\theta_j} \cdot \frac{d\mu_j(\theta_j)}{d\theta_j} \]  
(2.5.4)
\[ = N_j \mu_{2j}(\theta_j)/\theta_j^2 \]  
(2.5.5)
\[ I_{jj'} = 0 \quad \text{if} \quad j \neq j' \]  
(2.5.6)

where \(\mu_{2j}(\theta_j)\) is the variance of the \(j\)-th gpdf.

2.5.2 The hypothesis of homogeneity is
\[ H_0: \quad \theta_1 = \theta_2 = \ldots = \theta_k \, . \]

If the hypothesis \(H_0\) is true, the common value may be denoted by \(\theta\) and the efficient score and the information with respect to \(\theta\) are given by:
\[ \psi(\theta) = \frac{N}{\theta} [\bar{x} - \sum_{j=1}^{k} N_j \mu_j(\theta)/N] \]  
(2.5.7)

where
\[ \bar{x} = \frac{\sum_{j=1}^{k} N_j \bar{x}_j/N}{j=1} \, . \]
and

\[ I(\theta) = \frac{1}{\theta} \sum_{j=1}^{k} \frac{N_j \mu_j(\theta)}{N_j} \]  \quad (2.5.8)

\[ = \frac{1}{\theta^2} \sum_{j=1}^{k} N_j \mu_2 j(\theta). \]  \quad (2.5.9)

To solve the equation \( \psi(\hat{\theta}) = 0 \) for the maximum likelihood estimate \( \hat{\theta} \), one starts with an approximation \( \theta_0 \) and derives a better approximation \( \theta_1 \) from the formula:

\[ \theta_1 = \theta_0 + N[\bar{x} - \sum_{j=1}^{k} \frac{N_j \mu_j(\theta_0)}{N_j}] / \sum_{j=1}^{k} N_j \frac{d\mu_j(\theta)}{d\theta} \theta_0 \]  \quad (2.5.10)

or

\[ \theta_1 = \theta_0[1 + N[\bar{x} - \sum_{j=1}^{k} \frac{N_j \mu_j(\theta_0)}{N_j}] / \sum_{j=1}^{k} N_j \mu_2 j(\theta_0)]. \]  \quad (2.5.11)

2.5.3 A test of the homogeneity hypothesis \( H_0 \) is then given by the statistic

\[ X^2_{k-1} = \sum_{j=1}^{k} \frac{[\psi_j(\hat{\theta})]^2}{I_{jj}(\hat{\theta})} \]  \quad (2.5.12)

\[ = \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\hat{\theta})]^2 / \hat{\theta} \frac{d\mu_j(\theta)}{d\theta} \theta=\hat{\theta} \]  \quad (2.5.13)

\[ = \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\theta)]^2 / \mu_2 j(\theta) \]  \quad (2.5.14)

which is asymptotically distributed as a Chisquare with \( (k-1) \) degrees of freedom, if \( H_0 \) is true.
2.6 Estimation for a gpdf with Two Parameters

Consider a gpdf with two parameters taking the form:

\[ \text{Prob}\{X = x\} = \frac{a_x(\lambda)\theta^x}{f(\theta, \lambda)} \quad x \in T, \quad (2.6.1) \]

where \( T \) is the range of the gpdf and the generating function

\[ f(\theta, \lambda) = \sum_{x \in T} a_x(\lambda)\theta^x, \quad (2.6.2) \]

such that \( f(\theta, \lambda) \) is positive and bounded for all admissible values of the two parameters \( \theta \) and \( \lambda \), and the non-negative coefficients \( a_x(\lambda) \) now depend on \( x \) and \( \lambda \). The binomial and negative binomial distributions are special cases of (2.6.1) when they are considered to be the distributions with two parameters.

2.6.1 To estimate \( \theta \) and \( \lambda \) on the basis of a sample \( x_i \) (\( i = 1, 2, \ldots, N \)) of size \( N \) from (2.6.1), the logarithm of likelihood function is

\[ \log L = \text{constant} + \sum_{i=1}^{N} x_i \log \theta + \sum_{i=1}^{N} \log a_{x_i}(\lambda) \]

\[ - N \log f(\theta, \lambda). \]

The "efficient score" for \( \theta \) is then

\[ \psi_1 = \psi_1(\theta, \lambda) = \frac{\partial}{\partial \theta}[\log L] = \frac{N}{\theta} - \mu(\theta, \lambda) \quad (2.6.3) \]

and the likelihood equation \( \psi_1(\hat{\theta}, \hat{\lambda}) = 0 \) reduces to

\[ \bar{x} = \mu(\hat{\theta}, \hat{\lambda}) \quad (2.6.4) \]

which is the same as the first-moment equation.
The "efficient score" for $\lambda$ is

$$\psi_2 = \psi_2(\theta, \lambda) = \frac{\partial}{\partial \lambda} \log L$$

$$= N \left[ \sum_{i=1}^{N} \frac{d}{d\lambda} \log a_{x_1}(\lambda) / N \right. \left. - \frac{d}{d\lambda} \log f(\theta, \lambda) \right] \quad (2.6.5)$$

and the estimating equation $\psi_2(\hat{\theta}, \hat{\lambda}) = 0$ becomes

$$\frac{d}{d\lambda} \log f(\hat{\theta}, \lambda) = \sum_{i=1}^{N} \frac{d}{d\lambda} \log a_{x_1}(\lambda) / N . \quad (2.6.6)$$

This, however, is not a moment equation. The second moment equation will be

$$s^2 = \mu_2(\theta, \lambda) \quad (2.6.7)$$

where

$$s^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 / N .$$

Thus, unlike gpsd's of the form (1.1.4) with single parameter, gpsd's given by (2.6.1) with two parameters do not yield identical "moment" and "maximum likelihood" estimates.

2.6.2 The elements of the "information matrix"

$$\mathbf{j} = \begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} \quad (2.6.8)$$

are given by

$$I_{11} = -E \left( \frac{\partial \psi_1}{\partial \theta} \right) = \frac{N}{\theta} \left( \frac{\partial \mu}{\partial \theta} \right) \quad (2.6.9)$$

$$I_{12} = -E \left( \frac{\partial \psi_1}{\partial \lambda} \right) = -E \left( \frac{\partial \psi_2}{\partial \theta} \right)$$

$$= \frac{N}{\theta} \frac{\partial \mu}{\partial \lambda} \quad (2.6.10)$$
\[ I_{22} = -E\left( \frac{\partial^2}{\partial \lambda^2} \psi \right) \]

\[ = N \left[ \frac{\partial^2}{\partial \lambda^2} \log f(\Theta, \lambda) - h(\lambda) \right] \quad (2.6.11) \]

where

\[ h(\lambda) = E\left[ \frac{d^2}{d\lambda^2} \log a_x(\lambda) \right] \]

\[ = \frac{1}{f(\Theta, \lambda)} \left[ \frac{\partial^2}{\partial \lambda^2} f(\Theta, \lambda) \right] - E\left[ \frac{d}{d\lambda} \log a_x(\lambda) \right]^2. \]

The asymptotic "dispersion matrix" of the estimates \( \Theta, \lambda \) obtained by solving (2.6.4) and (2.6.6) is then given by

\[
\begin{pmatrix}
\text{var} (\Theta) & \text{cor} (\Theta, \lambda) \\
\text{cor} (\Theta, \lambda) & \text{var} (\lambda)
\end{pmatrix} = \Psi^{-1} \quad (2.6.12)
\]

If instead of \( \Theta \) and \( \lambda \), \( \mu = \mu(\Theta, \lambda) \) and \( \lambda \) are regarded as the parameters, the maximum likelihood estimates of \( \mu \) and \( \lambda \) are asymptotically uncorrelated: this follows from (2.6.10).
CHAPTER III

3.0 SIMPLE METHODS OF ESTIMATION FOR
A CLASS OF DISCRETE DISTRIBUTIONS

In Sections 2.1, 2.2 and 2.3, we discussed the method of max-
mum likelihood for estimation on the basis of samples from gpsd's which
are either complete, truncated or censored. The method, though "effi-
cient", generally involves heavy computation. Moreover, it does not
yield an unbiased estimate in several cases. In this chapter, we con-
sider some other methods of estimation and investigate their important
properties. All these are easy to compute, and it also turns out that
some of them provide unbiased estimates.

3.1 Estimation by the Ratio Method for a gpsd
[Range T finite and $T = (c, c+1, \ldots, c+k = d)$ with positive probabilities]

3.1.1 Consider the gpsd (1.1.4) with range $T$ finite and $T =
(c, c+1, \ldots, c+k = d)$ with positive probabilities, that is, the coefficients
$a_x > 0$ for all $x \in T$. To be explicit, the gpsd that we consider here is
of the form:

$$P_x = \text{Prob} \{X = x\} = \frac{a_x \theta^x}{f(\theta)}$$  \hspace{1cm} (3.1.1)

where

$x \in T = (c, c+1, \ldots, c+k = d)$, $d$ finite

$$f(\theta) = \sum_{x=c}^{d} a_x \theta^x$$  \hspace{1cm} (3.1.2)

and

$a_x > 0$ for $x \in T$. 

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Let
\[ g_r(x) = \frac{a_{x-r}}{a_x} \quad x \in T \quad (3.1.3) \]
with \( r \) being an integer such that \( x-r \in T \). Then
\[
\sum_{x=u}^{v} g_r(x)P_x = \sum_{x=u}^{v} a_{x-r} \theta^x / f(\theta) \\
= \theta^r \sum_{x=u-r}^{v-r} a_x \theta^x / f(\theta) \\
= \theta^r \sum_{x=u-r}^{v-r} P_x \quad (3.1.4)
\]
where \( u \) and \( v \) are arbitrary with \( c+r \leq u \leq v \leq d \). From (3.1.4), we get the identity
\[
\theta^r = \frac{\sum_{x=u}^{v} g_r(x)P_x}{\sum_{x=u-r}^{v-r} P_x} \quad (3.1.5)
\]
which can be made use of in problems of estimation. In a sample of size \( N \), if \( n_x \) is the observed frequency for \( x \), then since \( E(n_x) = NP_x \), the statistic
\[
\sum_{x=u}^{v} g_r(x)n_x / \sum_{x=u-r}^{v-r} n_x \quad (3.1.6)
\]
may be taken as an estimate of \( \theta^r \) for admissible values of \( r=1,2, \ldots \). Since \( u \) and \( v \) are arbitrary, the same method is applicable for estimation in truncated and censored g.s.d.'s also, provided that their range contains a subset of consecutive integers. We call these estimates "ratio estimates."

It is interesting to note that methods given by Plackett (1953) and Moore (1952, 1954) for estimating \( \theta \) in truncated Binomial and Poisson
distributions come out as special cases of the method we suggest here. The method which we call the ratio method is applicable not merely for estimating \( \theta \), but also for its integral powers and for any gbsd of this section, truncated or censored.

The ratio estimate is not generally unbiased or efficient, but is always easy to compute. In certain cases (see Section 3.2), however, unbiased estimates can be obtained by the ratio method. In other cases, such as those in this section, the bias is generally of the order \( \frac{1}{N} \) discussed below.

3.1.2 Consider the following ratio estimate of \( \theta \) for gbsd

\[
(3.1.1) \quad \Phi' = \frac{t_1}{t_2}
\]

where

\[
(3.1.8) \quad t_1 = \sum_{x=c+1}^{d} \left( \frac{a_{x-1}}{a_x} \right) n_x
\]

and

\[
(3.1.9) \quad t_2 = \sum_{x=c}^{d-1} n_x
\]

Then, writing

\[
(3.1.10) \quad E(t_2) = N \sum_{x=c}^{d-1} P_x = N(1-P_d) = NP, \text{ say,}
\]

where

\[
(3.1.11) \quad P = 1 - P_d,
\]

we have

\[
(3.1.12) \quad E(t_1) = NP \Phi.
\]

Let

\[
(3.1.13) \quad t_1 - E(t_1) = \delta t_1 \text{ and } t_2 - E(t_2) = \delta t_2.
\]
Then
\[ \Phi' = \frac{t_1}{t_2} = \Phi(1 + \frac{\delta t_1}{NP})(1 + \frac{\delta t_2}{NP})^{-1}. \]

Since the deviations \( \delta t_1, \delta t_2 \) are stochastically of order \( N^{1/2} \), we get on expansion
\[ \Phi' = \Phi[1 + \frac{\delta t_1}{NP} - \frac{\delta t_2}{NP} - \frac{(\delta t_1)(\delta t_2)}{N^2P^2\Phi} + \frac{(\delta t_2)^2}{N^2P^2}]. \quad (3.1.14) \]

neglecting terms of order higher than \( \frac{1}{N} \). Thus, to this order of approximation,
\[ E(\Phi') = \Phi[1 + \frac{E(\delta t_2)^2}{N^2P^2} - \frac{E(\delta t_1)(\delta t_2)}{N^2P^2\Phi}]. \quad (3.1.15) \]

Now a little computation gives
\[ E(\delta t_2)^2 = NP(1-P) \quad (3.1.16) \]
and
\[ E(\delta t_1)(\delta t_2) = N\Phi[P(1-P) - P_d-1]. \quad (3.1.17) \]

Thus
\[ E(\Phi') = \Phi + \frac{1}{N} \left( \frac{\Phi P_d-1}{P^2} \right). \quad (3.1.18) \]

from which we get the magnitude of the bias in \( \Phi' \), to order \( \frac{1}{N} \),
\[ b(\Phi') = \frac{1}{N} \left( \frac{\Phi P_d-1}{P^2} \right) 
= \left( \frac{\Phi P_d-1}{N(1-P_d)^2} \right)^2. \quad (3.1.19) \]

3.1.3 The variance of \( \Phi' \) correct to terms of order \( \frac{1}{N} \) is
\[ \text{Var} (\Phi') = \frac{1}{N^2P^2}[E(\delta t_1)^2 + \Phi^2 E(\delta t_2)^2 
- 2\Phi E(\delta t_1)(\delta t_2)]. \quad (3.1.20) \]
Now

\[ E(\delta t_1)^2 = N(D - P^2 \phi^2) \quad (3.1.21) \]

where

\[ D = \sum_{x=c+1}^{d} \left( \frac{a_{x-1}}{a_x} \right)^2 P_x \quad (3.1.22) \]

Thus, to order \( \frac{1}{N} \)

\[ \text{Var} (\phi^1) = \frac{1}{NF^2} [D - P \phi^2 + 2 \phi^2 P_{d-1}] \quad (3.1.23) \]

3.1.4 One simple estimate suggested by the identity

\[ \phi = \frac{a_x}{a_{x+1}} \cdot \frac{P_{x+1}}{P_x} \quad (3.1.24) \]

is given by

\[ m = \frac{a_x}{a_{x+1}} \cdot \frac{n_{x+1}}{n_x} \quad (3.1.25) \]

For this estimate, to terms of order \( \frac{1}{N} \):

\[ b(m) = \frac{1}{N} \frac{\phi}{P_x} \quad (3.1.26) \]

and

\[ \text{Var} (m) = \frac{\phi (1 + b_x \phi)}{Nb_x P_x} \quad (3.1.27) \]

where

\[ b_x = \frac{a_{x+1}}{a_x} \]

It is suggested in (3.1.26) that the order of the amount of bias for \( \phi \) is only \( \frac{1}{N} \). Also (3.1.26) and (3.1.27) suggest jointly that one may use the modal class for estimation with advantage.
3.2 Unbiased Estimation by the Ratio Method for a gpsd

[Range $T$ infinite and $T = (c, c+1, \ldots \infty)$ with positive probabilities]

It is easy to demonstrate that the ratio method discussed in Section 3.1 gives the unique unbiased estimate of $\Theta$, linear in frequencies, for a gpsd with range $T$ infinite and $T = (c, c+1, \ldots \infty)$ with positive probabilities. For, consider the gpsd

$$P_x = \text{Prob}\{X = x\} = \frac{a_x \Theta^x}{f(\Theta)} \quad x = c, c+1, \ldots \infty \quad (3.2.1)$$

where

$$f(\Theta) = \sum_{x=c}^{\infty} a_x \Theta^x \quad (3.2.2)$$

and

$$a_x > 0 \text{ for all } x = c, c+1, \ldots \infty.$$

3.2.1 Now, if in a sample of size $N$ from gpsd (3.2.1), the frequency of $x$ is $n_x$ and we want an unbiased estimate for $\Theta$ of the type linear in $n_x$, we should be able to demonstrate the existence of a function of $x$, $t(x)$, such that, denoting the corresponding estimate

$$\hat{\Theta} = \sum_{x=c}^{\infty} t(x)n_x \quad (3.2.3)$$

we must have $E(\hat{\Theta}) = \Theta$ for all $\Theta$ in the parameter space of (3.2.1).

That is

$$N \sum_{x=c}^{\infty} t(x) \frac{a_x \Theta^x}{f(\Theta)} = \Theta$$

or

$$N \sum_{x=c}^{\infty} t(x) a_x \Theta^x = \sum_{x=c}^{\infty} a_x \Theta^{x+1}.$$
Since this is an identity in \( \Theta \), equating coefficients of corresponding powers of \( \Theta \), we get

\[
t(x) = 0 \quad \text{for } x = c
\]
\[
= \frac{1}{N} \left( \frac{a_{x-1}}{a_x} \right) \quad \text{for } x = c+1, c+2, \ldots \infty.
\]

3.2.2 Thus, the unique unbiased estimate of \( \Theta \) linear in the frequencies comes out to be the ratio estimate \( \Theta' \), because

\[
\hat{\Theta} = \frac{1}{N} \sum_{x=c+1}^{\infty} \left( \frac{a_{x-1}}{a_x} \right) n_x
\]
\[
= \frac{1}{N} \sum_{x=c+1}^{\infty} \left( \frac{a_{x-1}}{a_x} \right) n_x
\]
\[
= \frac{1}{N} \sum_{x=c}^{\infty} n_x
\]
\[
= \Theta'.
\]

The exact variance of this estimate is

\[
\sigma^2(\Theta') = \frac{1}{N} \left[ \sum_{x=c+1}^{\infty} \left( \frac{a_{x-1}}{a_x} \right)^2 P_x - \Theta'^2 \right].
\] (3.2.4)

An unbiased estimate of \( \sigma^2(\Theta') \) is

\[
[ \sum_{x=c+1}^{\infty} \left( \frac{a_{x-1}}{a_x} \right)^2 n_x - N\Theta'^2 ]/N(N-1)
\] (3.2.5)

the proof of which is almost immediate once one recognizes that \( \Theta' \) is the mean of \( N \) independent identically distributed random variables \( Y_i \) with probability distribution given by (for \( i = 1, 2, \ldots N \))

\[
\text{Prob} \{ Y_i = 0 \} = P_c
\]

and

\[
\text{Prob} \{ Y_i = \frac{a_{x-1}}{a_x} \} = P_x \quad \text{for } x = c+1, c+2, \ldots \infty.
\]
One can compare \( \sigma^2(\theta^*) \) with the asymptotic variance \( \text{Var}(\hat{\theta}) \) of the maximum likelihood estimate of \( \theta \) and the efficiency of the ratio estimate \( \theta^* \) can be computed. Of course,

\[
\sigma^2(\theta^*) > \text{Var}(\hat{\theta})
\]

from the Cramer-Rao Information limit to the variance; but this comparison is not quite valid because the maximum likelihood estimate is not generally unbiased.

Lastly, one may establish that

\[
\theta^r_r = \frac{1}{N} \sum_{x=c+r}^{\infty} \frac{a_{x-r}}{a_x} n_x \tag{3.2.6}
\]

is the only unbiased estimate of \( \theta^r \) (\( r \) an integer) which is a linear function of the frequencies.

3.3 Estimation by the Two-Moments Method for a gpdf

[Range \( T = (c,c+1,...d) \), \( d \) finite or \( T = (c,c+1,...\infty) \) with positive probabilities]

Consider the gpdf (1.1.4) with finite or infinite range \( T = (c,c+1,...d) \) with positive probabilities; that is, consider the gpdf

\[
P_x = \text{Prob}\{X = x\} = \frac{a_x \theta^x}{f(\theta)} \tag{3.3.1}
\]

where

\[
x \in T = (c,c+1,...d), \ d \text{ finite or infinite}
\]

\[
f(\theta) = \sum_{x=c}^{d} a_x \theta^x \quad a_x > 0 \tag{3.3.2}
\]

3.3.1 For this distribution, it is easy to see that

\[
\mu = \theta G_{01} + c P_c \tag{3.3.3}
\]
\[ m_2 = \mu + \Theta G_{11} + c(c-1)P_c \]  \hspace{1cm} (3.3.4)

where

\[ G_{ij} = \sum_{x=c}^{d-1} x^i \left[ \frac{(x+1)a_{x+1}}{a_x} \right]^j p_x . \]  \hspace{1cm} (3.3.5)

Further, from (3.3.3) and (3.3.4), we have

\[ \frac{m_2 - \mu - \Theta G_{11}}{\mu - \Theta G_{01}} = c-1 \quad \text{when } c \neq 0 \]  \hspace{1cm} (3.3.6)

which when solved for \( \Theta \) gives the identity

\[ \Theta = \frac{m_2 - c\mu}{G_{11} - (c-1)G_{01}} \quad \text{when } c \neq 0 . \]  \hspace{1cm} (3.3.7)

From (3.3.3), we have the identity

\[ \Theta = \frac{\mu}{G_{01}} \quad \text{when } c = 0 . \]  \hspace{1cm} (3.3.8)

### 3.3.2

The identities (3.3.7) and (3.3.8) can be made use of in estimating \( \Theta \). One has only to compute

\[ S_1 = \sum_{x=c}^d x^i n_x \quad i = 1,2 \]  \hspace{1cm} (3.3.9)

and

\[ S_{ij} = \sum_{x=c}^{d-1} x^i \left[ \frac{(x+1)a_{x+1}}{a_x} \right]^j n_x \quad i = 0,1; j = 1 \]  \hspace{1cm} (3.3.10)

from the sample, and then

\[ t = \frac{S_2 - cS_1}{S_{11} - (c-1)S_{01}} \quad \text{when } c \neq 0 \]  \hspace{1cm} (3.3.11)

or

\[ t = \frac{S_1}{S_{01}} \quad \text{when } c = 0 \]  \hspace{1cm} (3.3.12)
can be taken as an estimate for \( \Theta \). Because we use the first two moments for the estimation of the single parameter, we call the estimate \( t \) as "two-moments estimate" and the method as "two-moments method."

### 3.3.3 Proceeding along the same lines as in Section 2.6, one gets to terms of order \( \frac{1}{N} \),

\[
T(t) = E(t) - \Theta = \frac{1}{NG^2} (\Theta \sigma_{22} - \sigma_{12})
\]

and

\[
\text{Var}(t) = \frac{1}{N} \left[ \sigma_{11} - 2\Theta \sigma_{12} + \Theta^2 \sigma_{22} \right]
\]

where (i) for \( c \neq 0 \),

\[
\begin{align*}
G &= G_{11} - (c-1)G_{01} \\
\sigma_{11} &= (m_1^2 - \mu^2) + c^2(m_2 - \mu^2) - 2c(m_3 - \mu m_2) \\
\sigma_{12} &= (G_{31} - m_2 G_{11}) - c(G_{21} - \mu G_{11}) - (c-1)(G_{21} - m_2 G_{01}) \\
&+ c(c-1)(G_{11} - \mu G_{01}) \\
\sigma_{22} &= (G_{22} - G_{11}^2) + (c-1)^2(G_{02} - G_{01}^2) - 2(c-1)(G_{12} - G_{11} G_{01})
\end{align*}
\]

and (ii) for \( c = 0 \),

\[
\begin{align*}
G &= G_{01} \\
\sigma_{11} &= m_2 - \mu^2 \\
\sigma_{12} &= G_{11} - \mu G_{01} \\
\sigma_{22} &= G_{02} - G_{01}^2 .
\end{align*}
\]

### 3.4 Estimation by the Two-Moments Method for a Truncated \text{gpd}

Consider the \text{gpd} (3.3.1) truncated to

\[ T^* = (c^*, c^* + 1, \ldots, d^*), \quad d^* \neq d \text{ when } d \text{ finite}. \]
The truncated gpd can be written as
\[ P_x^* = \text{Prob} \{ X^* = x \} = \frac{a_x e^{\theta x}}{f^*(\theta)} \quad x \in T^* \]  
(3.4.1)

where
\[ f^*(\theta) = \sum_{x=c^{-}}^{d'} a_x e^{\theta x} . \]  
(3.4.2)

3.4.1 For this distribution, it is easy to see that
\[ \mu^* - \Theta H_{01} = c^- P_{c^-}^* - (d'^{-}+1)P_{d'}^* \]  
(3.4.3)

and
\[ m_{2}^* - \mu^* - \Theta H_{11} = c^- (c^- -1) P_{c^-}^* - d'^{-} (d'^{-}+1) P_{d'}^* \]  
(3.4.4)

where
\[ H_{i,j} = \sum_{x=c^-}^{d'} x^i \frac{(x+1)a_x+1}{a_x} P_x^* . \]  
(3.4.5)

3.4.2 For estimation purposes, we consider following four mutually exclusive and exhaustive cases:

Case (1) \( c^- = 0 \) and \( d' \) Finite
Case (2) \( c^- = 0 \) and \( d' \) Infinite
Case (3) \( c^- \neq 0 \) and \( d' \) Infinite
Case (4) \( c^- \neq 0 \) and \( d' \) Finite.

Case (1): \( c^- = 0 \) and \( d' \) Finite

From (3.4.3) and (3.4.4), we have the identity
\[ \Theta = \frac{m_{2}^* - (d'^{-}+1)\mu^*}{H_{11} - d'^{-}H_{01}} \]  
(3.4.6)

which we utilize to estimate \( \Theta \). We have only to compute
\[ S_i = \sum x^i n_x \quad i = 1, 2 \]  
(3.4.7)
and

\[ h_{ij} = \sum_{x=0}^{d'} x^i \left( \frac{(x+1)a_{x+1}}{a_x} \right)^j a_x \quad i = 0,1; \ j = 1 \quad (3.4.8) \]

from the sample and then

\[ t^* = \frac{S_2 - (d' + 1)S_1}{h_{11} - d'h_{01}} \quad (3.4.9) \]

can be taken as an estimate for \( \varphi \). The estimate \( t^* \) makes use of the (additional) information that the sample is taken from some known \( \text{gpd} \) and truncated to the one under consideration. The estimate \( t \) of Section (2.8) does not require, and hence, does not make use of this information. The formula for the bias and variance of \( t^* \) can be written down to order \( \frac{1}{N} \) as:

\[ b(t^*) = \frac{1}{NH^2} (\varphi \sigma_{22} - \sigma_{12}^*) \quad (3.4.10) \]

and

\[ \text{Var} \ (t^*) = \frac{1}{NH^2} (\sigma_{11}^* - 2\varphi \sigma_{12}^* + \varphi^2 \sigma_{22}^*) \quad (3.4.11) \]

where

\[ H = h_{11} - d'h_{01} \]

\[ \sigma_{11}^* = (m_{11}^* - m_{22}^*) + (d' + 1)^2 (m_{22}^* - \mu_{22}^2) - 2(d' + 1)(m_{33}^* - \mu_{33}^* m_{22}^*) \]

\[ \sigma_{12}^* = (H_{31} - m_{22}^* h_{11}) - (d' + 1)(H_{21} - \mu^* h_{11}) \]

\[ - d' (H_{21} - m_{22}^* h_{01}) + d'(d' + 1)(H_{11} - \mu^* h_{01}) \]

and

\[ \sigma_{22}^* = (H_{22} - H_{11}^2) + d' (H_{02} - H_{01}^2) - 2d' (H_{12} - H_{01} H_{11}). \]
Case (2) and Case (3): \( c' = 0, d' \) Infinite and \( c' \neq 0, d' \) Infinite

It can be easily verified in these cases that

\[
H_{ij} = G_{ij}
\]

and, hence,

\[
t^* = t.
\]

Thus, we have the same treatment as in Section 3.3.

Here we also observe that if we allow \( d' = d \) -- even when \( d \) is finite -- and use \( a_{d+1} = 0 \) formally, we again get \( H_{ij} = G_{ij} \)

and \( t^* = t \). This observation is specially important in the case of the binomial distribution.

Case (4): \( c' \neq 0 \) and \( d' \) Finite

It may be noted that the \( t^* \) estimate is not available in this case. However, the \( t \) estimate still works, and the estimate of \( \varphi \) can thus be obtained by employing two moments.

3.5 An Upper Bound for Bias Per Unit Standard Error for Ratio Estimates and Two-Moments Estimates

We first establish a general result true for the bias of an estimate of a certain type. Let the probability distribution, from which a sample \( x_1, x_2, \ldots, x_n \) is drawn, be a general distribution of a random variable \( X \) with a single parameter \( \varphi \).

3.5.1 Let \( t_1 \) and \( t_2 \) be two statistics based on the sample such that

\[
\frac{E(t_1)}{E(t_2)} = \frac{E[t_1(x_1, x_2, \ldots, x_n)]}{E[t_2(x_1, x_2, \ldots, x_n)]} = \varphi
\]

for all \( \varphi \) in the parameter space of the given distribution.
Consider the estimate \( s = \frac{t_1}{t_2} \) to estimate \( \Theta \). To find the bias in \( s \) per unit standard error of \( s \), we have

\[
\text{Cov} (s, t_2) = E(st_2) - E(s) E(t_2)
\]

\[
= E(t_1) - E(s) E(t_2)
\]

\[
= [\Theta - E(s)] E(t_2).
\] (3.5.1)

Now

\[
|\text{cov}(s, t_2)| \leq \sigma(s) \sigma(t_2)
\] (3.5.2)

where \( \sigma \) denotes the standard error. Therefore, from (3.5.1) we have

\[
\left| \frac{E(s) - \Theta}{\sigma(s)} \right| \leq \left| \frac{\sigma(t_2)}{E(t_2)} \right| = \left| \text{c.v.}(t_2) \right|
\] (3.5.3)

where \( \text{c.v.}(t_2) \) is the coefficient of variation of \( t_2 \). Thus, for the bias in \( s \) we have

\[
\left| \frac{b(s)}{\sigma(s)} \right| \leq \left| \text{c.v.}(t_2) \right|
\] (3.5.4)

In particular, when \( t_2 \) is a constant, we have an unbiased estimate for \( \Theta \).

It may be noted that the ratio-estimates and two-moments estimates for the parameter \( \Theta \) of gpdf's, which we discussed earlier, are estimates of the actual type of the estimate \( s \) that we have discussed in this section. Hence, the result in (3.5.4) also applies to them.

### 3.6 Estimation for a Truncated gpdf with a Finite Range of Consecutive Integers, Maximum Unknown

We have discussed methods of estimation in relation to gpdf's with known range. Sometimes, however, one has to estimate the parameter \( \Theta \) even when the range is not completely known, as well as when one is interested in estimating the maximum of a finite range. The problem
of estimation can be solved in such cases by using the "Ratio Method" and "Two-Moments Method" simultaneously. This is taken up in this section.

3.6.1 On the basis of a sample of size $N$ with frequency $n_x$ for $x (0 \leq x \leq L, \sum n_x = N)$ drawn from the gpdf given by the truncated gpdf (3.4.1),

$$P_x^* = \text{Prob} \{ X^* = x \} = \frac{a_x \theta^x}{f^*(\theta)} \quad x = 0, 1, 2, \ldots d'$$  \hspace{1cm} (3.6.1)

for which

$$f^*(\theta) = \sum_{x=0}^{d'} a_x \theta^x \quad a_x > 0$$  \hspace{1cm} (3.6.2)

where $d'$ is not known; to estimate $\theta$, we choose the ratio estimate

$$\theta' = \frac{L}{\sum_{x=0}^{L-1} (x+1)n_x/L} \sum_{x=0}^{L-1} n_x \cdot$$  \hspace{1cm} (3.6.3)

The advantage with $\theta'$ is that besides its simplicity, it does not require the knowledge of $d'$.

3.6.2 To estimate $d'$, however, the identity

$$\theta = \frac{m_2^* - (d' + 1)\mu^*}{H_{11} - d' H_{01}}$$  \hspace{1cm} (3.6.4)

gives

$$d' = \frac{m_2^* - \mu^* - \theta H_{11}}{\mu^* - \theta H_{01}}$$  \hspace{1cm} (3.6.5)

where $\mu^*$ and $m_2^*$ are the first two moments about the origin of (3.6.1)

and

$$H_{1j} = \sum_{x=0}^{d} x^j \left[ \frac{(x+1)a_x}{a_x} \right] P_x^*$$  \hspace{1cm} (3.6.6)
Therefore, the estimate of $d'$ can be obtained as

$$
\hat{d} = \frac{S_2 - S_1 - \theta' h_{11}}{S_1 - \theta' h_{01}}
$$

(3.6.7)

where $\theta'$ is given by (3.6.3) and

$$
S_i = \sum_{x=0}^{L} x^i n_x \quad i = 1,2
$$

(3.6.8)

and

$$
h_{ij} = \sum_{x=0}^{L} x^i \left[ \frac{(x+1 \alpha x + 1)}{\alpha x} \right]^j n_x.
$$

(3.6.9)
CHAPTER IV

4.0 ESTIMATION PROBLEMS FOR THE BINOMIAL DISTRIBUTION

4.1 Introduction

The gpdf defined by (1.1.4) becomes

\[ \text{Prob} \{ X = x \} = \binom{n}{x} \theta^x / (1 + \theta)^n \]  \hspace{1cm} (4.1.1)
\[ x = 0, 1, 2, \ldots n \]

when \( f(\theta) = (1 + \theta)^n \). Writing \( \theta = \pi / (1 - \pi) \), (4.1.1) gives the probability law for \( X \) as:

\[ \text{Prob} \{ X = x \} = \binom{n}{x} \pi^x (1 - \pi)^{n-x} \]  \hspace{1cm} (4.1.2)
\[ x = 0, 1, 2, \ldots n \]

the well-known form of the binomial distribution.

The important properties of (4.1.2) can be summarily stated as follows:

\[ M(t) = (1 - \pi + \pi e^t)^n. \]  \hspace{1cm} (4.1.3)

The first two central moments and the coefficients \( \beta_1, \beta_2 \) are of the form:

\[ \mu = n \pi \]
\[ \mu_2 = n \pi (1 - \pi) \]
\[ \beta_1 = (1 - 2\pi)^2 / n\pi(1 - \pi) \]
\[ \beta_2 = 3 + \left\{ 1 - 6\pi(1 - \pi) \right\} / n\pi(1 - \pi). \]

The recurrence relations reduce to:

\[ \mu_{r+1} = \pi(1 - \pi) \left[ \frac{d}{d\pi} + n\pi \mu_{r-1} \right] \]  \hspace{1cm} (4.1.5)
derived first by Romanovsky (1925) and

\[ K_{r+1} = \pi (1-\pi) \frac{dK_r}{d\pi} \]  \hfill (4.1.6)

deduced by Frisch (1925) and rediscovered by Haldane (1940).

The distribution function \( B(r, \pi, n) \) defined by

\[ B(r, \pi, n) = \sum_{x=0}^{r} b(x, \pi, n) \]  \hfill (4.1.7)

can be reduced to

\[ B(r, \pi, n) = I_{1-\pi} (n-r, r+1) \]  \hfill (4.1.8)

where

\[ I_x (m,n) = \frac{1}{B(m,n)} \int_0^1 u^{m-1} (1-u)^{n-1} \, du \]

for which extensive tables have been edited by K. Pearson. Romig has extensively tabulated \( b(x, \pi, n) \) and \( B(r, \pi, n) \) for the range of arguments \( - = 0.01 \) (0.01) 0.50, \( n = 50(5) 100 \) and Applied Mathematics Series 6, 1950, gives \( b(x, \pi, n) \) for \( n = 2 (1) 49 \). The ordnance corps tables (1952) give values of \( 1 - B(r, \pi, n) \) for \( \pi = 0.01 \) (0.01) 0.50, \( n = 2 (1) 150 \). For large \( n \) and small \( \pi \), one can use tables of Poisson or Normal Probabilities, because:

\[ \lim_{n \to \infty, \pi \to 0} \frac{b(x, \pi, n)}{n \pi} = p(x, \mu) \]  \hfill (4.1.9)

where

\[ p(x, \mu) = e^{-\mu} \frac{\mu^x}{x!} \]

and

\[ \lim_{n \to \infty} (B(r, \pi, n) = \Phi(Z), \]  \hfill (4.1.10)

\[ n \to \infty \]
where
\[ Z = \frac{r + 1/2 - np}{\sqrt{np(1 - \pi)}} \]
and
\[ \Phi(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} \, du. \]

On the basis of single observation on \( X \), \( X = x \), the maximum likelihood estimate for \( \pi \) is given by \( \hat{\pi} = x/n \), (\( n \) known). One has \( E(\hat{\pi}) = \pi \) with \( \text{Var}(\hat{\pi}) = \pi(1-\pi)/n \).

On the basis of a random sample \( x_i \) (\( i = 1, 2, \ldots N \)) of size \( N \) from (1.2), the maximum likelihood estimate for \( \pi \) is given by
\[ \hat{\pi} = \frac{\overline{x}}{n} \]
where
\[ \overline{x} = \frac{1}{N} \sum_{i=1}^{N} x_i / N. \]

(4.1.11) provides an unbiased estimate for the parameter \( \pi \) with
\[ \text{Var}(\hat{\pi}) = \pi(1-\pi)/nN. \]

4.2 Estimation from a Sample for a Singly Truncated Binomial Distribution

Fisher (1936) and Haldane (1932, 1938) discussed uses of the truncated binomial distribution. For instance, in problems of human genetics, in estimating the proportion of albino children produced by couples capable of producing albinos, sampling has necessarily to be restricted to families having at least one albino child. Finney (1949) has cited some more applications. Fisher and Haldane derived the maximum likelihood procedure to estimate the parameter \( \pi \). Moore
(1954) suggested a simple "ratio-estimate" based on an identity between binomial probabilities. For a slightly different problem, Rider (1955) suggested an alternative estimation procedure which uses first two moments.

We present in this section some numerical tables to facilitate the heavy computation involved in evaluating the maximum likelihood estimate of \( \pi \) from a sample from singly truncated binomial distribution. The estimates given by Moore and Rider have been derived from the general results discussed in Chapter III. The efficiency and the amount of bias of these estimates are investigated in certain special cases.

The probability law of the binomial distribution truncated at \( c \) on the left can be written as

\[
\begin{align*}
B^* (x, \pi, n) &= \left( B^* (c, \pi, n) \right)^{-1} \frac{n!}{x!} \pi^x (1-\pi)^{n-x} \\
x &= c, c+1, \ldots, n.
\end{align*}
\] (4.2.1)

where

\[
B^* (r+1, \pi, n) = 1 - B (r, \pi, n).
\] (4.2.2)

The first two moments about the origin of (4.2.1), then, are

\[
\mu^* = \mu^* (c, \pi, n) = n \pi \cdot \frac{B^* (c-1, \pi, n-1)}{B^* (c, \pi, n)} \quad (4.2.3)
\]

and

\[
\frac{m^*}{2} = \frac{m^*}{2} (c, \pi, n) = \mu^* (c, \pi, n) \left\{ 1 + \frac{\mu^* (c-1, \pi, n-1)}{\mu^* (c, \pi, n)} \right\} \quad (4.2.4)
\]

The case of truncation to the right can be dealt with in a similar way by replacing \( \pi \) by \( 1 - \pi \) and the truncation point \( c \) by \( n - c \).
4.2.1 To estimate \( \pi \) by likelihood on the basis of a random sample \( x_i \) (1=1, 2, \ldots N) of size \( N \) from (4.2.1), results derived by the general approach in Section 2.2 can be written down as follows with proper substitutions in this particular case.

The likelihood equation for \( \pi \) is

\[
\bar{x} = \hat{\mu}^*
\]  

(4.2.5)

where

\[
\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}
\]

and \( \hat{\mu}^* \) is defined by (4.2.3)

Denoting this estimate as \( \hat{\mu} \), its asymptotic variance is given by

\[
\text{Var} (\hat{\mu}) = \frac{\pi (1 - \pi)}{N} \frac{d^2 \mu^*}{d \pi^2}
\]

(4.2.6)

\[
= \left( \pi (1-\pi) \right)^2 \frac{N \mu^*}{\mu_2^*}
\]

(4.2.7)

where \( \mu_2^* \) is the variance of \( \mu^* \).

As the equation (4.2.5) does not readily give an algebraic solution, one may use an iterative process of solution. However, (4.2.5) suggests that if tables be made available for means \( \mu^* \)'s for sufficiently close values of \( \pi \), one can have a ready solution. The practical case of importance is \( c = 1 \) and sometimes \( c = 2 \). For the case \( c = 1 \),

\[
\mu^* = n \pi / B^*(1, \pi, n) = \frac{n \pi}{1 - (1-\pi)^n}
\]

so that the likelihood equation becomes

\[
\frac{\bar{x}}{n} = \frac{\pi}{1 - (1-\pi)^n}
\]

(4.2.8)
and the expression (4.2.6) for the asymptotic variance reduces to

\[
\text{Var} (\hat{\pi}) = \frac{\pi(1-\pi)}{N} \cdot \frac{1-(1-\pi)^n}{n} \left[ 1 + \frac{n\pi (1-\pi)^{n-1}}{1 - (1-\pi)^n} \right]^{-1}
\] (4.2.9)

a result first derived by Fisher (1936).

Here we present in Table I the values of \(\mu^*/n\) for the binomial distribution truncated on the left at \(c = 1\) for values of \(\pi\) spaced at suitable intervals. For the case \(c = 2\), we present Table II for values of \(\pi\) at intervals of 0.01. Suitable charts based on these tables may be also of great help in facilitating the procedure of estimation.

These tables can be used to compute \(\text{Var} (\hat{\pi})\) by using either Formula (4.2.6) or Formula (4.2.7). In case (4.2.6) is used, \(\Delta \frac{\mu^*}{\pi}\) can be approximated by the finite difference ratio \(\Delta \frac{\mu^*}{\pi}\). This approximation is expected to be good since the tabular \(\Delta \pi\) interval is small. In case Formula (4.2.7) is used, the relationship for use is

\[
\mu^*_2 (c, \pi, n) = \mu^* (c, \pi, n) \left[ 1 + \left( \mu^*(c-1, \pi, n-1) - \mu^* (c, \pi, n) \right) \right].
\] (4.2.10)

4.2.2 For a slightly different problem, where, in a sample from a complete binomial distribution, the frequencies in some lowest classes are missing, Rider (1955) suggested a method of estimation, which uses first two moments of the complete binomial and leads to a linear equation.

The method of two-moments is also applicable in the usual problem of estimation from a sample from singly truncated binomial and forms a particular case of the general method discussed in
Section 3.4. Proceeding on those lines, one gets in this case

\[ \theta = \frac{\pi}{1 - \pi} = \frac{m^* - cu^*}{H_{11} - (c-1)H_{01}} \]  \hspace{1cm} (4.2.11)

where \( \mu^* \) and \( m^*_2 \) are defined by (4.2.3) and (4.2.4) respectively, and \( H_{11} \) and \( H_{01} \) reduce to

\[ H_{11} = n\mu^* - m^*_2 \]
\[ H_{01} = n - \mu^* \]

(4.2.11) gives then

\[ \pi = \frac{m^*_2 - cu^*}{(n-1)\mu^* - n(c-1)} \]  \hspace{1cm} (4.2.12)

so that, on the basis of a random sample of size \( N \) with \( n_x \) as the frequency of \( x \) drawn from (4.2.1), the estimate for \( \pi \) can be written as

\[ t = \frac{S_2 - cs_1}{(n-1)s_1 - n(c-1)N} \]  \hspace{1cm} (4.2.13)

where

\[ s_1 = \sum x^n_x \]

and

\[ s_2 = \sum x^2 n_x \]

It is obvious that (4.2.13) is quite simple and that a great deal of computational labour can be saved if (4.2.13) is used instead of (4.2.5). On the other hand, the estimate obtained from (4.2.13) is likely to be inefficient. It is important, therefore, to investigate the loss in efficiency due to the use of (4.2.13) instead of (4.2.5).
To find the asymptotic variance of the two-moments estimate \( t \) of \( \pi \), one gets on some simplification,

\[
\text{Var} (t) = \frac{1}{NH^2} \left( \sigma^*_{11} + \pi^2 \sigma^*_{22} - 2\pi \sigma^*_{12} \right) \tag{4.2.14}
\]

where

\[
H = (n-1) \mu^* - n (c-1)
\]

\[
\sigma^*_{11} = (m^*_4 - \frac{m^*_2}{2}) + c^2 (m^*_2 - \mu^2) - 2c(m^*_3 - \mu m^*_2)
\]

\[
\sigma^*_{22} = (n-1) \left( m^*_2 - \mu^2 \right)
\]

and

\[
\sigma^*_{12} = (n-1) \left( m^*_3 - \mu m^*_2 \right) - c(m^*_3 - \mu^2)
\]

where \( m^*_r \) is the \( r \)-th theoretical moment of \((4.2.1)\) about the origin. Thus,

\[
\text{Var} (t) = \frac{1}{N \{ (n-1)\mu^* - n (c-1) \}^2} \left[ (m^*_4 - \frac{m^*_2}{2}) + \{(n-1)c + \pi \}^2 (m^*_3 - \mu) 
\right.
\]

\[
- 2 \{(n-1)c + \pi \} (m^*_3 - \mu m^*_2) \right]. \tag{4.2.15}
\]

The asymptotic efficiency of \( t \) is then given by

\[
\text{Eff} (t) = \frac{\text{Var} (\hat{\pi})}{\text{Var} (t)}. \tag{4.2.16}
\]

The special cases of some importance in genetics are \( c = 1 \) and \( \pi = 1/4, 1/2 \) or \( 3/4 \). The efficiency of the Two-Moments Estimate (TM) relative to the Maximum Likelihood Estimate (ML) in these cases is tabulated on the following page.
TABLE 4.2.1

ASYMPTOTIC EFFICIENCY OF TM FOR $c = 1$

<table>
<thead>
<tr>
<th></th>
<th>$\pi = 1/4$</th>
<th>$1/2$</th>
<th>$3/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>.925</td>
<td>.875</td>
<td>.875</td>
</tr>
<tr>
<td>4</td>
<td>.871</td>
<td>.818</td>
<td>.859</td>
</tr>
<tr>
<td>5</td>
<td>.817</td>
<td>.795</td>
<td>.870</td>
</tr>
<tr>
<td>6</td>
<td>.809</td>
<td>.789</td>
<td>.886</td>
</tr>
<tr>
<td>7</td>
<td>.781</td>
<td>.794</td>
<td>.901</td>
</tr>
<tr>
<td>8</td>
<td>.766</td>
<td>.803</td>
<td>.913</td>
</tr>
<tr>
<td>9</td>
<td>.755</td>
<td>.814</td>
<td>.923</td>
</tr>
<tr>
<td>10</td>
<td>.749</td>
<td>.823</td>
<td>.931</td>
</tr>
</tbody>
</table>

Close investigation of the above table shows that the efficiency of TM in case of $\pi = 1/2$ and $\pi = 3/4$ decreases in the beginning with $n$, reaches a minimum and then increases with increasing values of $n$. For $\pi = 1/4$, however, the efficiency decreases throughout. Let us compute, therefore, the efficiency of TM for higher values of $n$. The following gives the results obtained for $n = 11(1)15$.

<table>
<thead>
<tr>
<th></th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>.746</td>
</tr>
<tr>
<td>12</td>
<td>.744</td>
</tr>
<tr>
<td>13</td>
<td>.745</td>
</tr>
<tr>
<td>14</td>
<td>.747</td>
</tr>
<tr>
<td>15</td>
<td>.750</td>
</tr>
</tbody>
</table>
Thus, in case of $\pi = 1/4$ also, the efficiency reaches a minimum and then increases with increasing $n$. It is interesting to note that in all these cases the efficiency of TM has reached the minimum at $n = 3/\pi$.

4.2.3 Following the general approach discussed in Section 3.1, a simple estimate for $\pi$ can be obtained in the case of singly truncated binomial distribution (4.2.1). In this case, $a_{x-1}/a_x = x/(n-x+1)$ and since $Q = \pi/(1-\pi)$, we have the following "ratio-estimate" for $\pi$:

$$\pi' = \frac{t_1}{t_1 + t_2} \quad (4.2.17)$$

where

$$t_1 = \sum_{x=c+1}^{n} \frac{x n_x}{n-x+1}$$

and

$$t_2 = \sum_{x=c}^{n-1} n_x .$$

When $c = 1$, i.e. when only "zero" values are truncated, the estimate takes the form suggested by Moore (1954):

$$\pi' = \frac{t_1}{t_1 + t_2} \quad (4.2.18)$$

where

$$t_1 = \sum_{x=2}^{n} \frac{x n_x}{n-x+1}$$

and

$$t_2 = \sum_{x=1}^{n-1} n_x .$$
To investigate the efficiency of $\pi'$ given by (4.2.17) its asymptotic variance can be written down as:

$$\text{Var} (\pi') = \left( \frac{1 - \pi}{NP^2} \right)^2 \left[ (1 - \pi)^2 D - P \pi^2 + 2 \pi^2 P_{n-1} \right]$$

(4.2.19)

where

$$P = \sum_{n=c}^{n-1} b^*(x, \pi, n)$$

and

$$D = \sum_{x=c+1}^{n} \left( \frac{x}{n-x+1} \right) b^*(x, \pi, n)$$

$$P_{n-1} = b^* (n-1, \pi, n).$$

Also the asymptotic variance of the maximum likelihood estimate $\hat{\pi}$ obtained from (4.2.5) is given by

$$\text{Var} (\hat{\pi}) = \frac{\pi (1-\pi)^2}{N\mu^*}$$

Where $\mu^*$ is the variance of (4.2.1)

Therefore the asymptotic efficiency of $\pi'$ takes the form:

$$\text{Eff}(\pi') = \frac{P^2}{\mu^2} \left( \frac{1 - \pi}{\pi} \right)^2 \left[ D - P + 2 P_{n-1} \right]^{-1}$$

(4.2.20)
In particular, when \( c = 1 \)

\[
P = 1 - \frac{z^n}{1-(1-\pi)^n}
\]

\[
\mu_2^* = \frac{n\pi(1-\pi)}{1-(1-\pi)^n} \left( 1 + \frac{n\pi(1-\pi)^{n-1}}{1-(1-\pi)^n} \right)
\]

\[
P_{n-1} = \frac{n\pi^{n-1}(1-\pi)}{1-(1-\pi)^n}
\]

and

\[
D = \sum_{x=2}^{n} \left( \frac{2}{n-x+1} \right) \binom{n}{x} \pi^x(1-\pi)^{n-x} / [1-(1-\pi)^n] \]

reduces to

\[
D = \left( \frac{\pi}{1-\pi} \right) \left[ \frac{n+1}{1-(1-\pi)^n} \left( 1-\pi^n \right) E \left( \frac{1}{x}, n, 1-\pi \right) \right. \\
\left. - \frac{(1-\pi)^n}{n} \right] - P
\]

where

\[
E \left( \frac{1}{x}, n, \pi \right) = \sum_{x=1}^{n} \frac{1}{x} \binom{n}{x} \pi^x (1-\pi)^{n-x} / \left[ 1 - (1-\pi)^n \right]
\]

and is tabulated by Grab and Savage (1954).

The special cases of some importance in genetics are \( c = 1 \) and \( \pi = 1/4, 1/2 \) or 3/4. The efficiency of the Ratio-Estimate (R) relative to the Maximum Likelihood Estimate (ML) in these cases is tabulated and shown on the following page.
TABLE 4.2.2

ASYMPTOTIC EFFICIENCY OF R FOR c = 1

<table>
<thead>
<tr>
<th>n</th>
<th>π = 1/4</th>
<th>1/2</th>
<th>3/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>.924</td>
<td>.875</td>
<td>.875</td>
</tr>
<tr>
<td>4</td>
<td>.909</td>
<td>.769</td>
<td>.772</td>
</tr>
<tr>
<td>5</td>
<td>.919</td>
<td>.715</td>
<td>.664</td>
</tr>
<tr>
<td>6</td>
<td>.933</td>
<td>.694</td>
<td>.565</td>
</tr>
<tr>
<td>7</td>
<td>.947</td>
<td>.693</td>
<td>.523</td>
</tr>
<tr>
<td>8</td>
<td>.952</td>
<td>.705</td>
<td>.481</td>
</tr>
<tr>
<td>9</td>
<td>.956</td>
<td>.723</td>
<td>.435</td>
</tr>
<tr>
<td>10</td>
<td>.959</td>
<td>.776</td>
<td>.388</td>
</tr>
</tbody>
</table>

Close investigation of the above table shows that the efficiency of R in case of π = 1/4 and π = 1/2 decreases in the beginning with n, reaches a minimum and then increases with increasing values of n. For π = 3/4, however, the efficiency decreases throughout for n = 3(1)10.

4.2.4 For c = 1, we have separately discussed the Two-Moments Estimate and the Ratio-Estimate for π. To make a comparative study of these two equally simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.4 and 3.1, one gets, to order 1/N, the amount of bias of t (TM) and π' (R) as follows:

\[
b(t) = \frac{1}{N} \frac{\mu^2 + \mu'\sigma^2 - m^2}{(n-1)\mu'^2} = \frac{B(t)}{N}
\]

and

\[
b(\pi') = \frac{(1-\pi)^2}{N^2} \left[ \frac{\pi^2}{p} + \frac{(\pi-2\pi^2)p_n - (1-\pi)^2}{n-1} \right] = \frac{B(\pi')}{N}.
\]
The table on the following page gives $B(t)$, $B(\pi')$ and also a relative efficiency of $t$ over $\pi'$ for $c = 1$ and $\pi = 1/4, 1/2$ and $3/4$. The relative efficiency is given by Rel. Eff = Var $(\pi')/\text{Var}(t)$.

Let us also study the amount of bias relative to the standard error of the two estimates for some sample size say 100. The following table gives the bias as a percentage of standard error (100 $|b|$/S.E.) for both TM and R for $c = 1$ and $\pi = 1/4, 1/2$ and $3/4$.

**TABLE 4.2.4**

<table>
<thead>
<tr>
<th>n</th>
<th>$\pi = 1/4$</th>
<th>$\pi = 1/2$</th>
<th>$\pi = 3/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TM</td>
<td>R</td>
<td>TM</td>
</tr>
<tr>
<td>3</td>
<td>6.34</td>
<td>5.07</td>
<td>7.11</td>
</tr>
<tr>
<td>4</td>
<td>6.94</td>
<td>4.05</td>
<td>6.21</td>
</tr>
<tr>
<td>5</td>
<td>7.03</td>
<td>3.39</td>
<td>6.48</td>
</tr>
<tr>
<td>6</td>
<td>7.24</td>
<td>2.94</td>
<td>5.84</td>
</tr>
<tr>
<td>7</td>
<td>7.12</td>
<td>2.61</td>
<td>5.66</td>
</tr>
<tr>
<td>8</td>
<td>6.97</td>
<td>2.37</td>
<td>5.31</td>
</tr>
<tr>
<td>9</td>
<td>6.46</td>
<td>2.18</td>
<td>4.99</td>
</tr>
<tr>
<td>10</td>
<td>6.39</td>
<td>2.02</td>
<td>4.28</td>
</tr>
</tbody>
</table>

Table 4.2.3 shows that both TM and R are underestimates of $\pi$. A closer investigation, however, brings out that the bias to order $1/N$ is in general considerably smaller for R. Also, Table 4.2.3 shows that whereas for $\pi = 1/2$ and $\pi = 3/4$, R is less efficient than TM, it is more efficient when $\pi = 1/4$. Thus, a closer study of the relative efficiency of the two estimates is necessary. However, Table 4.2.3 suggests for $n = 3(1)10$ that the Ratio-Estimate may be used to estimate
<table>
<thead>
<tr>
<th>n</th>
<th>( N(\text{Amount of Bias to Order } 1/N) )</th>
<th>Var(R)</th>
<th>Var(TM)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( TM )</td>
<td>( R )</td>
<td></td>
</tr>
<tr>
<td><strong>Case (1) ( \pi = 1/4 )</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-.2412</td>
<td>-.1927</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>-.2152</td>
<td>-.1227</td>
<td>.958</td>
</tr>
<tr>
<td>5</td>
<td>-.1896</td>
<td>-.0861</td>
<td>.889</td>
</tr>
<tr>
<td>6</td>
<td>-.1715</td>
<td>-.0648</td>
<td>.867</td>
</tr>
<tr>
<td>7</td>
<td>-.1535</td>
<td>-.0510</td>
<td>.824</td>
</tr>
<tr>
<td>8</td>
<td>-.1379</td>
<td>-.0420</td>
<td>.804</td>
</tr>
<tr>
<td>9</td>
<td>-.1187</td>
<td>-.0355</td>
<td>.790</td>
</tr>
<tr>
<td>10</td>
<td>-.1097</td>
<td>-.0301</td>
<td>.781</td>
</tr>
<tr>
<td><strong>Case (2) ( \pi = \sqrt{2} )</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-.2717</td>
<td>-.1458</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>-.1940</td>
<td>-.1307</td>
<td>1.065</td>
</tr>
<tr>
<td>5</td>
<td>-.1748</td>
<td>-.1184</td>
<td>1.111</td>
</tr>
<tr>
<td>6</td>
<td>-.1398</td>
<td>-.1062</td>
<td>1.138</td>
</tr>
<tr>
<td>7</td>
<td>-.1230</td>
<td>-.0943</td>
<td>1.146</td>
</tr>
<tr>
<td>8</td>
<td>-.1063</td>
<td>-.0833</td>
<td>1.139</td>
</tr>
<tr>
<td>9</td>
<td>-.0934</td>
<td>-.0736</td>
<td>1.126</td>
</tr>
<tr>
<td>10</td>
<td>-.0748</td>
<td>-.0651</td>
<td>1.053</td>
</tr>
<tr>
<td><strong>Case (3) ( \pi = 3/4 )</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-.1763</td>
<td>-.0820</td>
<td>1.000</td>
</tr>
<tr>
<td>4</td>
<td>-.1290</td>
<td>-.0833</td>
<td>1.112</td>
</tr>
<tr>
<td>5</td>
<td>-.1004</td>
<td>-.0893</td>
<td>1.543</td>
</tr>
<tr>
<td>6</td>
<td>-.0818</td>
<td>-.0962</td>
<td>1.569</td>
</tr>
<tr>
<td>7</td>
<td>-.0689</td>
<td>-.0931</td>
<td>1.724</td>
</tr>
<tr>
<td>8</td>
<td>-.0595</td>
<td>-.0928</td>
<td>1.900</td>
</tr>
<tr>
<td>9</td>
<td>-.0524</td>
<td>-.0915</td>
<td>2.124</td>
</tr>
<tr>
<td>10</td>
<td>-.0468</td>
<td>-.0896</td>
<td>2.398</td>
</tr>
</tbody>
</table>
the parameter $\pi$ of a binomial distribution truncated at $c = 0.1$,
especially when $\pi$ is near to $1/4$ or less; whereas Two-Moments Estimate
may be preferred when $\pi$ is near to $1/2$ or more.

4.2.5 The detailed computation procedure of evaluating the
three types of estimates discussed above will be illustrated with
reference to K. Pearson's data on albinism in man. The table below
gives the number of families ($n_x$) each of five children having exactly
$x$ albino children in the family, ($x = 1, 2, 3, 4, 5$).

<table>
<thead>
<tr>
<th>Number of albinos in family ($x$)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of families ($n_x$)</td>
<td>25</td>
<td>23</td>
<td>10</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

If $\pi$ is the probability for a child to be an albino, we may
accept the truncated binomial model:

$$
\frac{n}{x^x (1-\pi)^{n-x}} \frac{1 - (1-\pi)^R}{1 - (1-\pi)^R}
$$

$x = 1, 2, \ldots n$.

for the probability of $x$ albinos in a family of $n$. Here $n = 5$, and
the problem is to estimate $\pi$ on the basis of the data given in the
table above.

**Maximum Likelihood Estimate**: From the table, we get

$$
N = 60
$$

$$
S_x = \sum x n_x = 110
$$

$$
\bar{x} = S_x / N = 1.8333
$$
so that \( \bar{x}/n = 0.366667 \). Referring to Table I for \( n = 5 \), we find the following:

\[
\begin{array}{cc}
\pi & \mu*/n \\
0.30 & 0.360607 \\
0.31 & 0.367474
\end{array}
\]

The maximum likelihood estimate is given by that value of \( \pi \) for which \( \mu*/n = 0.366667 \). By linear interpolation, we thus get

\[
\hat{\pi} = 0.30 + \frac{0.366667 - 0.360607}{0.367474 - 0.360607} (0.31 - 0.30)
\]

\[
= 0.3088 .
\]

The variance of this estimate is estimated from the formula

\[
\text{Var} (\hat{\pi}) = \frac{\pi (1 - \pi)}{N \left( \frac{\partial \mu*}{\partial \pi} \right)} .
\]

\( (\frac{\partial \mu*}{\partial \pi}) \) can be obtained approximately from the tables by taking differences instead of derivatives. Thus,

\[
(\frac{\partial \mu*}{\partial \pi}) \approx \frac{5 \times (0.367474 - 0.360607)}{0.31 - 0.30} = 3.4335 .
\]

Hence,

\[
\text{Var} (\hat{\pi}) \approx \frac{0.3088 \times 0.6912}{60 \times 3.4335} = 0.0010361 .
\]

Thus the standard error of \( \hat{\pi} \) is given approximately by:

\[
\text{S.E.} (\hat{\pi}) = \sqrt{(0.0010361)} = 0.03219 .
\]

This is as far as linear interpolation in the tables will go. If we want to carry the approximate solution of the likelihood equation
further, we start with $\pi_o = 0.3088$ as the first approximation and compute, the theoretical mean

$$\mu^* = \frac{n\pi_o}{1 - (1-\pi_o)^n} = \frac{5 \times 0.3088}{1 - (0.6912)^5}$$

$$= 1.83330$$

and

$$\frac{d\mu^*}{d\pi_o} = \frac{-\mu^*}{\pi_o(1-\pi_o)} \left[ 1 + (n-1) \pi_o - \mu^* \right]$$

$$= \frac{1.83330}{0.3088 \times 0.6912} \left( 1 + 4 \times 0.3088 - 1.83330 \right)$$

$$= 3.4520.$$

It is interesting to compare the exact value of $\frac{d\mu^*}{d\pi}$ viz., 3.4520 at $\pi = 0.3088$ with the approximation obtained by differencing in the tables, viz., 3.4335. The next approximation is then given by:

$$\pi_1 = \pi_o + (\bar{X} - \mu^*) / (\frac{d\mu^*}{d\pi})$$

$$= 0.3088 + (1.83333 - 1.83330) / 3.4520$$

$$= 0.308809$$

which does not affect the fourth place. The variance of the estimate is

$$\text{Var}(\hat{\pi}) = \frac{\pi(1-\pi)}{N} / (\frac{d\mu^*}{d\pi})$$

$$= \frac{0.3088 \times 0.6912}{60} / 3.4520$$

$$= 0.0010305$$

so that the standard error is $\text{S.E.}(\hat{\pi}) = \sqrt{0.0010305} = 0.03210.$

The agreement of this with the previous estimate obtained directly from the tables is remarkable, and these latter computations
are really unnecessary, especially in view of the somewhat large standard error.

It is believed that linear interpolation in the tables would be generally adequate for all practical purposes and the second cycle of approximation would not be necessary.

**Two-Moments Estimate:** To compute this estimate for \( \pi \), we require in addition the value of \( S_2 = \sum x^2 \frac{n_x}{n} = 248 \). Then the estimate is

\[
\begin{align*}
t &= \frac{1}{n-1} \left[ \frac{S_2}{S_1} - 1 \right] \\
&= \frac{1}{4} \left[ \frac{248}{110} - 1 \right] = 0.3136.
\end{align*}
\]

To compute the variance of \( t \) we require:

\[
\begin{align*}
\mu^*_n &= \frac{\mu n}{1 - (1 - \pi)^n} = 1.95280 \\
m^*_2 &= \mu^*[ (n-1)\pi + 1 ] = 4.40239 \\
m^*_3 &= \mu^*[ (n-1)\pi+1 + (n-1)\pi \{ (n-2)\pi + 2 \} ] \\
&= 11.60615
\end{align*}
\]

and

\[
\begin{align*}
m^*_4 &= \mu^*[ (n-1)\pi+1 + 3(n-1)\pi \{ (n-2)\pi+2 \} \\
&\quad + (n-1)\pi. (n-2)\pi \{(n-3)\pi+3 \} ]
\end{align*}
\]

all evaluated by taking 0.3136 as the estimate for \( \pi \). The variance of \( t \) is estimated from the formula

\[
\text{Var} (t) = \frac{1}{N \{ (n-1)\mu^* \}^2} \left[ (m^*_4 - m^*_2^2) + (n-1)\pi+1 \{ (n-2)\pi+2 \} (m^*_4 - m^*_2^2) \right. \\
&\left. - 2 \{ (n-1)\pi+1 \} (m^*_3 - m^*_2) \right] \\
&= 0.0012066
\]

so that the standard error is \( \text{S.E.} (t) = 0.03474 \).
Ratio Estimate: The ratio estimate for \( \pi \) is given by

\[
\pi' = \frac{t_1}{t_1 + t_2}
\]

where

\[
t_1 = \sum_{x=2}^{n} \left( \frac{x}{n-x+1} \right) n_x
\]

and

\[
t_2 = \sum_{x=1}^{n-1} n_x.
\]

Here \( t_2 = 59 \) and \( t_1 \) can be computed from the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{x}{n-x+1} )</th>
<th>( n_x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, \( t_1 = 28.50 \).

The ratio estimate is obtained as \( \pi' = \frac{28.50}{28.50 + 59} = 0.3257 \).

To compute the variance of \( \pi' \), we require

\[
P = 1 - \frac{\pi^n}{1-(1-\pi)^n} = 0.99574
\]

\[
P_{n-1} = \frac{n(1-\pi)}{\pi} (1-P) = 0.04408
\]

\[
\mu_2 = \frac{n \pi}{1-(1-\pi)^n} \left[ (n-1)\pi + 1 - \frac{n \pi}{1-(1-\pi)^n} \right] = 0.86052,
\]
and

\[ D = \left( \frac{\pi}{1-\pi} \right) \left[ \frac{n+1}{1-(1-\pi)^n} \right] \left\{ (1 - \pi^n) \ E \left( \frac{1}{X}, n, 1-\pi \right) - \frac{(1-\pi)^n}{n} \right\} - P \]

\[ = 0.56156 . \]

The quantity \( E(\frac{1}{X}, n, 1-\pi) \) was obtained from the table by Grab and Savage, which gives for \( n = 5, \)

\[ \begin{array}{ll}
1 - \pi & E(\frac{1}{X}, n, 1-\pi) \\
65 & 0.35465 \\
70 & 0.32183
\end{array} \]

By interpolation, \( E(\frac{1}{X}, n, 1-\pi) = 0.33870 \) taking \( \pi' = 0.3257 \) as the estimate for \( \pi \) throughout. Then the variance of \( \pi' \) is estimated from the formula

\[ \text{Var} (\pi') = \frac{P^2}{\mu^*} \left[ \left( \frac{1-\pi}{\pi} \right)^2 D - P + 2P \left( \frac{n-1}{n} \right) \right]^{-1} \]

\[ = 0.0013410 \]

so that the standard error of \( \pi' \) is S.E. (\( \pi' \)) = 0.03662. The following table summarizes the results obtained:

<table>
<thead>
<tr>
<th>Estimate</th>
<th>Value</th>
<th>Variance</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML</td>
<td>0.3088</td>
<td>0.0010305</td>
<td>0.03210</td>
</tr>
<tr>
<td>TM</td>
<td>0.3136</td>
<td>0.0012066</td>
<td>0.03474</td>
</tr>
<tr>
<td>R</td>
<td>0.3257</td>
<td>0.0013410</td>
<td>0.03662</td>
</tr>
</tbody>
</table>
4.3 Homogeneity and Combined Estimation for Singly Truncated Binomial Distributions

While sampling for studies in albinism, observations are available simultaneously from families with varying family-size. In such situations, one may be required to examine if the distributions (in the case of albinism: families) are homogeneous in respect of the parameter $\pi$ and if so, to make a combined estimate of $\pi$.

4.3.1 On the basis of a random sample of size $N = \sum_{j=1}^{k} N_j$ from $k$ singly truncated binomial distributions characterized by the probability law:

$$\left[ B^* \left( c, \pi, n_j \right) \right]^{-1} \binom{n_j}{x} \pi^x (1-\pi)^{n_j-x} \quad (4.3.1)$$

$x = 1, 2, \ldots n_j$

$j = 1, 2, \ldots k$, following Section 2.5.

The $j$-th "efficient score" is

$$\psi_j = \frac{N_j}{\pi_j (1-\pi_j)} \left[ \frac{1}{\pi_j} - \mu_j(\pi_j) \right] \quad (4.3.2)$$

where $\mu_j(\pi_j)$ is the mean of the $j$-th distribution.

The elements of the information matrix are

$$I_{jj} = \frac{N_j}{\pi_j (1-\pi_j)} \frac{d\mu_j}{d\pi_j} = \frac{N_j \mu_{2j}(\pi_j)}{[\pi_j (1-\pi_j)]^2} \quad (4.3.3)$$

$I_{jj} = 0 \quad j \neq j'$

where $\mu_{2j}(\pi_j)$ is the variance of the $j$-th distribution.

The hypothesis of homogeneity is $H_0: \pi_1 = \pi_2 = \ldots = \pi_k$. 
If the hypothesis $H_0$ is true, the common value may be denoted by $\pi$ and the efficient score and the information with respect to $\pi$ are given by:

$$\psi = \frac{N}{\pi(1-\pi)} \left[ \bar{x} - \frac{1}{N} \sum_{j=1}^{k} N_j \mu_j(\pi) \right]$$  \hspace{1cm} (4.3.4)$$

where

$$\bar{x} = \frac{1}{N} \sum_{j=1}^{k} N_j \bar{x}_j$$  \hspace{1cm} (4.3.5)$$

and

$$I = \frac{1}{\pi(1-\pi)} \sum_{j=1}^{k} N_j \frac{d\mu_j}{d\pi}$$  \hspace{1cm} (4.3.6)$$

$$= \frac{1}{[\pi(1-\pi)]^2} \sum_{j=1}^{k} N_j \mu_{2j}(\pi)$$  \hspace{1cm} (4.3.7)$$

To solve the equation $\psi = 0$ for $\pi$, we may start with an approximation $\pi_0$ and derive a better approximation $\pi_1$ from the formula

$$\pi_1 = \pi_0 + N \left[ \bar{x} - \frac{1}{N} \sum_{j=1}^{k} N_j \mu_j(\pi_0) \right] / \sum_{j=1}^{k} N_j \left( \frac{d\mu_j}{d\pi} \right) \pi_0$$  \hspace{1cm} (4.3.8)$$

Or

$$\pi_1 = \pi_0 + N\pi_0(1-\pi_0) \left[ \bar{x} - \frac{1}{N} \sum_{j=1}^{k} N_j \mu_j(\pi_0) \right] / \sum_{j=1}^{k} N_j \mu_{2j}(\pi_0)$$  \hspace{1cm} (4.3.9)$$

and repeat this process of iteration till sufficient accuracy is attained. This maximum likelihood estimate will be denoted by $\hat{\pi}$.

A test of the homogeneity hypothesis $H_0$ is then given by the statistic

$$\chi^2_{k-1} = \sum_{j=1}^{k} \psi_j^2(\hat{\pi}) / I_{jj}(\hat{\pi})$$

$$= \frac{1}{\pi(1-\pi)} \sum_{j=1}^{k} \left[ N_j [\bar{x}_j - \mu_j(\pi)]^2 / \frac{d\mu_j}{d\pi} \right] / \pi$$

$$= \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\pi)]^2 / \mu_{2j}(\pi)$$  \hspace{1cm} (4.3.10)$$

$$= \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\pi)]^2 / \mu_{2j}(\pi),$$  \hspace{1cm} (4.3.11)$$
which is asymptotically distributed as a Chi-square with \((k-1)\) df if \(H_0\) is true.

4.3.2 We shall illustrate the computational technique and the use of the tables with reference to the problem of estimating the proportion of albino children from K. Pearson's (1913) data quoted below:

<table>
<thead>
<tr>
<th>No. of children in family</th>
<th>No of families</th>
<th>Total number of albino children in the families</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n_j)</td>
<td>(N_j)</td>
<td>(T_j)</td>
</tr>
<tr>
<td>2</td>
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<td>16</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Total \(N = 411\) \(T = 864\)

To get the first approximation, we compute the values of \(\bar{x}_j/n_j = T_j/N_j n_j\) and referring to the tables or to the charts obtain for each household-size an estimate of \(\pi_j\). This is done in columns (3) and (4) of Table 4.3.1.

We find the estimates clustering around \(\pi_0 = 0.30\) which value we take as the starting point of our computations.

The next step is to read off from the tables the mean values \(\mu_j^* (\pi_0)\) and the difference-ratios \(\delta_j = [\Delta \mu_j^* / \Delta \pi]_0\) for different values of \(n_j\)'s. These are shown in columns (5) and (6) of Table 4.3.1.


<table>
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<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
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<td>$T_j$</td>
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<td>$x_j/n_j$</td>
<td>$\pi_j$</td>
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<tr>
<td>Total</td>
<td>411</td>
<td>864</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Weighted average: $2.10219$

$\pi = 0.3082$  $\chi^2_{13} = 19.558$

** From Table I **

** Computed directly from the formulae **

$$\mu_j(\pi) = \frac{n_j \pi}{1 - (1 - \pi) n_j}; \quad \frac{d\mu_j}{d\pi} = \frac{\mu_j}{\pi(1 - \pi)} [1 + (n_j - 1) \pi - \mu_j]$$
The details of the computation process are the same as on the illustrative example in Section 4.2.

The next step is to compute the weighted averages.

\[ \bar{x} = \frac{\sum N_j \bar{x}_j}{N} = 2.10219 \]

\[ \bar{\mu} = \frac{\sum N_j \mu_j}{N} = 2.06462 \]

and

\[ \bar{\delta} = \frac{\sum N_j \delta_j}{N} = 4.57352. \]

Then next approximation to the maximum likelihood estimate is thus:

\[ \hat{\pi}_1 = \pi_0 + \frac{\bar{x} - \bar{\mu}}{\bar{\delta}} = 0.3082 \]

the same as obtained by Haldane (1938) who wrote down the likelihood equation in the form

\[ \frac{T}{\pi} = \sum \frac{N_j N_j}{\lambda(1-\pi) N_j} \]

and solved it directly by iteration. One single computation in our case is thus sufficient. The variance of the estimate is approximately given by:

\[ V(\hat{\pi}) = \frac{\pi(1-\pi)}{N \bar{\delta}} \]

\[ = \frac{0.3082 \times 0.6918}{411 \times 4.57352} = 0.000113 \]

so that \(\text{S.E.}(\hat{\pi}) = 0.0106\). The values for S.E. obtained by Haldane is 0.0107 which is the same for all practical purposes. To test if the proportion of albino children is the same in families of different sizes, we have to compute

\[ \chi^2_{13} = \frac{1}{\hat{\pi}(1-\hat{\pi})} \sum N_j \left[ \frac{\bar{x}_j - \mu^*(\hat{\pi})}{\frac{d\mu}{d\pi}} \right]^2. \]
But since the maximum likelihood estimate $\hat{\pi} = 0.3082$ does not differ very much from our starting approximation $\pi_0 = 0.30$ we may use the approximation

$$\frac{d\mu_j}{d\pi}_\pi \sim \frac{\Delta\mu_j}{\Delta\pi}_\pi = \delta_j$$

which are already computed. Thus

$$\chi^2_{13} = \frac{1}{0.3082 \times 0.6918} \sum_j N_j \frac{(x_j - \mu_j)^2}{\delta_j}$$

$$= \frac{4.1701}{0.3082 \times 0.6918} = 19.558$$

which with $14-1 = 13$ degrees of freedom is not significant. The families of different sizes can thus be regarded as homogeneous in respect of the proportion of albino children and the common proportion is $0.3082 \pm 0.0106$.

### 4.4 Estimation from a Sample for a Doubly Truncated Binomial Distribution

In studying albinism, sampling may be restricted to only those families which contain at least one albino child and also at least one non-albino, Finney (1949), giving thus rise to samples from doubly truncated binomial distribution. We discuss the case of general truncation here and present some numerical tables to facilitate the heavy computation involved in evaluating the maximum likelihood estimate of $\pi$ from a sample from a binomial of which only extremes are truncated. The simple "ratio-estimate" is also derived and its efficiency is investigated for this special case of practical importance.

#### 4.4.1 The probability law of a doubly truncated binomial
with truncation points, say, at \( c \) and \( d \) can be written as:

\[
\begin{align*}
\text{b* (x, } & \pi, n) = \left[\text{B* (c, } d, \pi, n)\right]^{-1}, \\
(\frac{n}{x}) & \pi^{x} (1-\pi)^{n-x} \\
x & = c, c+1, ... d
\end{align*}
\]  

(4.4.1)

where

\[
\text{B* (c, } d, \pi, n) = \text{B (d, } \pi, n) - \text{B (c -1, } \pi, n). \quad (4.4.2)
\]

The first two moments about the origin is (4.4.1) are

\[
\mu^{*} = \mu^{*}(c, d, \pi, n) = n \pi \cdot \text{B*(c-l, d-l, } \pi, n-1)/\text{B*(c, d, } \pi, n) 
\]  

(4.4.3)

and

\[
\mu_{2}^{*} = \mu_{2}^{*}(c, d, \pi, n) = \mu^{*}(c, d, \pi, n).  \\
\quad [1 + \mu^{*}(c-1, d-1, \pi, n-1)].  
\]  

(4.4.4)

To estimate \( \pi \) on the basis of a random sample of size \( N \) with frequency \( n_{x} \) for \( x \) drawn from (4.4.1), the likelihood equation for \( \pi \) can be written down as:

\[
\frac{n}{\pi} = \frac{\hat{\mu}^{*}}{n} 
\]  

(4.4.5)

and the asymptotic variance of the estimate \( \hat{\pi} \) obtained from (4.4.5) is

\[
\text{Var} (\hat{\pi}) = \frac{\pi(1-\pi)}{N} / \frac{d\mu^{*}}{d\pi} 
\]  

(4.4.6)

\[
= \frac{\left[\pi (1-\pi)\right]^2}{N \mu_{2}^{*}}. 
\]  

(4.4.7)

To facilitate the solution of (4.4.5) in the special case when \( c = 1 \) and \( d = n-1 \) i.e extreme observations truncated, we present tables at suitable intervals of \( \pi \) for \( \mu^{*}/n \) which reduces in this case to:

\[
\mu^{*}/n = \mu^{*}(1, n-1, \pi, n) = (n - \pi) / [1 - (1-\pi) n]. 
\]  

(4.4.8)
An approximate value of \( \text{Var}(\hat{\pi}) \) may be obtained by getting
\[
\frac{\Delta \mu^*}{\Delta \pi} \sim \Delta \mu^* \text{ from tables of } \mu^* \text{ or else to get the exact value, one can use }
\[
\mu^*_2(1,n-1,\pi,n) = \mu^*_2(1,n-1,\pi,n) [1 + \mu^*_2(0,n-2,\pi,n-1) - \mu^*_2(1,n-1,\pi,n)]
\]
where values of \( \mu^*_2(0,n-2,\pi,n-1) \) can be obtained from Table I for single truncation.

4.4.2 Unlike cases of single truncation, it may be of interest to note that the two-moments estimate is not available here. This follows from Section 3.4, Case 4.

4.4.3 Following the general approach discussed in Section 3.1, the simple "ratio-estimate" for \( \pi \) can be written down for \( c = 1 \) and \( d = n-1 \) as:
\[
\pi' = \frac{t_1}{t_1 + t_2}
\]
where
\[
t_1 = \sum_{x=2}^{n-1} \left( \frac{x n x}{(n-x+1)} \right)
\]
and
\[
t_2 = \sum_{x=1}^{n-2} n_x
\]
The asymptotic variance of \( \pi' \) then takes the form:
\[
\text{Var}(\pi') = \frac{(1-\pi)^2}{N \pi^2} [(1-\pi)^2 D - P_x^2 + 2 \pi^2 P_{n-2}]
\]
where
\[
P_x = \binom{n}{x} \pi^x (1-\pi)^{n-x} / [1-\pi^n - (1-\pi)^n]
\]
\[
P = \sum_{x=1}^{n-2} P_x
\]
and
\[ D = \sum_{x=2}^{n-1} \left( \frac{x}{n-x+1} \right)^2 p_x \]

which reduces, in this case, to
\[
D = \left( \frac{\pi}{1-\pi} \right) \left( \frac{n+1}{n} \right) \left[ (1-\pi) \sum_{x=1}^{n} \frac{x}{n} E\left(\frac{1}{x}, n, 1-\pi\right) - \frac{P_{n-1}}{n-1} + \frac{P_n}{n} \right] - p
\]

where
\[
E\left(\frac{1}{x}, n, \pi\right) = \sum_{x=1}^{n} \frac{1}{x} x^\pi (1-\pi)^{n-x} / [n-(1-\pi)^n] .
\]

### 4.5 Simultaneous Estimation of Both Parameters of a Binomial Distribution

The binomial distribution has essentially two parameters \(\pi\) and \(n\) of which \(n\) is usually known and only \(\pi\) has to be estimated. However, certain cases might arise in which \(n\) is unknown, and both \(n\) and \(\pi\) have to be estimated. For instance, while experimenting with a radioactive substance, in addition to the mean number (\(\mu = n\pi\)) of disintegrating atoms, it may perhaps be of interest to know the number (\(n\)) of atoms capable of disintegration for the substance in fixed intervals of time for some specified solid angle and fit a model correspondingly.

4.5.1 To estimate \(\pi\) and \(n\) on the basis of a random sample of size \(N\) with observed frequency \(n_x\) for \(x \left(\sum n_x = N\right)\) drawn from (4.2.1) with \(n\) unknown, following Section 2.6, the moment-estimates are given by
\[
\bar{x} = n\pi \quad (4.5.1)
\]
and
\[
S^2 = n\pi (1-\pi) , \quad (4.5.2)
\]
where \( \bar{x} = \sum x_n x / N \) and \( s^2 = \sum n_x (x - \bar{x})^2 / N \).

The likelihood equations reduce to

\[
\bar{x} = \hat{n} \hat{\pi}
\]

and

\[
\sum_{r \geq 0} \frac{T_{r+1} + N \log (1 - \hat{\pi})}{\hat{n} - r} = 0
\]

where

\[
T_{r+1} = \sum_{x \geq r} n_x
\]

Eliminating \( \hat{\pi} \) from (4.5.3) and (4.5.4) we have to solve for \( \hat{n} \), the equation:

\[
\sum_{r \geq 0} \frac{T_{r+1}}{\hat{n} - r} + N \log (1 - \bar{x}) = 0
\]

The elements of the information matrix are:

\[
I_{11} = nN/\pi(1-\pi)
\]

\[
I_{12} = N/(1-\pi)
\]

\[
I_{22} = E[\sum T_{r+1}/(n-r)^2] = N \sum [1-B(r,\pi,n)]/(n-r)^2.
\]

We note that \( n \) is a discrete parameter and also the range of (4.2.1) depends on \( n \). The properties of the estimates are therefore not known, but in the (4.5.1) and (4.5.2) worked out, however, fairly accurate results are obtained.

4.5.2 To estimate \( \pi \) and \( n \) on the basis of a random sample of size \( N \) with observed frequency \( n_x \) for \( x \) \((\sum n_x = N)\) drawn from a truncated binomial, say given by (4.4.1) the moment-equations are:

\[
\bar{x} = \mu
\]

and

\[
s^2 = m^2
\]
where \( \bar{x} = \frac{\sum x_i}{N} \), \( S^2 = \frac{\sum x_i^2 - \bar{x}^2}{N} \) and \( \pi^* \) and \( \mu^* \) are defined by (4.4.3) and (4.4.4) respectively.

The "efficient scores" for \( \pi \) and \( n \) reduce to:

\[
\psi_1 = \frac{N}{\pi(1-\pi)} \left( \bar{x} - \mu^* \right)
\]

and

\[
\psi_2 = \sum_{r\geq0} \frac{T_{r+1}}{n-r} + N \log (1-\pi) - N \frac{\partial B^*/B^*}{\partial n}
\]

where \( B^* \)'s are defined by (4.4.2).

The likelihood equations then become

\[
\bar{x} = \mu^*
\]

and

\[
\psi_2 = 0
\]

The elements of "information matrix" are

\[
I_{11} = \frac{N}{\pi(1-\pi)} \cdot \frac{\partial \mu^*}{\partial n}
\]

\[
I_{12} = \frac{N}{\pi(1-\pi)} \cdot \frac{\partial \mu^*}{\partial n}
\]

and

\[
I_{22} = \left[ \frac{\partial^2 B^*/B^*}{\partial n^2} \right] + E \left[ \sum_{r\geq0} \frac{T_{r+1}}{(n-r)^2} \right]
\]

(4.5.11) and (4.5.12) may be solved for estimation, approximating

\[
\frac{\partial}{\partial n} B(r, \pi, n) \approx \frac{\Delta B}{\Delta n}
\]

and getting \( \frac{\Delta B}{\Delta n} \) from binomial tables where \( B \) is defined by (4.1.7).
However, exact values of $\frac{\partial}{\partial n} B(r, \pi, n)$ and $\frac{\partial^2}{\partial n^2} B(r, \pi, n)$ which we shall call "Incomplete Dibeta and Tribeta Function" respectively, can be obtained as follows:

$$4.5.3 \quad \frac{\partial}{\partial n} B(r, \pi, n) = \frac{\partial}{\partial n} I_{1-\pi} (n-r, r+1)$$

$$= I_{1-\pi} (n-r, r+1) \left[ E_z(n-r, r+1) \right]_{z=1}^{z=1-\pi}$$

$$\text{(4.5.14)}$$

where $I$'s are incomplete beta functions, and

$$E_z(n, m) = \frac{\int_0^1 u^{n-1} (1-u)^{m-1} \log u \, du}{\int_0^1 u^{n-1} (1-u)^{m-1} \, du} \quad (4.5.15)$$

which means the expected value of $\log u$ when $u$ follows a beta distribution truncated on the right at $z$, with parameters $n$ and $m$. $E_z(n, m)$ can be reduced to

$$E_z(n, m) = \log z - \frac{1}{I_z(n, m)} \sum_{r=0}^{m-1} I_z(n+r, m-r) \frac{1}{n+r} \quad (4.5.16)$$

In particular,

$$E_1(n, m) = \sum_{r=0}^{m-1} \frac{1}{n+r} \quad (4.5.17)$$

(4.5.16) suggests that the values of "Incomplete Dibeta Function" can be exactly obtained by using tables of "Incomplete Beta Function" which are extensively tabulated.

To obtain "Incomplete Tribeta Function"

$$\frac{\partial^2}{\partial n^2} B(r, \pi, n) = \frac{\partial}{\partial n} I_{1-\pi} (n-r, r+1) \left[ E_z(n-r, r+1) \right]_{z=1}^{z=1-\pi}$$

$$\text{(4.5.18)}$$
We get after some simplification of R.H.S. of (4.5.18),

\[
\frac{\partial^2}{\partial n^2} B(r, \pi, n) = I_{1-\pi}^{1}(n-r, r+1) \left( [E_z(n-r, r+1)]^2 \right. \\
+ \left. [V_z(n-r, r+1)]^2 \right)
\]  

(4.5.19)

where \( V_z(n,m) \) is the variance of \( \log u \) when \( u \) follows a beta distribution with parameters \( n \) and \( m \) truncated on the right at \( z \).

\( V_z(n,m) \) can be obtained from

\[
V_z(n,m) = E_z^2(n,m) - [E_z(n,m)]^2
\]  

(4.5.20)

where \( E_z^2(n,m) \) can be reduced to

\[
E_z^2(n,m) = (\log z)^2 - \frac{2}{I_z(n,m)} \sum_{r=0}^{m-1} \frac{I_z(n+r, m-r)}{n+r} E_z(n+r, n-r).
\]  

(4.5.21)

In particular,

\[
E_1^2(n,m) = 2 \sum_{r=0}^{m-1} \left( \sum_{r=0}^{m-1} \frac{1}{n+r} \right)/(n+r).
\]  

(4.5.22)

4.5.4 The computation procedure for simultaneous estimation will be illustrated with reference to two examples: one on radioactive disintegrations and the second one on throwing of dice.

Example 4.5.1 The first two columns of the following table give data collected by Rutherford and others, showing the number (\( n_x \)) of intervals of time, each of 7.5 seconds, during which the number (\( x \)) of \( \alpha \) particles omitted from a certain radioactive substance.
Data: Rutherford and Geiger: Radioactive Disintegration

<table>
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<tr>
<th>No. of α particles</th>
<th>No. of intervals</th>
<th>$T_x = \sum_{s} n_s$</th>
<th>$T_X$</th>
<th>$n = 77$</th>
<th>$n = 79$</th>
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<td>$n_x$</td>
<td>$T_x$</td>
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<td></td>
<td></td>
</tr>
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<td>0</td>
<td>57</td>
<td>2608</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td></td>
</tr>
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<td>30.1026</td>
<td></td>
</tr>
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<td>25.5195</td>
<td></td>
</tr>
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<td>18.9474</td>
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<td>12.1067</td>
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</tr>
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<td>6.7568</td>
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</tr>
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<td>0.6056</td>
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<td>16</td>
<td>0.2353</td>
<td>0.2286</td>
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<td>0.0896</td>
<td>0.0870</td>
<td></td>
</tr>
<tr>
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<td>2</td>
<td>0.0303</td>
<td>0.0294</td>
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</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>2608</strong></td>
<td><strong>134.2995</strong></td>
<td><strong>131.0065</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\sum x n_x = 10094$ \quad N \log n - \log(n \cdot x)$

$x = 3.870$ \quad 134.4390 \quad 130.9693

$\frac{\sum x n_x}{N} = 48650$ \quad \psi

$s^2 = 3.676$ \quad -0.1395 \quad +0.0372

$n = 77 \quad \hat{n} = 78$

Moment Estimates: Following Section 4.5, we have for this data,

$N = 2608$

$\bar{x} = \frac{\sum xn_x}{N} = 3.870$

$s^2 = \frac{\sum(x-\bar{x})^2 n_x}{N} = 3.676$

so that the estimate for the mean number $\mu$ of $\alpha$-particles emitted per interval is

$\mu = \bar{x} = 3.870$
and the number \( n \) of particles capable of disintegration for the
substance during the interval of 7.5 seconds is estimated by

\[
    n = \frac{x^2}{\bar{x} - \bar{x}^2} = 77.
\]

**Maximum likelihood estimates**: The estimate for the mean number of \( \alpha \)-
particles per interval remains the same, namely \( \hat{\mu} = 3.870 \). To get
the estimate \( \hat{n} \) of \( n \), starting with the moment estimate \( n = 77 \), we
solve the equation:

\[
    \psi(n) = \sum_{r \geq 0} \frac{T_{r+1}}{n-r} - N \left[ \log n - \log (n-x) \right] = 0.
\]

For \( n = 77 \), we have \( N \left[ \log n - \log (n-x) \right] = 134.4390 \). From column
4 of the above table, we have for \( n = 77 \),

\[
    \sum_{r \geq 0} \frac{T_{r+1}}{n-r} = \sum_{x \geq 1} \frac{T_x}{n-x+1} = 134.2995
\]

\[
    \therefore \psi(77) = 134.2995 - 134.4390 = -0.1395.
\]

Let us try next \( n = 79 \), say. Now, \( N \left[ \log n - \log (n-x) \right] = 130.9693 \)
and column 5 of the above table gives for \( n = 79 \),

\[
    \sum_{r \geq 0} \frac{T_{r+1}}{n-r} = 131.0065
\]

\[
    \therefore \psi(79) = 0.0372
\]

Thus, whereas \( \psi(77) \) is negative, \( \psi(79) \) is positive and therefore
the likelihood estimate for \( n \) is \( \hat{n} > 77 \) and \( < 79 \).

\[
    \therefore \hat{n} = 78.
\]

**Example 4.5.2** The first two columns of the following table give data,
due to Weldon, that show the results of throwing \( n \) dice 4096 times, a
throw of 4, 5 or 6 being called a success. \( x \) denotes the number of
successes and \( n_x \) the frequency of \( x \).
<table>
<thead>
<tr>
<th>Successes</th>
<th>Frequency</th>
<th>$T_x = \sum n_x$</th>
<th>$\frac{T_x}{n = 12}$</th>
<th>$\frac{T_x}{n = 13}$</th>
<th>$\frac{T_x}{n = 11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$n_x$</td>
<td>$T_x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>4096</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>4096</td>
<td>371.7273</td>
<td>315.0769</td>
<td>372.3636</td>
</tr>
<tr>
<td>2</td>
<td>60</td>
<td>4089</td>
<td>371.7273</td>
<td>340.7500</td>
<td>408.9000</td>
</tr>
<tr>
<td>3</td>
<td>198</td>
<td>4029</td>
<td>402.9000</td>
<td>366.2727</td>
<td>447.6666</td>
</tr>
<tr>
<td>4</td>
<td>430</td>
<td>3831</td>
<td>425.6667</td>
<td>383.1000</td>
<td>478.8750</td>
</tr>
<tr>
<td>5</td>
<td>731</td>
<td>3401</td>
<td>425.1250</td>
<td>377.8888</td>
<td>485.8571</td>
</tr>
<tr>
<td>6</td>
<td>948</td>
<td>2670</td>
<td>381.4285</td>
<td>333.7500</td>
<td>445.0000</td>
</tr>
<tr>
<td>7</td>
<td>847</td>
<td>1722</td>
<td>287.0000</td>
<td>245.0000</td>
<td>344.4000</td>
</tr>
<tr>
<td>8</td>
<td>536</td>
<td>875</td>
<td>175.0000</td>
<td>145.8333</td>
<td>218.7500</td>
</tr>
<tr>
<td>9</td>
<td>257</td>
<td>339</td>
<td>84.7500</td>
<td>67.8000</td>
<td>113.0000</td>
</tr>
<tr>
<td>10</td>
<td>71</td>
<td>82</td>
<td>27.3333</td>
<td>20.5000</td>
<td>41.0000</td>
</tr>
<tr>
<td>11</td>
<td>11</td>
<td>11</td>
<td>5.5000</td>
<td>3.6666</td>
<td>11.0000</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>4096</strong></td>
<td><strong>2927.7641</strong></td>
<td><strong>2600.6383</strong></td>
<td><strong>3366.8123</strong></td>
<td></td>
</tr>
</tbody>
</table>

$$\sum n_x = 25145 \quad \quad \quad N \log n - \log (n-\bar{x})$$

$$\bar{x} = 6.139 \quad \quad \quad 2935.1030 \quad 2617.7000 \quad 3344.9000$$

$$\sum x^2 n_x = 166367$$

$$s^2 = 2.930 \quad \quad \quad -7.3389 \quad -17.0617 \quad + 21.9123$$

$$n = 12 \quad \quad \quad \hat{n} = 12$$

**Moment Estimates:** Following Section 4.5, we have for Example (4.5.2)

$$N = 4096$$

$$\bar{x} = 6.139$$

$$s^2 = 2.930$$

so that the estimate for the number of dice thrown is given by

$$n = \frac{x^2}{\bar{x} - s^2} = 12 \quad ,$$
and the estimate of the proportion of successes ($\pi$) is

$$\pi = \frac{\bar{x}}{n} = \frac{6.139}{12} = 0.5116$$

**Maximum likelihood estimates:** To get firstly the estimate $\hat{n}$ of $n$, starting with the moment-estimate $n = 12$, we solve the equation:

$$\psi(n) = \sum_{r \geq 0} \frac{T_{r+1}}{n-r} - N[\log n - \log (n-\bar{x})] = 0 \cdot$$

For $n = 12$, we have $N[\log n - \log (n-\bar{x})] = 2935.1030$. From column 4 of the above table, we have for $n = 12$,

$$\sum_{r \geq 0} \frac{T_{r+1}}{n-r} = \sum_{x \geq 1} \frac{T_x}{n-x+1} = 2927.7641$$

$$\therefore \psi(12) = 2927.7641 - 2935.1060$$

$$= -7.3389$$

Let us try next $n = 13$, say, to see if $\psi(13)$ is near zero. We get by proceeding as before, $\psi(13) = -17.0617$ which is further from zero than $\psi(12)$. Therefore, we try $n = 11$. We have then $\psi(11) = 21.9123$ which indicates that $\hat{n} = 12$. The estimate of $\pi$ is then obtained by

$$\hat{\pi} = \frac{\bar{x}}{\hat{n}} = 0.5116.$$ (Note: Weldon had thrown 12 dice).
5.0 ESTIMATION PROBLEMS FOR THE POISSON DISTRIBUTION

5.1 Introduction

The gpdf defined by (1.1.4) becomes

\[ \text{Prob}\{ X = x \} = \frac{1}{x!} e^x / e^\theta \quad x = 0, 1, 2, \ldots \infty \]  \hspace{1cm} (5.1.1)

when f(\theta) = e^\theta. Writing \( \mu = \theta \), (1.1.4) gives the probability law for X as:

\[ \text{Prob}\{ X = x \} = p(x, \mu) = e^{-\mu} \cdot \frac{\mu^x}{x!}, \quad x = 0, 1, 2, \ldots \infty \]  \hspace{1cm} (5.1.2)

the well-known form of the Poisson distribution.

The important properties of (5.1.2) can be summarily stated as follows:

\[ M(t) = e^{\mu(e^t - 1)}. \]  \hspace{1cm} (5.1.3)

The first two central moments and the coefficients \( \beta_1, \beta_2 \) are of the form:

\[ \mu = \mu \]
\[ \mu_2 = \mu \]
\[ \beta_1 = 1/\mu \]
\[ \beta_2 = 3 + 1/\mu. \]  \hspace{1cm} (5.1.4)

The recurrence relation connecting the central moments is

\[ \mu_{r+1} = \mu \left( \frac{d\mu_r}{d\mu} + r\mu_{r-1} \right) \]  \hspace{1cm} (5.1.5)

and all the cumulants are equal

\[ K_r = \mu \quad r = 1, 2, \ldots \]  \hspace{1cm} (5.1.6)

It is well-known that the equality of all cumulants is necessary and sufficient for a probability distribution to become Poisson. For
any gpsd to be Poisson, however, the equality of first two cumulants
only is necessary and sufficient (Chapter I, Theorem 3).

The distribution function \( P(r, \mu) \) defined by

\[
P(r, \mu) = \sum_{x=0}^{r} p(x, \mu)
\]

(5.1.7)
can be reduced to

\[
P(r, \mu) = I_\mu(r+1)
\]

(5.1.8)
where

\[
I_x(r) = \int_{0}^{x} e^{-u} u^{r-1} du / \int_{0}^{\infty} e^{-u} u^{r-1} du
\]

is the incomplete gamma integral tabulated by K. Pearson. Molina (1947)
has extensively tabulated \( p(x, \mu) \) and \( 1-P(r, \mu) \) for the range of argument
\( \mu = 0.001 \ (0.0001) .01(0.01).30(.1) 15(1) 100 \). Kitagawa (1952) has
also edited "Tables of Poisson Distribution." For large \( \mu \), we have
the normal approximation given by:

\[
\lim_{\mu \to \infty} P(r, \mu) = \phi(z)
\]

(5.1.9)
where

\[
z = (r + 1/2 -\mu)/\sqrt{\mu}
\]

and

\[
\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u/2} u^{2} du.
\]

On the basis of a random sample \( x_i (i = 1, 2, \ldots N) \) of size \( N \)
from (5.1.2), both the moments and maximum likelihood estimate for \( \mu \)
is given by

\[
\hat{\mu} = \bar{x}
\]

(5.1.10)
where

\[
\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \ (5.1.10) \text{ provides an unbiased estimate of } \mu
\]

with \( \text{Var} (\hat{\mu}) = \mu/N \).
5.2 Estimation from a Sample for Truncated Poisson Distribution

Problems of estimation in a truncated Poisson distribution with known truncation points have been discussed by various authors. The case of truncation on the left has been considered by David and Johnson (1948) who gave the maximum likelihood estimate, by Plackett (1953) who gave a simple and highly efficient ratio-estimate, and by Rider who used first two moments. Truncation on the right has been discussed by Tippett (1932), Bliss (1948), and Moore (1952). Tippett derived the maximum likelihood solution, Bliss developed an approximation to it, and Moore suggested a simple ratio estimate. Double Truncation has been studied by Moore (1954) and Cohen (1954). Moore gave ratio-estimates, while Cohen provided likelihood equations.

Neat and compact equations for estimation by the method of maximum likelihood (which has been shown to be identical with the method of moments, in general, for single-parameter gpdf's) can be derived from the general approach discussed in Chapter II. We present numerical tables and some suitable charts to facilitate the solution of these equations in certain special cases. The estimates given by Rider and Moore have been derived from the general results discussed in Chapter III. The efficiency and the amount of bias of these estimates are investigated in some cases. Problem of estimation has been also considered for single truncation with unknown truncation point.

5.2.1 The probability law of the singly truncated Poisson Distribution with truncation point on the right at \( \hat{d} \) can be written as:

\[
p^\star(x, \mu) = [P(d, \mu)]^{-1} e^{-\mu} \frac{\mu^x}{x!} \quad (5.2.1)
\]

\[x = 0, 1, 2, \ldots \hat{d}\]

where \( P(r, \mu) \) is defined by (5.1.7).
The first four moments about the origin of (5.2.1) can be written down as:
\[ \mu^* = \mu^* (d, \mu) = \mu P(d-1, \mu) / P(d, \mu) \]  
(5.2.2)
\[ m_{2}^* = m_2^*(d, \mu) = \mu^*(d, \mu) [1 + \mu^*(d-1, \mu)] \]  
(5.2.3)
\[ m_{3}^* = m_3^*(d, \mu) = \mu^*(d, \mu) [1 + 2\mu^*(d-1, \mu) + m_2^*(d-1, \mu)] \]

and
\[ m_{4}^* = m_4^*(d, \mu) = \mu^*(d, \mu) [1 + 3\mu^*(d-1, \mu) + 3m_2^*(d-1, \mu)] \]
\[ + m_2^*(d-1, \mu) \].

5.2.2 To estimate \( \mu \) on the basis of a random sample \( x_i (i = 1, 2, \ldots N) \) of size \( N \) from (5.2.1), the results derived by the general approach in Chapter II can be written down as follows.

The likelihood equation for \( \mu \) is
\[ \bar{x} = \hat{\mu}^* \]  
(5.2.4)
where
\[ \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \]  
and \( \mu^* \) is defined by (5.2.2).

Denoting this estimate as \( \hat{\mu} \), the asymptotic variance is given by
\[ \text{Var} (\hat{\mu}) = \frac{\mu^*/N}{\left( \frac{d\mu^*}{d\mu} \right)} \]  
(5.2.5)
\[ = \frac{\mu^2}{N\mu^*_2} \]  
(5.2.6)
where \( \mu^*_2 \) is the variance of (5.2.1). Equation (5.2.4) suggests that if tables be made available for means \( \mu^* \)'s for sufficiently close values of \( \mu \), we can have a ready solution of (5.2.4). We present in
Table IV a numerical table for the arguments $\mu = 0.0(1)4.9$ and $d = 4(1)10$.

This table can be used to compute $\text{Var} (\hat{\mu})$ by using Formula (5.2.5) or Formula (5.2.6). In case (5.2.5) is used, $\frac{d\mu^*}{d\mu}$ can be approximated by the finite difference ratio $\frac{\Delta \mu^*}{\Delta \mu}$. In the event Formula (5.2.6) is used, the relationship for use is

$$\mu_2^*(d, \mu) = \mu^*(d, \mu)[1 + \mu^*(d-1, \mu) - \mu^*(d, \mu)]. \quad (5.2.7)$$

5.2.3 The method of two moments is applicable in the usual problem of estimation from a sample from a Poisson distribution singly truncated on the right, and forms a particular case of the general method discussed in Section 3.4 of Chapter III. Proceeding on those lines, one gets in this case

$$\phi = \mu = \frac{m_2^* - (d+1)\mu^*}{H_{11} - dH_{01}} \quad (5.2.8)$$

where $\mu^*$ and $m_2^*$ are defined by (5.2.3) and (5.2.4), respectively, and $H_{11}$ and $H_{01}$ reduce to

$$H_{11} = \mu^*$$

$$H_{01} = 1.$$

Then (5.2.8) gives

$$\mu = \frac{m_2^* - (d+1)\mu^*}{\mu^* - d} \quad (5.2.9)$$

so that, on the basis of a random sample of size $N$ with $n_x$ as the frequency of $x$ drawn from (5.2.1), the estimate for $\mu$ can be written as

$$t = \frac{S_2 - (d+1)S_1}{S_1 - dN}, \quad (5.2.10)$$
where
\[ S_1 = \sum x_n \quad \text{and} \quad S_2 = \sum x^2_n. \]

To find the asymptotic variance of the two-moments estimate (TM) given by (5.2.10), one gets on simplification,

\[ \text{Var} \ (t) = \frac{1}{NH^2} \left( \sigma_{11}^* + \mu^2 \sigma_{22}^* - 2\mu \sigma_{12}^* \right) \]  (5.2.11)

where
\[ H = \mu^* - d \]

\[ \sigma_{11}^* = (m_4^* - m_2^*2) + (d+1)^2(m_2^* - \mu^*2) \]
\[ - 2(d+1)(m_2^* - \mu^*m_2^*) \]  (5.2.12)

\[ \sigma_{22}^* = m_2^* - \mu^*2 \]

and

\[ \sigma_{12}^* = (m_3^* - \mu^*m_2^*) - (d+1)(m_2^* - \mu^*2) \]

where \( m_r^* \) is the \( r \)-th theoretical moment of (5.2.1) about origin.

Thus,
\[ \text{Var} \ (t) = \frac{1}{N(\mu^*-d)^2} \left[ (m_4^* - m_2^*2) + (\mu+d+1)^2(m_2^* - \mu^*2) \right. \]
\[ - 2(\mu+d+1)(m_2^* - \mu^*m_2^*) \].  (5.2.13)

The asymptotic efficiency of \( t \) is then given by

\[ \text{Eff} \ (t) = \frac{\text{Var} \ (\bar{X})}{\text{Var} \ (t)}. \]

The following table gives the asymptotic efficiency of \( t \) relative to \( \mu \) for values of \( d = 5 \) with \( \mu = .25, .5(.5) 2.5 \), and \( d = 10 \) with \( \mu = .5(.5)5. \)
TABLE 5.2.1

EFFICIENCY OF TM

<table>
<thead>
<tr>
<th>μ</th>
<th>.25</th>
<th>.50</th>
<th>1.00</th>
<th>1.50</th>
<th>2.00</th>
<th>2.50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case (i) d = 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eff.</td>
<td>.978</td>
<td>.954</td>
<td>.904</td>
<td>.867</td>
<td>.850</td>
<td>.838</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>μ</th>
<th>.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case (ii) d = 10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eff.</td>
<td>.989</td>
<td>.988</td>
<td>.960</td>
<td>.942</td>
<td>.920</td>
<td>.897</td>
<td>.874</td>
<td>.855</td>
<td>.835</td>
<td>.815</td>
</tr>
</tbody>
</table>

Thus, the asymptotic efficiency of TM is never less than 82 percent in the above cases, and one may therefore use TM to estimate μ in such problems.

5.2.4 Following the general approach discussed in Section 3.1 of Chapter III, a simple ratio-estimate can be obtained for μ of (5.2.1). In this case, \( a_{x-1}/a_x = x \), and since \( \Theta = \mu \), the ratio-estimate for \( \mu \) takes the form

\[
\mu' = \frac{\sum x n_x}{\sum n_x} \left( \frac{d}{d-1} \right)
\]  \hspace{1cm} (5.2.14)

as first suggested by Moore (1954).

The asymptotic variance of the "Ratio-estimate" (\( R \)) given by (5.2.14), can be obtained as

\[
\text{Var} (\mu') = \frac{1}{NPR^2} (D - \mu^2 + 2\mu^2_{d-1}), \hspace{1cm} (5.2.15)
\]
where, in this case,

\[ P = \sum_{0}^{d-1} p^*(x,\mu) \]

\[ D = \sum_{0}^{d} x^2 \ p^*(x,\mu) = m^* \]

and

\[ P_{d-1} = p^*(d-1,\mu). \]

The asymptotic efficiency of \( \mu' \) is then given by

\[ \text{Eff} \ (\mu') = \text{Var} \ (\hat{\mu}) / \text{Var} \ (\mu'). \]

The following table gives the asymptotic efficiency of \( \mu' \) relative to \( \hat{\mu} \) for values of \( d = 5 \) with \( \mu = .25, .5(5)2.5 \), and \( d = 10 \) with \( \mu = .5(5)4.5 \).

### TABLE 5.2.2

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>.25</th>
<th>.50</th>
<th>1.00</th>
<th>1.50</th>
<th>2.00</th>
<th>2.50</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case (i) d = 5</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eff.</td>
<td>.999</td>
<td>.990</td>
<td>.979</td>
<td>.967</td>
<td>.951</td>
<td>.923</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Case (ii) d = 10</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eff.</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>.999</td>
<td>.992</td>
<td>.981</td>
<td>.894</td>
<td>.817</td>
</tr>
</tbody>
</table>

Thus, \( R \) seems to be highly efficient on the whole and its efficiency always exceeds 82 percent in Table 5.2.2.
5.2.5 So far, we have separately discussed the Two Moments Estimate and the Ratio Estimate for \( \mu \). To make a comparative study of these two simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.1 and 3.3 of Chapter III, one gets to order \( \frac{1}{N} \) the amount of bias of \( t(TM) \) and \( \mu'(R) \) as follows:

\[
 b(t) = \frac{(\mu \sigma_{22}^* - \sigma_{12}^*)}{NH^2} = \frac{B(t)}{N} \tag{5.2.16}
\]

where \( H, \sigma_{22}^* \) and \( \sigma_{12}^* \) are defined by (5.2.12) and

\[
 b(\mu') = \frac{\mu P_{d-1}}{NP^2} = \frac{B(\mu')}{N} \tag{5.2.17}
\]

The relative efficiency of \( R \) over \( TM \) is given by

\[
 \text{Rel.Eff.} = \frac{\text{Var}(t)}{\text{Var}(\mu')}.
\]

Table 5.2.3 gives \( B(t), B(\mu') \) and Rel. Eff. for values of \( d = 5 \) with \( \mu = .25, .5(.5)2.5 \) and \( d = 10 \) with \( \mu = .5(.5)5 \). Table 5.2.3 shows that both TM and R are over-estimates of \( \mu \). A closer investigation, however, brings out that the bias to order \( \frac{1}{N} \) is always considerably smaller for R. Also, R is more efficient than TM. Thus, we conclude that one may prefer the Ratio Estimate to the Two-Moments Estimate for \( \mu \) of the Poisson distribution singly truncated on the right because of its simplicity, small bias and high efficiency.

5.2.6 The probability law of the singly truncated Poisson distribution with truncation point on the left at \( c \) can be written as:

\[
p^*(x, \mu) = \left[ \frac{P^*(c, \mu)}{x!} \right] e^{-\mu} \frac{\mu^x}{x!} \quad \text{for} \quad x = c, c+1, \ldots \infty
\tag{5.2.18}
\]
TABLE 5.2.3

COMPARISON BETWEEN TM AND R

<table>
<thead>
<tr>
<th>μ</th>
<th>N(Amount of Bias to order 1/N)</th>
<th>Var(TM)</th>
<th>Var(R)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TM</td>
<td>R</td>
<td></td>
</tr>
<tr>
<td>.25</td>
<td>.0526</td>
<td>.0003</td>
<td>1.022</td>
</tr>
<tr>
<td>.50</td>
<td>.1111</td>
<td>.0008</td>
<td>1.048</td>
</tr>
<tr>
<td>1.00</td>
<td>.2498</td>
<td>.0015</td>
<td>1.134</td>
</tr>
<tr>
<td>1.50</td>
<td>.4260</td>
<td>.0719</td>
<td>1.187</td>
</tr>
<tr>
<td>2.00</td>
<td>.6507</td>
<td>.1977</td>
<td>1.199</td>
</tr>
<tr>
<td>2.50</td>
<td>.9181</td>
<td>.4461</td>
<td>1.210</td>
</tr>
</tbody>
</table>

Case (i) d = 5

| .5  | .0526                          | .0000   | 1.011  |
| 1.0 | .1111                          | .0000   | 1.012  |
| 1.5 | .1765                          | .0003   | 1.041  |
| 2.0 | .2500                          | .0004   | 1.062  |
| 2.5 | .3333                          | .0022   | 1.087  |
| 3.0 | .4284                          | .0081   | 1.115  |
| 3.5 | .5547                          | .0231   | 1.144  |
| 4.0 | .6640                          | .0536   | 1.170  |
| 4.5 | .8093                          | .1063   | 1.243  |
| 5.0 | .9786                          | .1876   | 1.226  |

Case (ii) d = 10
## TABLE 5.2.4

**BIAS AS A PERCENTAGE OF STANDARD ERROR FOR N = 100**

<table>
<thead>
<tr>
<th>μ</th>
<th>Bias/ S.E. X 100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TM</td>
</tr>
<tr>
<td></td>
<td>Case (i) d = 5</td>
</tr>
<tr>
<td>.25</td>
<td>1.0412</td>
</tr>
<tr>
<td>.50</td>
<td>1.5339</td>
</tr>
<tr>
<td>1.00</td>
<td>2.3566</td>
</tr>
<tr>
<td>1.50</td>
<td>3.9107</td>
</tr>
<tr>
<td>2.00</td>
<td>3.9114</td>
</tr>
<tr>
<td>2.50</td>
<td>4.3275</td>
</tr>
<tr>
<td></td>
<td>Case (ii) d = 10</td>
</tr>
<tr>
<td>.5</td>
<td>.7402</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1043</td>
</tr>
<tr>
<td>1.5</td>
<td>1.4120</td>
</tr>
<tr>
<td>2.0</td>
<td>1.7150</td>
</tr>
<tr>
<td>2.5</td>
<td>2.0202</td>
</tr>
<tr>
<td>3.0</td>
<td>2.3352</td>
</tr>
<tr>
<td>3.5</td>
<td>2.7484</td>
</tr>
<tr>
<td>4.0</td>
<td>3.0121</td>
</tr>
<tr>
<td>4.5</td>
<td>3.3599</td>
</tr>
<tr>
<td>5.0</td>
<td>3.8477</td>
</tr>
</tbody>
</table>
where

\[ P^*(c, \mu) = 1 - P(c-1, \mu). \]  \hspace{1cm} (5.2.19)

The first two moments about origin of (5.2.1) can be written down as

\[ \mu^* = \mu^*(c, \mu) = \mu \frac{P^*(c-1, \mu)}{P^*(c, \mu)} \]  \hspace{1cm} (5.2.20)

and

\[ m_2^* = m_2^*(c, \mu) = \mu^*(c, \mu) \left[ 1 + \mu^*(c-1, \mu) \right]. \]  \hspace{1cm} (5.2.21)

To estimate \( \mu \) on the basis of a random sample \( x_i \) \((i = 1, 2, \ldots, N)\) of size \( N \) from (5.2.18), results derived by the general approach in Chapter II can be written down as follows.

The likelihood equation for \( \mu \) is

\[ \bar{x} = \hat{\mu}^* \]  \hspace{1cm} (5.2.22)

where

\[ \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \] and \( \mu^* \) is defined by (5.2.20).

Denoting this estimate as \( \hat{\mu} \), the asymptotic variance is given by

\[ \text{Var} (\hat{\mu}) = \frac{\mu^2}{N} \left( \frac{d \mu^*}{d \mu} \right) \]  \hspace{1cm} (5.2.23)

\[ = \frac{\mu^*}{N\mu_2^*} \]  \hspace{1cm} (5.2.24)

where \( \mu_2^* \) is the variance of (5.2.18).

It is suggested by (5.2.22) that if tables be made available for means \( \mu^* \)'s for sufficiently close values of \( \mu \), we can have a ready solution of (5.2.22). For a case of special significance when \( c = 1 \), i.e., when only zero observations are truncated, (5.2.22) becomes

\[ \bar{x} = \mu/(1-e^{-\mu}) \]  \hspace{1cm} (5.2.25)
and the asymptotic variance given by (5.2.23) reduces to

\[ \text{Var} (\hat{\mu}) = \frac{\mu}{N} \left( 1 - e^{-\mu} \right)/(\mu + 1 - \frac{\mu}{1 - e^{-\mu}}). \]  

(5.2.26)

Here, we present in a somewhat extensive table, Table V, values of \( \mu^*(1,\mu) \) for the Poisson distribution truncated on the left at \( c = 1 \) for values of \( \mu \) spaced at suitable intervals. A chart based on this table is also given to facilitate the procedure of estimation.

This table can be used to compute \( \text{Var} (\hat{\mu}) \) by using Formula (5.2.23) or Formula (5.2.24). In case Formula (5.2.23) is used, \( \frac{d\mu^*}{d\mu} \) can be approximated by the finite difference ratio \( \frac{\Delta \mu^*}{\Delta \mu} \). In case Formula (5.2.24) is used, the relationship for use is

\[ \mu^*_2(1,\mu) = \mu^*(1,\mu)[1 + \mu - \mu^*(1,\mu)]. \]  

(5.2.27)

Tables for \( \mu^*(c,\mu) \) of the Poisson distribution truncated on the left have been also given for various values of \( c \) and \( \mu \).

5.2.7 For a slightly different problem, where in a sample from a complete Poisson distribution, the frequencies for some lowest "counts" are missing, Rider (1953) suggested a method of estimation which uses first two moments of the complete Poisson and leads to a linear equation.

The method of two-moments is also applicable in the usual problem of estimation from a sample from singly truncated Poisson and forms a particular case of the general method discussed in Section 3.4 of Chapter III. Proceeding on these lines, one gets the estimate for \( \mu \) in this case as

\[ t = \frac{S_2 - cS_1}{S_1 - (c-1)N} \]  

(5.2.28)
where

\[ S_1 = \sum x n_x \quad \text{and} \quad S_2 = \sum x^2 n_x. \]

To find the asymptotic variance of the two-moments estimate \( t \) of \( \mu \), one gets on simplification,

\[
\text{Var} (t) = \frac{1}{N[\mu^* - (c-1)]^2} \left[ (m^*_4 - m^*_2)^2 + (\mu+c)^2 (m^*_2 - \mu^*_2) \right. \\
\left. - 2(\mu+c)(m^*_2 - \mu^*_m^*) \right].
\]

The asymptotic efficiency of \( t \) is then given by

\[
\text{Eff} (t) = \frac{\text{Var} (\bar{\mu})}{\text{Var} (t)}.
\]

The case of single truncation on the left at \( c = 1 \) is of practical importance. David and Johnson (1952) studied the efficiency for this particular case. The following is the table of \( \text{Eff} (t) \) computed by them.

**TABLE 5.2.5**

**EFFICIENCY OF TM FOR \( c = 1 \)**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>.5</th>
<th>1</th>
<th>1.5</th>
<th>2.0*</th>
<th>2.5</th>
<th>3.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Eff.</strong></td>
<td>.87</td>
<td>.80</td>
<td>.75</td>
<td>.73</td>
<td>.71</td>
<td>.71</td>
<td>.72</td>
</tr>
</tbody>
</table>

Source: David and Johnson (1952)

Thus, the efficiency of TM is not less than 70 percent for \( c = 1 \) with \( \mu = .5(.5)4.0. \)

5.2.8 Following the general approach discussed in Section 3.1 of Chapter III, a simple ratio-estimate for \( \mu \) can be obtained in the case of singly truncated Poisson distribution (5.2.1). In this case,
\[ a_{x-1}/a_x = x \text{ and since } \Phi = \mu, \text{ we have the following "ratio-estimate" for } \mu: \]
\[ \mu' = \sum_{x=c+1}^{\infty} \frac{x n_x}{N} \quad (5.2.30) \]

when \( c = 1 \), i.e., when only "zero" counts are truncated, the estimate takes the form suggested by Plackett (1953):
\[ \mu' = \sum_{x=2}^{\infty} \frac{x n_x}{N}. \quad (5.2.31) \]

The unique unbiased estimate of \( \mu \) linear in the frequencies (ibid., Section 3.2 of Chapter III), is provided in (5.2.30). The exact variance of this estimate is
\[ \sigma^2(\mu') = \frac{1}{N} \left[ \sum_{x=c+1}^{\infty} x^2 F_x - \mu^2 \right] \quad (5.2.32) \]

and an unbiased estimate of \( \sigma^2(t) \) is
\[ \hat{\sigma}^2 = \left( \sum_{x=c+1}^{\infty} x^2 n_x - N \mu'^2 \right) / N(N-1) \quad (5.2.33) \]

when \( c = 1 \), (5.2.32) reduces to
\[ \sigma^2(\mu') = \frac{1}{N} [\mu + \mu^2/(e^\mu - 1)] \quad (5.2.34) \]

first derived by Plackett (1953). Plackett computed also the efficiency of \( \mu' \) in this special case. The following table gives the efficiencies of \( \mu' \) relative to \( \hat{\mu} \).

It can be shown that the efficiency of \( \mu' \) never falls below 0.9536, the minimum value being attained when \( \mu = 1.355 \) (Plackett).
TABLE 5.2.6

EFFICIENCY OF R FOR c = 1

<table>
<thead>
<tr>
<th>μ</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eff.</td>
<td>.9693</td>
<td>9559</td>
<td>9539</td>
<td>9586</td>
<td>9662</td>
<td>9743</td>
<td>9815</td>
<td>9872</td>
</tr>
</tbody>
</table>

Source: Plackett (1953)

5.2.9 So far, we have separately discussed the Two Moments Estimate and the Ratio Estimate for μ. To make a comparative study of these two simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.1 and 3.3 one gets to order 1/N the amount of bias of t (TM) as follows:

\[ b(t) = (\mu \sigma_{22}^* - \sigma_{12}^*)/NH^2 = \frac{B(t)}{N} \]  \hspace{1cm} (5.2.35)

where \( H = \mu^* - (c-1) \)

\[ \sigma_{22}^* = m_2^* - \mu^* \]

and \( \sigma_{12}^* = (m_3^* - \mu^* m_2^*) - c (m_2^* - \mu^*) \).

For \( \mu' \), however, one has

\[ b(\mu') = 0. \]  \hspace{1cm} (5.2.36)

The relative efficiency of R over TM is given by

\[ \text{Rel. Eff} = \frac{\text{Var}(t)}{\text{Var}(\mu')}. \]

The following table gives bias and relative efficiency of TM and R for \( \mu = .5(.5)^4 \). Thus, we conclude that one may prefer Ratio Estimate to estimate \( \mu \) of the Poisson distribution singly truncated on the left at \( c = 1 \) because of its simplicity, unbiasedness and high efficiency.
### TABLE 5.2.7

**BIAS AND RELATIVE EFFICIENCY OF TM AND R**

for $c = 1$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$N (\text{Amount of Bias to order } l/N)$</th>
<th>$\text{Var(TM)}$</th>
<th>$\text{Var(R)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>-.3935</td>
<td>.0000</td>
<td>1.11</td>
</tr>
<tr>
<td>1.0</td>
<td>-.6321</td>
<td>.0000</td>
<td>1.19</td>
</tr>
<tr>
<td>1.5</td>
<td>-.6373</td>
<td>.0000</td>
<td>1.27</td>
</tr>
<tr>
<td>2.0</td>
<td>-.8647</td>
<td>.0000</td>
<td>1.31</td>
</tr>
<tr>
<td>2.5</td>
<td>-.9179</td>
<td>.0000</td>
<td>1.36</td>
</tr>
<tr>
<td>3.0</td>
<td>-.9502</td>
<td>.0000</td>
<td>1.37</td>
</tr>
<tr>
<td>3.5</td>
<td>-.9698</td>
<td>.0000</td>
<td>1.37</td>
</tr>
<tr>
<td>4.0</td>
<td>-.9817</td>
<td>.0000</td>
<td>1.37</td>
</tr>
</tbody>
</table>

5.2.10 Some cases are likely to arise in which one is aware of the type of truncation, but does not know the point at which truncation occurs. For instance, when lots of manufactured items come for acceptance to a consumer from the producer who has earlier censored items having more than, say, $d$ defects, consumer has to draw a sample or samples from a singly truncated Poisson population with unknown truncation point on the right. Estimation of $\mu$ and $d$ thus becomes essential before setting up any acceptance sampling plan.

On the basis of a sample $x_i$ ($i = 1, 2, \ldots, N$) of size $N$ with observed frequency $n_x$ for $x$ ($0 < x < R$, $\sum n_x = N$) drawn from

$$
p^*(x, \mu, d) = (P(d, \mu))^{-1} e^{-\mu} \cdot \frac{\mu^x}{x!}
$$

(5.2.37)

$x = 0, 1, 2, \ldots, d$
to estimate $\mu$, we choose the Ratio-estimate

$$\mu' = \sum_{x=0}^{R} \frac{x n_x}{\sum_{x=0}^{R-1} n_x}. \quad (5.2.38)$$

The advantage with $\mu'$ is that besides its simplicity, it does not need the knowledge of the truncation point.

To estimate $d$, the identity

$$\mu = \frac{m_2^* - (d+1) \mu^*}{\mu^* - d}$$

gives

$$d = \frac{m_2^* - (\mu+1) \mu^*}{\mu^* - \mu}. \quad \quad \quad \quad \quad (5.2.39)$$

Therefore the estimate of $d$ can be obtained as:

$$d' = \frac{S_2 - (\mu'+1) S_1}{S_1 - \mu'N}$$

where $\mu'$ is given by (5.2.38), $S_1 = \Sigma x n_x$ and $S_2 = \Sigma x^2 n_x$.

5.2.11 The detailed computation procedure of evaluating the three types of estimates discussed above is illustrated with reference to data collected by Varley (1949) to study population balance in the Knapweed Gall-fly. The table below gives the number of flower-heads ($n_x$) each having exactly $x$ gall-cells ($x = 1, 2, \ldots$)

<table>
<thead>
<tr>
<th>Number of gall-cells in a flower-head ($x$)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of flower-heads ($n_x$)</td>
<td>287</td>
<td>272</td>
<td>196</td>
<td>79</td>
<td>29</td>
<td>20</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Assuming the truncated Poisson model:

\[
\frac{\mu^x}{x! (e^{\mu-1})} \quad x = 1, 2, \ldots
\]

for the probability of \( x \) gall-cells in a flower-head, the problem is to estimate \( \mu \) on the basis of the given data.

Maximum Likelihood Estimate: For the data, we get

\[
N = 886 \\
S_1 = 2023
\]

so that \( \bar{x} = 2.2833 \). Referring to Table V for \( c = 1 \), we find the following:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \mu^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>2.2342</td>
</tr>
<tr>
<td>2.0</td>
<td>2.3130</td>
</tr>
</tbody>
</table>

The maximum likelihood estimate is given by that value of \( \mu \) for which \( \mu^* = 2.2833 \). By linear interpolation we thus get

\[ \hat{\mu} = 1.9623. \]

The variance of this estimate is estimated from the formula:

\[ \text{Var}(\hat{\mu}) = \frac{\mu^2}{N \mu_2^*} \]

where \( \mu_2^* = \mu^*(1 + \mu - \mu^*) \).

On computation, \( \mu_2^* = 2.2833 \times (1 + 1.9623 - 2.2833) = 1.5504 \)

and so,

\[ \text{Var}(\hat{\mu}) = \frac{(1.9623)^2}{886 \times 1.5504} = 0.002803. \]
Thus, the standard error of $\mu$ is given by

$$ \text{S.E.} (\hat{\mu}) = \sqrt{0.002803} = 0.0529.$$

Two Moments Estimate: To compute this estimate of $\mu$, we require in addition the value of

$$ S_2 = \sum x^2 n_x = 6027 $$

Then the estimate is

$$ t = \frac{S_2}{S_1} - 1 = \frac{6027}{2023} - 1 = 1.9792. $$

To compute the variance of $t$, taking $1.9792$ as the estimate for $\mu$, we have

$$ \mu^* = \frac{\mu - \epsilon}{e^{\mu} - 1} = 2.2965 $$

$$ m_2^* = \mu^*(1+\mu) = 6.8417 $$

$$ m_2 = m_2^* - \mu^2 = 1.5678 $$

$$ m_3^* = \mu(\mu^*+m_2^*) + (1+\mu) \mu^* $$

$$ = 22.7571 $$

and $m_4^* = \mu[m^* + m_2^* + 2\mu^* + m_3^* - 2\mu(1 + \mu)] + (m_3^* - \mu m_2^*)(1 + 2\mu)$$

$$ = 91.6896. $$

The variance of $t$ is then estimated from the formula:

$$ \text{Var}(t) = \frac{1}{N \mu^*} \left[ (m_4^* - m_2^2) + (1 + \mu)^2 \cdot \mu^* - 2(1 + \mu)(m_3^* - \mu m_2^*) \right] $$

$$ = 0.003600, $$

so that the standard error is

$$ \text{S.E.}(t) = 0.0546. $$

The following table summarizes the results obtained:
<table>
<thead>
<tr>
<th>Estimate</th>
<th>Value</th>
<th>Variance</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>ML</td>
<td>1.9623</td>
<td>0.002803</td>
<td>0.0529</td>
</tr>
<tr>
<td>TM</td>
<td>1.9792</td>
<td>0.003600</td>
<td>0.0600</td>
</tr>
<tr>
<td>R</td>
<td>1.9594</td>
<td>0.002982</td>
<td>0.0546</td>
</tr>
</tbody>
</table>

5.3 Estimation from a Sample for a Censored Poisson Distribution

Moore (1952) and Cohen (1954) discussed the problem of estimation of \( \mu \) from a censored sample of the Poisson distribution. Moore gave a simple ratio-estimate and Cohen derived likelihood equations for both singly and doubly censored samples. In this section, we provide a neat and compact likelihood equation for estimation. The amount of bias involved in estimating \( \mu \) by the sample mean after pooling observations of higher counts has been investigated. A suitable chart is provided to suggest as to when one should resort to a finer method of estimation.

5.3.1. Suppose that in a random sample of size \( N \) from (5.1.2), we have a record of the number \( n_1 \) of observations in the right tail defined by \( (\geq c) \) and of the \( n^* \) observations \( x_i (i = 1, 2, \ldots, n^*) \), \( x_i < c \) so that \( N = n^* + n_1 \). Results for estimation derived by the general approach in Section 2.3 can be written down in this case as follows:

The efficient score for \( \mu \) is

\[
\psi(\mu) = \frac{1}{\mu} \left\{ n^* \bar{x}^* - (N\mu - n_1v_1) \right\}
\]  

(5.3.1)

where \( \bar{x} = \frac{n^*}{n} \sum_{i=1}^{n^*} x_i/n^* \) and \( v_1 \) is the mean of the Poisson distribution truncated on the left at \( c \).
Thus, the likelihood equation for estimating \( \mu \) is

\[
n^* \bar{x}^* = N\mu - n_1 v_1
\]

(5.3.2)

The asymptotic variance of the estimate \( \mu \) derived from (5.3.2) is \( 1/I(\mu) \) where

\[
I(\mu) = \frac{N}{\mu} (1 - P \frac{d v_1}{d \mu})
\]

(5.3.3)

where \( P \) is the probability of the right tail.

Equation (5.3.2) does not readily give an algebraic solution and the iterative process of solution has to be resorted to. To facilitate the process of estimation we use Table V. to obtain values of means for values of \( \mu \) and \( c \) spaced at suitable intervals.

5.3.2 A simple estimate of \( \mu \) from a censored sample under consideration can be obtained as

\[
m = \frac{\sum x_i + cn^*}{N}
\]

(5.3.4)

which, though always an underestimate, may at times, when the magnitude of the bias is small may be useful.

The relative bias is

\[
b = \frac{E(m) - \mu}{\mu} = \frac{P(\mu^* - c)}{\mu}
\]

(5.3.5)

where \( \mu^* \) is the mean of the left tail.

Here we present in Table 5.3.1 the values of the relative bias expressed as a percentage for various values of \( \mu \) and \( c \) spaced at suitable intervals. Chart 4 based on this table may be also of help to suggest as to when one should resort to a finer method of estimation. Also, before
TABLE 5.3.1

\[ -b = \frac{P(\mu^* - c)}{\mu} \]

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>.000372</td>
<td>.000326</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
</tr>
<tr>
<td>1.0</td>
<td>.004348</td>
<td>.00688</td>
<td>.000096</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
</tr>
<tr>
<td>1.5</td>
<td>.016609</td>
<td>.003723</td>
<td>.000752</td>
<td>.000013</td>
<td>.000000</td>
<td>.000000</td>
<td>.000000</td>
</tr>
<tr>
<td>2.0</td>
<td>.037570</td>
<td>.011444</td>
<td>.002962</td>
<td>.000695</td>
<td>.000147</td>
<td>.000028</td>
<td>.000005</td>
</tr>
<tr>
<td>2.5</td>
<td>.068365</td>
<td>.024781</td>
<td>.007907</td>
<td>.002296</td>
<td>.000597</td>
<td>.000141</td>
<td>.000030</td>
</tr>
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<td>.002270</td>
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<td>.052955</td>
<td>.028736</td>
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<td>.261078</td>
<td>.178163</td>
<td>.114771</td>
<td>.069693</td>
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<tr>
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<td>.480000</td>
<td>.384046</td>
<td>.296118</td>
<td>.219109</td>
<td>.155060</td>
<td>.104685</td>
</tr>
</tbody>
</table>
Chart 5.3.1 Chart for the Relative Bias of the "Pooled Estimate" for $c = 4(1)10$
starting an experiment on "Poisson-counts", Table 5.3.1 and Chart 5.3.1 may be of great help to an experimenter in deciding the value of the count beyond which he need only record not the individual counts, but only the total frequency.

Equation of estimation in other types of censored samples may be written down on the same lines as above.

5.4 Estimation With Doubtful Observations

While sampling from a Poisson distribution (5.1.2), cases arise in which one is doubtful if the observations in the "zero-class" really come from the Poisson population. For instance, while recording data on counts of minute particles in an experiment, one is doubtful if the "zero-counts" sometimes occur because of failure in the working of the counter. Before actually estimating the parameter, therefore, one has to test if the zero-counts conform to the Poisson distribution under consideration, which, incidentally also tests if the counter works throughout or not.

5.4.1 We take the model for the distribution as

$$\text{Prob}\{X = x\} = \begin{cases} \beta e^{-\beta} & \text{for } x = 0 \\ \frac{(1-\beta)^x e^{\mu-1}}{x!} & \text{for } x = 1,2,\ldots,\infty \end{cases}$$ \hspace{1cm} (5.4.1)

On the basis of a sample of size $N$ with $n_1$ zero-counts and $n^*$ non-zero counts $x_i$ ($i = 1, 2, \ldots n^*$, $n^* = N-n_1$) one gets the efficient scores for $\beta$ and $\mu$ as:

$$\psi_1 = \frac{n_1}{\beta} - \frac{n^*}{1 - \beta}$$ \hspace{1cm} (5.4.2)
and

$$\psi_2 = \frac{n^*}{\mu} (\bar{x}^* - \mu^*)$$  \hspace{1cm} (5.4.3)$$

where $$\bar{x}^* = \frac{\sum_{i=1}^{n^*} x_i}{n^*}$$ and $$\mu^* = \mu/(1-e^{-\mu})$$

The estimates therefore are given by

$$\hat{\beta} = \frac{n_1}{N}$$  \hspace{1cm} (5.4.4)$$

and

$$\hat{\mu}^* = \bar{x}^*$$  \hspace{1cm} (5.4.5)$$

Now, the hypothesis $$H_o$$ of interest is

$$\beta = e^{-\mu}$$  \hspace{1cm} (5.4.6)$$

obviously the estimate of $$\mu$$ under $$H_o$$ is given by

$$\hat{\mu}_o = \bar{x}$$

where

$$\bar{x} = \frac{\sum_{i=1}^{n^*} x_i}{N}$$  \hspace{1cm} (5.4.7)$$

Therefore, the estimate of $$\beta$$ under $$H_o$$ becomes

$$\hat{\beta}_o = e^{-\hat{\mu}_o}$$  \hspace{1cm} (5.4.8)$$

and the Chi-square criterion with 1 df given by (2.4.16) reduces to

$$\chi^2 = \chi_0^2 (1 + \frac{1}{V_o})$$  \hspace{1cm} (5.4.9)$$

where

$$\chi_0^2 = \frac{N \left( \frac{n_1}{N} - \hat{\beta}_o \right)^2}{\hat{\beta}_o (1 - \hat{\beta}_o)}$$

and

$$V_o = \frac{1 - \hat{\beta}_o}{\hat{\mu}_o} - 1$$
Illustrative Example: Let us illustrate the method of estimation with doubtful observations with reference to data given by Scrace. The problem is connected with the number of dust nuclei in the air and the data give the frequency distribution of the number of drops in a small volume of air that fall on to a stage in a chamber containing moisture and filter. The data are as follows:

<table>
<thead>
<tr>
<th>Number of dust nuclei ( (x) )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency ( (n_x) )</td>
<td>23</td>
<td>56</td>
<td>88</td>
<td>95</td>
<td>73</td>
<td>40</td>
<td>17</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

For the full distribution, we have

\[
N = 400
\]

\[
S_1 = 1170
\]

and

\[
\bar{x} = 2.9250
\]

Scrace is of the opinion that this mean (and hence the estimate of the Poisson parameter given by \( \mu = \bar{x} \)) is slightly high in that a number of zero counts were wrongly rejected as being due to the apparatus not working. The zero-counts are doubtful and so before writing down the estimate, we shall have to test if these zero-counts conform to the full Poisson-data.

Under the null hypothesis \( H_0 \) that zero-counts conform to the full distribution, we have, following the results derived in Section 5.4,

\[
\hat{\mu}_0 = \bar{x} = 2.9250.
\]

Therefore under \( H_0 \) the estimate of \( \beta \), the proportion of zero-counts to the total number of counts, is given by

\[
\hat{\rho}_0 = e^{-\hat{\mu}_0} = e^{-2.9250}
\]

\[
= 0.53665
\]
To compute the Chi-square criterion given by (5.4.9) we have

\[ x_0^2 = \frac{N (\frac{n}{N} - \hat{\beta}_o)^2}{\hat{\beta}_o (1 - \hat{\beta}_o)} \]

\[ = \frac{400(\frac{23}{400} - 0.053665)^2}{0.053665 \times 0.946335} \]

\[ = 0.115839 \]

and

\[ V_0 = \frac{1 - \hat{\beta}_o}{\hat{\beta}_o \hat{\beta}_o} - 1 = 5.028763 \]

so that, the Chi-square criterion with 1 degree of freedom to test \( H_0 \) comes out to be

\[ x^2 = x_0^2 (1 + \frac{1}{V_0}) \]

\[ = 0.138874 \]

which is not significant, showing thereby that the zero-counts conform to the full Poisson distribution and the estimate of \( \mu \) (obtained to be \( \hat{\mu}_o = 2.925 \)) computed from the full data is statistically quite legitimate. This conclusion brings out that the rejections of zero-counts were rightly judged by the experimenter, rightly because the counter had really failed to work then.
CHAPTER VI

6.0 ESTIMATION PROBLEMS FOR THE NEGATIVE BINOMIAL DISTRIBUTION

6.1 Introduction

The gpdf with two parameters defined by (2.6.1) becomes

\[ \text{Prob} \{ X = x \} = (\lambda + \frac{x - 1}{x}) \theta^x / (1 - \theta)^{-\lambda} \]

\[ x = 0, 1, 2, \ldots \infty \]  \hspace{1cm} (6.1.1)

when

\[ f(\lambda, \theta) = (1 - \theta)^{-\lambda}, \hspace{0.5cm} \lambda > 0. \]

Writing \( k = \lambda \) and \( \frac{\mu}{\mu + k} = \theta \), (6.1.1) gives the probability for \( X \) as:

\[ \text{Prob} \{ X = x \} = y(x, \mu, k) = \left( \frac{k+x-1}{x} \right) \left( \frac{k}{\mu+k} \right) x \]

\[ x = 0, 1, 2, \ldots \infty \]  \hspace{1cm} (6.1.2)

a well-known form of the negative binomial distribution.

The important properties of (6.1.2) can be summarily stated as follows:

\[ M(t) = [1 - \frac{\mu}{K} (e^t - 1)]^{-k}. \]  \hspace{1cm} (6.1.3)

The first two central moments are given by

\[ \mu = \mu \]

\[ \mu_2 = \frac{\mu}{k} (\mu + k) \]  \hspace{1cm} (6.1.4)

and the coefficients \( \beta_1, \beta_2 \) take the form:

\[ \beta_1 = \frac{1}{\mu} + \frac{1}{k} \left( 3 + \frac{\mu}{\mu+k} \right) \]

\[ \beta_2 = 3 + \frac{1}{\mu_2} + \frac{6}{k} \]  \hspace{1cm} (6.1.5)
The recurrence relation connecting central moments

\[ \mu_{r+1} = \mu_2 \left( \frac{d \mu_r}{d \mu} + r \mu_{r-1} \right) \tag{6.1.6} \]

does not seem to have been noticed before, this follows immediately
from (1.1.7).

The recurrence relation connecting cumulants is

\[ K_{r+1} = \mu_2 \frac{d K_r}{d \mu} \tag{6.1.7} \]

which is derived in a slightly different form by Guldberg (1935)
and Wishart (1949).

The distribution function \( Y(r, \mu, k) \) defined by

\[ Y(r, \mu, k) = \sum_{x=0}^{r} y(x, \mu, k) \tag{6.1.8} \]

can be reduced to

\[ Y(r, \mu, k) = I_x(k, r+1) \]

where

\[ \phi = \frac{k}{\mu + k} \tag{6.1.9} \]

and

\[ I_x(m,n) = \frac{1}{B(m,n)} \int_0^x u^{m-1} (1-u)^{n-1} \, du \]

for which extensive tables have been edited by K. Pearson. When
\( k \) is positive integer, one can use

\[ Y(r, \mu, k) = \sum_{x=k}^{r+k} \binom{r+k}{x} \left( \frac{k}{\mu+k} \right)^x \left( \frac{\mu}{\mu+k} \right)^{r+k-x} \]

\[ = 1 - B(k - 1, \phi, r+k) \tag{6.1.10} \]
where \( \phi \) is given by (6.1.9) and \( B(r, \pi, n) \) is the cumulative binomial probability defined in (4.1.7). For large \( k \), one can use tables of Poisson probabilities, because

\[
\lim_{k \to \infty} y(x, \mu, k) = p(x, \mu) \tag{6.1.11}
\]

where

\[
p(x, \mu) = e^{-\mu} \cdot \frac{\mu^x}{x!}.
\]

For large \( \mu \) and \( k \), however, we have the normal approximation given by

\[
\lim_{\mu \to \infty, k \to \infty} Y(r, \mu, k) = \Phi(z) \tag{6.1.12}
\]

where \( z = (r + 1/2 - \mu)/\sqrt{\mu} \)

and

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du.
\]

6.2 Estimation of Parameters of Complete Negative Binomial Distribution

The negative binomial distribution can be viewed as a compound Poisson distribution, Greenwood and Yule (1920), when the mean of the Poisson distribution follows the gamma distribution.

One finds, therefore, its applications in all fields wherever the data under consideration are too heterogeneous to be fitted by a Poisson distribution. For instance, Greenwood and Yule (1920) applied it to accident-data, Fisher (1941) and Bliss (1953) to biological data, Sichel (1951) to psychological data, whereas Wise (1946) found its use in an industrial sampling problem.
6.2.1 To estimate μ and k on the basis of a random sample of size N with observed frequency \( n_x \) for \( x (\sum n_x = N) \) drawn from \( (6.1.2) \), following Section 2.6, the moment-estimates are given by

\[
\bar{x} = \mu \quad (6.2.1)
\]

and

\[
S^2 = \frac{\mu}{k} (\mu + k) \quad (6.2.2)
\]

where

\[
\bar{x} = \frac{\sum x n_x}{N} \quad \text{and} \quad S^2 = \frac{\sum (x-\bar{x})^2}{N}.
\]

The likelihood equations reduce to

\[
\bar{x} = \hat{\mu} \quad (6.2.3)
\]

and

\[
\sum_{r \geq 0} \frac{T_{r+1}}{r+k} = N \log \left( \frac{\hat{\mu} + \hat{k}}{\hat{k}} \right) \quad (6.2.4)
\]

where \( T_{r+1} = \sum_{x > r} n_x \) which was first derived by Haldane (1941).

Eliminating \( \hat{\mu} \) from (6.2.3) and (6.2.4), we have to solve for \( \hat{k} \) the equation

\[
\sum_{r \geq 0} \frac{T_{r+1}}{r+k} = N \log \left( \frac{\bar{x} + \hat{k}}{\hat{k}} \right) \quad (6.2.5)
\]

The elements of the information matrix are:

\[
I_{11} = \frac{N}{\mu^2}
\]

\[
I_{12} = 0 \quad (6.2.6)
\]

and

\[
I_{22} = \sum_{r \geq 0} \left[ \frac{1 - Y(r, \mu, k)}{(k + r)^2} \right] + \frac{Nu}{k(\mu + k)}
\]
6.2.2 It is easy to see that the moment-estimate is identical with the likelihood-estimate for the mean \( \mu \) and is given by the sample mean \( \bar{x} \). The estimates for \( k \) are however different. The moment estimate for \( k \), though inefficient, is easy to compute, whereas the likelihood estimate, though efficient, is rather difficult to obtain. It is important to know, therefore, as to when one should finally proceed to obtain likelihood-estimate for \( k \). For large \( \mu \), Sichel (1951) showed the efficiency of the moment-estimate for \( k \) to be never less than 0.80. He also showed that the efficiency is minimum for large \( \mu \) when \( k = 5.5 \) and recommended that for values of \( k > 5.5 \), it is not necessary to estimate \( k \) by the arduous likelihood-estimate. Fisher (1941) examined the efficiency of the moment-estimates and concluded that if \( \mu \) is less than \( k/9 \) for any value of \( k \) or if \( k \) exceeds 18 for any value of \( \mu \), high efficiency is assured, for intermediate values, however, if the product \((1 + \frac{k}{\mu})(k + 2)\) exceeds 20, the efficiency is satisfactorily high.

6.3 Estimation of Parameters of a Truncated Negative Binomial Distribution

Sampford (1955) discussed an application of truncated negative binomial distribution. Taking \( k \) and \( \omega = \frac{k}{\mu+k} \) to be parameters, he gave methods to obtain "moment-estimates" and "likelihood-estimates" for \( \omega \) and \( k \) when zero-observations are truncated and also investigated the efficiency of the moment estimates so obtained. Rider (1955) used three moments and provided simple estimates for \( k \) and \( p = \frac{\mu}{k} \). Following the general approach discussed in Section 2.6, neat and compact likelihood equations can be made available.
The probability law of the negative binomial distribution of which zero-observations are truncated can be written as
\[ y^*(x, \mu, k) = \left[ 1 - \left( \frac{k}{\mu + k} \right)^k \right]^{x-1} \cdot \left( \frac{\mu}{\mu + k} \right)^x \left( \frac{k}{\mu + k} \right)^k \]
for \( x = 1, 2, 3, \ldots \infty \) (6.3.1)

The first two moments about the origin of (6.3.1) then, are
\[ \mu^* = \frac{\mu}{1 - \left( \frac{k}{\mu + k} \right)^k} \quad (6.3.2) \]
and
\[ m_2^* = \mu^* (1 + \mu + \frac{\mu}{k}) \quad (6.3.3) \]

6.3.1 To estimate \( \mu \) and \( k \) on the basis of a random sample of size \( N \) with frequency \( n_x \) for \( x(\sum n_x = N) \) drawn from (6.3.1) the moment equations can be written down as
\[ \bar{x} = \mu^* = \frac{\mu}{1 - \left( \frac{k}{\mu + k} \right)^k} \quad (6.3.4) \]
and
\[ S^2 = m_2^* = \mu^* (1 + \mu + \frac{\mu}{k}) \quad (6.3.5) \]
where
\[ \bar{x} = \frac{\sum n_x x}{N} \quad \text{and} \quad S^2 = \frac{\sum (x - \bar{x})^2}{N}. \]

Eliminating \( k \) from (6.3.4) and (6.3.5), one gets
\[ \mu = \bar{x} \left[ 1 - \left( \frac{\bar{x}}{S^2 - \bar{x} \mu} \right) \frac{\bar{x} \mu}{S^2 - \bar{x} \mu} \right]. \quad (6.3.6) \]

An estimate of \( \mu \) can be obtained by solving this single equation (6.3.6) by using an iterative method. It is easy to see that the
estimate of $k$ is also available during the process of iteration, because one has

$$k = \frac{\bar{x}_\mu}{s^2 - \bar{x}(\mu + 1)}$$  \hspace{1cm} (6.3.7)

6.3.2 Following the general approach discussed in Section 2.6, the likelihood equations for $\mu$ and $k$ in this case can be written down as:

$$\bar{x} = \hat{\mu}^*$$  \hspace{1cm} (6.3.8)

and

$$\sum_{r \geq 0} \frac{T_{r+1}}{k+r} = \frac{N\hat{\mu}^*}{\mu} \log \left( \frac{\hat{\mu} + k}{k} \right)$$  \hspace{1cm} (6.3.9)

where $\mu^*$ is defined by (6.3.2) and $T_{r+1} = \sum_{x > r} n_x$. The elements of the information matrix are:

$$I_{11} = N \mu_2 \frac{\partial \mu^*}{\partial \mu} = \frac{N\mu^*_2}{\mu^2}$$  \hspace{1cm} (6.3.10)

$$I_{12} = N \mu_2 \frac{\partial \mu^*}{\partial k} = -\frac{N\mu^*_2}{\mu^2} \cdot \frac{1}{\mu + k}$$  \hspace{1cm} (6.3.11)

and

$$I_{22} = \sum_{r \geq 0} \left[ \frac{1 - Y(r, \mu, k)}{(k+r)^2} + \frac{N}{\mu(\mu + k)} \frac{\mu^*}{2} \log k \frac{k}{\mu + k} \right]$$  \hspace{1cm} (6.3.12)

where $\mu_2$ and $\mu_2^*$ are the variances of (6.1.2) and (6.3.1) respectively.

6.3.3 Equating the first three sample-moments to the corresponding theoretical moments about origin of (6.3.1) Rider (1955) gets simple estimates for $\mu$ and $k$ as follows:

$$\mu' = \frac{2s_2^2 - s_1(s_1 + s_3)}{s_1(s_2 - s_1)}$$  \hspace{1cm} (6.3.13)
and

\[ k' = \frac{2s^2 - s(s + s_3)}{s_1(s_1 + s_3) + s_2(s_1 + s_2)} \]  \hspace{1cm} (6.3.14)

where

\[ s_i = \sum x_i n_x / N \quad i = 1, 2, 3. \]

### 6.4 Estimation when k is Known

The negative binomial distribution is essentially a distribution with two parameters. Cases, however, arise in which the parameter k is known and only the other parameter has to be estimated. This is taken up in this section.

When k is a positive integer, writing \( \phi = \frac{k}{u+k} \), the negative binomial law given by (6.1.2) becomes

\[ y(x, \phi, k) = \binom{k + x - 1}{x} \phi^k (1-\phi)^x \] \hspace{1cm} (6.4.1)

\[ x = 0, 1, 2, \ldots \infty \]

(6.4.1) can be looked upon as the probability law for the number X = k + x of independent successive trials required to get k successes when \( \phi \) is the probability of success at each trial and can be used in sampling problems in which sampling is done until a certain number k of "character-bearers" is available in the sample. It has been used by Haldane (1945) in biology and by Craig (1953) among others in industrial problems.

#### 6.4.1 On the basis of single observation on X, X = k+x, the likelihood estimate for \( \phi \) can be obtained as

\[ \hat{\phi} = \frac{k}{k+x} = \frac{k}{X} \] \hspace{1cm} (6.4.2)
It can be seen that $\hat{\phi}$ is biased for $\phi$. Following Section 3.2, however, a unique unbiased ratio-estimate for $\phi$ can be obtained. In this case, $a_{x-1}/a_x = x/(k+x-1)$ and since $\Theta = 1 - \phi$, we have the ratio-estimate for $\phi$ as

$$\phi' = \frac{k - 1}{k+x-1} = \frac{k - 1}{x - 1} \quad k > 1$$  \hspace{1cm} (6.4.3)

provided first by Haldane (1945) and shown to be the only unbiased estimate for $\phi$ by Girshick, Mosteller and Savage (1946).

To estimate $\phi$ on the basis of a random sample $X = k + x$ of size $N$ with frequency $n_x$ for $x$ drawn from (6.4.1), the following estimates are available:

**Maximum Likelihood Estimate:** Following Section 2.1, the likelihood equation (which is the same as moment equation) can be written down as $\bar{x} = \hat{\mu}$ where $\bar{x} = \sum x_n / N$ and $\mu$ is the mean of (6.4.1) given by

$$\mu = k(1-\phi)/\phi$$  \hspace{1cm} (6.4.4)

one gets the estimate for $\phi$, therefore, as

$$\hat{\phi} = \frac{k}{k+x}$$  \hspace{1cm} (6.4.5)

The asymptotic variance and the amount of bias to order $1/N$ of $\hat{\phi}$ can be easily obtained as

$$\text{Var} (\hat{\phi}) = \frac{\phi^2 (1 - \phi)}{N k}$$  \hspace{1cm} (6.4.6)

and

$$b(\hat{\phi}) = \frac{\phi(1 - \phi)}{N k}$$  \hspace{1cm} (6.4.7)
The following table gives the amount of bias and standard error of \( \hat{\phi} \) (ML) for \( k = 1, 2, 3 \) with \( \phi = .01, .05, .10, .25, .50, .75 \).

**TABLE 6.4.1**

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \phi )</th>
<th>Amount of Bias to order ( \frac{1}{N} )</th>
<th>Standard Error</th>
<th>( 100 \times \frac{\text{Bias}}{\text{S.E.}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.01</td>
<td>.000099</td>
<td>.000990</td>
<td>10.00</td>
</tr>
<tr>
<td></td>
<td>.05</td>
<td>.000475</td>
<td>.004873</td>
<td>9.75</td>
</tr>
<tr>
<td></td>
<td>.10</td>
<td>.000100</td>
<td>.009487</td>
<td>9.49</td>
</tr>
<tr>
<td></td>
<td>.25</td>
<td>.001875</td>
<td>.021660</td>
<td>8.66</td>
</tr>
<tr>
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<td>.50</td>
<td>.002500</td>
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<td>7.07</td>
</tr>
<tr>
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<td></td>
<td>.75</td>
<td>.000625</td>
<td>.021660</td>
<td>2.89</td>
</tr>
</tbody>
</table>

**Ratio Estimate:** Following Section 3.1, the unique unbiased linear estimate for \( \phi \) can be written down as

\[
\phi' = 1 - \frac{1}{N} \sum_{x=1}^{\infty} \frac{x}{k+x-1} n_x.
\]  

(6.4.8)

The exact variance of this ratio-estimate \( \phi' \) is

\[
\sigma^2(\phi') = \frac{1}{N} \left[ \sum_{x=1}^{\infty} \left( \frac{x}{k+x-1} \right)^2 y(x, \phi, k) - (1-\phi)^2 \right]
\]

(6.4.9)
and its unbiased estimate is given by

\[
\left[ \sum_{x=1}^{\infty} \left( \frac{x}{k+\alpha-1} \right)^2 n_x - N(1-\phi')^2 \right] / N(N-1).
\]

The efficiency of \( \phi' \) is then obtained as \( \text{Eff.}(\phi') = \text{Var}(\phi')/\sigma^2(\phi') \)

The following table gives the efficiency of \( \phi' \) for \( k = 1, 2 \) with \( \phi = .01, .05, .10, .25, .50, .75. \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>(.01)</th>
<th>(.05)</th>
<th>(.10)</th>
<th>(.25)</th>
<th>(.50)</th>
<th>(.75)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.010</td>
<td>.050</td>
<td>.100</td>
<td>.250</td>
<td>.500</td>
<td>.750</td>
</tr>
<tr>
<td>2</td>
<td>.136</td>
<td>.221</td>
<td>.289</td>
<td>.442</td>
<td>.647</td>
<td>.830</td>
</tr>
</tbody>
</table>

Also, if one notices that \( S_1 = \sum x n_x \) follows the negative binomial law with parameters \( \phi \) and \( Nk \), one more unbiased ratio-estimate can also be written down for \( \phi \) on the lines of the estimate given for single observation by (6.4.3) as,

\[
\phi'' = \frac{Nk - 1}{Nk + S_1 - 1}
\]

\[
= \frac{k - 1/N}{k + \bar{x} - 1/N}.
\]  \( \text{(6.4.10)} \)

The exact variance of \( \phi'' \) is then given by

\[
\sigma^2(\phi'') = \sum_{S_1=0}^{\infty} \left( \frac{Nk - 1}{Nk+S_1-1} \right)^2 y(S_1, \phi, Nk) - \phi^2. \]  \( \text{(6.4.11)} \)
Also the asymptotic variance of $\hat{\phi}''$ can be written down as

$$\text{Var}(\hat{\phi}'') = \frac{\phi^2(1-\phi)}{Nk} \cdot \frac{1}{(1 - \frac{1}{Nk})^2}$$

which indicates that $\hat{\phi}''$ is highly efficient. It is also unbiased and easy to compute. Hence it may be used with advantage to estimate $\phi$ because the maximum likelihood estimate $\hat{\phi}$ given by (6.4.5), though efficient, has some bias whereas the ratio-estimate $\hat{\phi}'$ given by (6.4.8), though unbiased, involves serious loss of efficiency.

6.4.2 Cases can arise in which one has to estimate $\phi$ on the basis of a sample drawn from truncated negative binomial distribution.

The negative binomial distribution (6.4.1) truncated say, on the left at $c$ can be written down as:

$$y^*(x, \phi, k) = \left[ Y^*(c, \phi, k) \right]^{-1} \left( \frac{k+x-1}{x} \right)^k \phi^k (1-\phi)^x$$

where $Y^*(r+1, \phi, k) = 1 - Y(r, \phi, k)$. The first two moments about origin of (6.4.13) are

$$\mu^* = \mu^*(c, \phi, k) = \frac{k(1-\phi)}{\phi} \cdot \frac{Y^*(c-1, \phi, k+1)}{Y^*(c, \phi, k)}$$

and

$$m_2^* = m_2^*(c, \phi, k) = \mu^*(c, \phi, k) \left[ 1 + \mu^*(c-1, \phi, k+1) \right]$$

To estimate $\phi$ on the basis of a random sample of size $N$ with frequency $n_x$ for $x$ drawn from (6.4.13), results derived by the
general approach in Section 2.2 can be written down as follows with proper substitutions in this particular case.

**Maximum Likelihood Estimate:** The likelihood equation for \( \phi \) is

\[
\bar{x}^* = \mu^* \tag{6.4.16}
\]

where \( \bar{x}^* = \sum \frac{x_n}{N} \) and \( \mu^* \) is defined by (6.4.14). Denoting this estimate as \( \hat{\phi} \), its asymptotic variance is given by

\[
\text{Var}(\hat{\phi}) = - \frac{(1 - \phi)}{N} \frac{d\mu^*}{d\phi} \tag{6.4.17}
\]

\[
= \frac{(1 - \phi)^2}{N\mu^*} \tag{6.4.18}
\]

where \( \mu^*_2 \) is the variance of (6.4.13).

As the equation (6.4.16) does not readily give an algebraic solution, one may use an iterative process of solution. However, one can have a ready solution of (6.4.16), if tables be made available for \( \mu \)'s for sufficiently close values \( \phi \). Here we present in Table VI values of \( \mu^* \) for the special case of \( c=1 \) with \( k=1, 2, 3 \) at suitable intervals of \( \phi \). When \( c=1, (6.4.16) \) reduces to

\[
\mu^* = \frac{k(1 - \phi)}{\hat{\phi}(1 - \phi^k)} \tag{6.4.19}
\]

This table can be used to compute \( \text{Var}(\hat{\phi}) \) by using either (6.4.17) or (6.4.18). In case (6.4.17) is used, \( \frac{d\mu^*}{d\phi} \) can be approximated by the finite difference ratio \( \frac{\Delta \mu^*}{\Delta \phi} \). In case Formula (6.4.18) is used, the relationship for use is

\[
\mu^*_2(c, \phi, k) = \mu^*(c, \phi, k) \left[ 1 + \mu^*(c-1, \phi, k+1) - \mu^*(c, \phi, k) \right].
\]
Ratio Estimate: The simple ratio estimate for $\phi$ of (6.4.13) can be written down as

$$\phi' = 1 - \frac{1}{N} \sum_{x=c+1}^{\infty} \frac{x}{(k+x-1)} n_x.$$  

(6.4.20)

(6.4.20) provides the unique unbiased linear function of frequencies for estimating $\phi$ and its exact variance is given by

$$\sigma^2(\phi') = \frac{1}{N} \left[ \sum_{x=c+1}^{\infty} \frac{x}{(k+x-1)}^2 y(x,\phi,k) - (1-\phi)^2 \right].$$  

(6.4.21)

We have already seen, however, for the complete negative binomial distribution that this type of ratio-estimate, though unbiased, involve serious loss of efficiency. In the case of truncated distribution also, therefore, it is expected to have a similar pattern of low efficiencies.

Two-Moments Estimate: Following Section 3.4, we get for the distribution (6.4.1)

$$\Theta = 1 - \phi = \frac{m^* - c \mu^*}{H_{11} - (c-1)H_{01}}$$  

(6.4.22)

where $\mu^*$ and $m^*$ are defined by (6.4.14) and (6.4.15) respectively and $H_{01}$ and $H_{11}$ reduce to

$$H_{01} = k + \mu^*$$

$$H_{11} = k \mu^* + m^*$$

(6.4.22) gives then

$$\phi = \frac{(k+1)\mu^* - (c-1)k}{m^* + (k+c+1)\mu^* - (c-1)k}$$  

(6.4.23)
so that a simple estimate for $\phi$ can be written down as

$$
t = \frac{(k+1) S_1 - (c-1)kN}{S_2 + (k-c+1)S_1 - (c-1)kN}
$$

(6.4.24)

where $S_i = \sum x^i n_x$ \quad $i = 1, 2$.

When $c = 1$, (6.4.24) reduces to

$$
t = \frac{(k+1) S_1}{S_2 + kS_1}
$$

(6.4.25)

The asymptotic variance of $t$ given by (6.4.25) reduces to

$$
\text{Var}(t) = \frac{1}{NH^2} (\sigma^*_{11} + \phi^2 \sigma^*_{22} - 2\phi \sigma^*_{12})
$$

(6.4.26)

where

$$
H = m^*_2 + k\mu^*
$$

$$
\sigma^*_{11} = (k+1)^2 \mu^*_2
$$

$$
\sigma^*_{22} = \frac{(m^*_1 - m^*_2)^2}{4} + k^2 \mu^*_2 + 2k (m^*_3 - \mu^* m^*)
$$

and

$$
\sigma^*_{12} = (k+1)[(m^*_3 - \mu^* m^*) + (k \mu^*_2)]
$$

where $m^*_r$ is the $r$-th theoretical moment and $\mu^*$ and $\mu^*_2$ are the mean and variance respectively of (6.4.13) with $c = 1$.

Also one gets to order $\frac{1}{N}$ the amount of bias of $t$ for $c = 1$ as

$$
b(t) = (\phi \sigma^*_{22} - \sigma^*_{12})/NH^2
$$

where $H, \sigma^*_{22}, \sigma^*_{12}$ are defined above.
6.5 Homogeneity and Combined Estimation, k Known

Consider a situation in which m different machines in a section of a factory are producing items of some kind. One has to examine if these different machines are homogeneous in respect of the quality (\phi) of the product as judged by the proportion defectives, and if so, to make a combined estimation of \phi.

On the basis of inverse binomial sampling applied to lots of fairly large number of items from a machine corresponding to each lot from every machine, one will be having an observation of number of items (k+x) that had to be inspected to get k defectives. Let \( N_j \) be the number of lots inspected from j-th machine, so that \( N = \sum_{j=1}^{m} N_j \) denotes the total number of lots inspected.

6.5.1 Thus, on the basis of a random sample of size \( N = \sum_{j=1}^{m} N_j \) from m distributions characterized by the probability law:

\[
(k+x-1) \binom{k}{k-1} \phi_j^k (1 - \phi_j)^x
\]  

\( j = 1, 2, \ldots, m \) where \( \phi_j \) denotes the quality of product from j-th machine. Following Section 2.5, we have the j-th "efficient-score"

\[
\psi_j = \frac{N_j}{1-\phi_j} (\bar{x}_j - \mu_j(\phi_j)) \tag{6.5.2}
\]

where \( \mu_j(\phi_j) \) is the mean of the j-th distribution.

The elements of the information matrix are

\[
I_{jj} = -\frac{N_j}{1-\phi_j} \frac{d\mu_j}{d\phi_j} = \frac{N_j\mu_2_j(\phi_j)}{(1-\phi_j)^2} \tag{6.5.3}
\]

\[
I_{jj'} = 0 \quad (j \neq j')
\]

where \( \mu_2_j(\phi_j) \) is the variance of the j-th distribution.

The hypothesis of homogeneity is \( H_0: \phi_1 = \phi_2 = \cdots = \phi_m \).
If the hypothesis $H_0$ is true, the value may be denoted by $\phi$ and the efficient score and the information with respect to $\phi$ are given by

$$\psi = \frac{N}{1-\phi} (\bar{x} - \mu)$$  \hspace{1cm} (6.5.4)

where

$$\bar{x} = \sum_{j=1}^{m} N_j \frac{\bar{x}_j}{N}$$

and

$$\mu = \sum_{j=1}^{m} N_j \mu_j(\phi)/N = \frac{k}{\bar{x}} (1 - \phi)$$

$$I = \frac{1}{1-\phi} \sum_{j=1}^{m} N_j \frac{d\mu_j}{d\phi}$$

$$= \frac{1}{(1-\phi)^2} \sum_{j=1}^{m} N_j \mu^2_j(\phi)$$

$$= \frac{NK}{\phi^2(1-\phi)}$$  \hspace{1cm} (6.5.5)

Solving $\hat{\psi} = 0$, we have $\hat{\phi} = \frac{k}{k+x}$  \hspace{1cm} (6.5.6)

A test of the homogeneity hypothesis $H_0$ is then given by the statistic

$$\chi^2 = \sum_{j=1}^{m} [\psi_j(\hat{\phi})]^2 / I_{jj}(\hat{\phi})$$  \hspace{1cm} (6.5.7)

$$= \sum_{j=1}^{m} N_j [\bar{x}_j - \mu_j(\hat{\phi})]^2 / \mu^2_j(\hat{\phi})$$  \hspace{1cm} (6.5.8)

which is asymptotically distributed as a chi-square with (m-1) degrees of freedom if $H_0$ is true.

6.5.2 We will illustrate the computation procedure with reference to data in the table below, which gives, for 10 (m=10) pairs of pages from Tippett's random sampling numbers, the observed distribution of the number (x) of even integers between two consecutive zeros (k=1) characterized by the model:

$$\phi(1-\phi)^x \quad x = 0$$
For the distribution of \( x \), we have to test if the pairs of pages are homogeneous in respect of \( \phi \) and if so, to get a combined estimate of \( \phi \).

**FREQUENCY FOR PAGES**

| No. of even integers (x) | Pair: 2 4 6 8 10 12 14 16 18 20 | \( j \) & \( j \) & \( j \) & \( j \) & \( j \) & \( j \) & \( j \) & \( j \) & \( j \) & \( j \) |
|-------------------------|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( x \)                | 3 5 7 9 11 13 15 17 19 21       |                 |                 |                 |                 |                 |                 |                 |                 |
| 0                      | 67 71 53 65 79 71 59 70 60 66   | 661             |                 |                 |                 |                 |                 |                 |                 |
| 1                      | 44 58 50 47 59 59 51 54 52 47   | 521             |                 |                 |                 |                 |                 |                 |                 |
| 2                      | 50 32 35 43 26 44 46 44 48 44   | 412             |                 |                 |                 |                 |                 |                 |                 |
| 3                      | 42 25 23 33 27 37 29 36 25 21   | 298             |                 |                 |                 |                 |                 |                 |                 |
| 4                      | 26 23 28 26 27 30 25 29 23 30   | 267             |                 |                 |                 |                 |                 |                 |                 |
| 5                      | 19 28 21 19 24 21 24 15 22 20   | 213             |                 |                 |                 |                 |                 |                 |                 |
| 6                      | 16 17 18 17 15 17 15 16 17 17   | 157             |                 |                 |                 |                 |                 |                 |                 |
| 7                      | 14 20 12 14 20 10 6 16 21 16   | 141             |                 |                 |                 |                 |                 |                 |                 |
| 8                      | 7 9 11 13 8 15 9 11 11 107     |                 |                 |                 |                 |                 |                 |                 |                 |
| 9                      | 5 5 9 11 7 9 5 5 8 73          |                 |                 |                 |                 |                 |                 |                 |                 |
| 10                     | 9 4 11 5 10 5 3 7 3 96         |                 |                 |                 |                 |                 |                 |                 |                 |
| 11                     | 6 9 7 7 4 9 2 6 6 73           |                 |                 |                 |                 |                 |                 |                 |                 |
| 12                     | 3 3 1 4 2 1 3 2 5 3 27         |                 |                 |                 |                 |                 |                 |                 |                 |
| 13                     | 1 2 2 4 5 3 3 2 3 28           |                 |                 |                 |                 |                 |                 |                 |                 |
| 14                     | 0 2 4 3 5 2 2 4 6 4 32         |                 |                 |                 |                 |                 |                 |                 |                 |
| 15                     | 4 3 2 1 2 2 6 2 3 1 26         |                 |                 |                 |                 |                 |                 |                 |                 |
| 16                     | 4 2 1 3 1 3 3 0 0 18           |                 |                 |                 |                 |                 |                 |                 |                 |
| 17                     | 1 2 2 2 3 2 0 0 1 2 15         |                 |                 |                 |                 |                 |                 |                 |                 |
| 18                     | 1 2 0 0 2 1 1 1 1 0 9          |                 |                 |                 |                 |                 |                 |                 |                 |
| 19                     | 2 0 2 1 2 0 2 2 1 13           |                 |                 |                 |                 |                 |                 |                 |                 |
| 20                     | 0 1 2 0 0 1 2 0 1 1 8          |                 |                 |                 |                 |                 |                 |                 |                 |
| 21                     | 0 0 1 1 0 0 0 0 0 0 2           |                 |                 |                 |                 |                 |                 |                 |                 |
| 22                     | 0 0 0 0 1 0 0 0 0 0 3           |                 |                 |                 |                 |                 |                 |                 |                 |
| 23                     | 0 0 0 0 0 2 1 3 0 1 7          |                 |                 |                 |                 |                 |                 |                 |                 |
| 24                     | 1 0 0 0 0 0 0 1 2 4            |                 |                 |                 |                 |                 |                 |                 |                 |
| 25                     | 0 0 1 0 0 0 0 1 0 0 2           |                 |                 |                 |                 |                 |                 |                 |                 |
| 26                     | 2 1 0 0 0 0 0 0 0 0 3           |                 |                 |                 |                 |                 |                 |                 |                 |
| 27                     | 0 0 0 0 0 0 0 0 0 0 0           |                 |                 |                 |                 |                 |                 |                 |                 |
| 28                     | 0 0 0 0 0 0 0 0 0 0 1           |                 |                 |                 |                 |                 |                 |                 |                 |

| Total                  | N_j 324 319 296 317 342 333 309 309 315 313 3177 | S_{1j} 1242 1210 1252 1222 1378 1194 1225 1138 1247 1223 | \( \bar{x}_j \) 3.8333 3.7931 4.2297 3.8549 4.0146 3.5856 3.9644 3.6828 3.9587 3.9073 |
For the above data, we have the following:

<table>
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<th>N_j</th>
<th>x̄_j</th>
<th></th>
<th>N_j</th>
<th>x̄_j</th>
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<td>6</td>
<td>333</td>
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<td>3.7931</td>
<td>7</td>
<td>309</td>
<td>3.9644</td>
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<tr>
<td>3</td>
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<td>8</td>
<td>309</td>
<td>3.6828</td>
</tr>
<tr>
<td>4</td>
<td>317</td>
<td>3.8549</td>
<td>9</td>
<td>315</td>
<td>3.9587</td>
</tr>
<tr>
<td>5</td>
<td>342</td>
<td>4.0146</td>
<td>10</td>
<td>313</td>
<td>3.9073</td>
</tr>
</tbody>
</table>

\[
N = \sum_{j=1}^{10} N_j = 3177
\]
\[
S_1 = \sum x_n x = 12326
\]
so that
\[
\bar{x} = \frac{S_1}{N} = 3.8798.
\]

The maximum likelihood estimate of \(\phi\) under the hypothesis of homogeneity is then given by
\[
\hat{\phi} = \frac{1}{1 + \bar{x}} = \frac{1}{4.8798} = 0.2049.
\]

To test if \(\phi\) is the same for different pairs of pages, we have to compute
\[
\sum N_j[x_j - \mu_j(\phi)]^2 = \sum N_j(x_j - \bar{x})^2 = 90.974
\]
and
\[
\mu_2(\phi) = \frac{(1-\phi)}{\phi^2} = \frac{\bar{x}}{\phi} = 18.935
\]
so that
\[
\chi^2 = \frac{\sum N_j (x_j - \bar{x})^2}{\mu_2(\phi)} = \frac{90.974}{18.935} = 4.8045
\]

which with 10-1 = 9 degrees of freedom is not significant. The pairs of pages of Tippett's random sampling numbers can thus be regarded as homogeneous in respect of the distribution of even integers between two consecutive zeros, as they should be expected.
CHAPTER VII

7.0 ESTIMATION PROBLEMS FOR THE LOGARITHMIC SERIES DISTRIBUTION

7.1 Introduction

The gpdf defined by (1.1.4) becomes

\[ \text{Prob} \{ X = x \} = \frac{\theta^x}{x \log (1-\theta)} \quad (7.1.1) \]

when \( f(\theta) = -\log(1-\theta) \).

Writing \( \alpha = \frac{1}{-\log(1-\theta)} \), (7.1.1) gives the probability law for \( X \) as:

\[ \text{Prob} \{ X = x \} = p(x, \theta) = \frac{\theta^x}{x} \quad (7.1.2) \]

a well-known form of the Logarithmic series distribution.

The first two moments about origin of (7.1.2) can be obtained as:

\[ \mu = \frac{\theta \theta}{1-\theta} \quad (7.1.3) \]

and

\[ m_2 = \frac{\mu}{1-\theta}. \quad (7.1.4) \]

7.2 Estimation from a Sample for Complete Logarithmic Series Distribution

Applications of logarithmic series distribution have been discussed among others by Fisher (1943), Williams (1943, 1944), Harrison (1945) and Kendall (1948). Problems of estimation, however, do not seem to have been thoroughly investigated. Following the general approach discussed in Chapters II and III, we provide in this section different estimates for the parameter \( \theta \) of the logarithmic series and investigate their efficiency and the amount of bias in
certain special cases. A table of mean-values of the logarithmic series is provided to obtain the maximum likelihood estimate with facility.

7.2.1 To estimate $\theta$ by likelihood on the basis of a random sample of size $N$ with frequency $n_x$ for $1 \leq x \leq \infty$, $\sum n_x = N$ drawn from (7.1.2), results derived by the general approach in Section 2.1 can be written down as follows:

The likelihood equation for $\theta$ is

$$\bar{x} = \hat{\mu}$$  \hspace{1cm} (7.2.1)

where $\bar{x} = \sum x_n / N$ and $\mu$ is defined by (7.1.3).

Denoting this estimate as $\hat{\theta}$, its asymptotic variance is given by

$$\text{Var}(\hat{\theta}) = \frac{\theta^2}{N\mu_2}$$  \hspace{1cm} (7.2.2)

where $\mu_2$ is the variance of (7.1.2). (7.2.1) suggests that if a table be made available for means $\mu$'s for sufficiently close values of $\theta$, we can have a ready solution of (7.2.1). Here, we present a numerical Table VII for the argument $\theta = .01 (.01) .99$. This table can be used to compute $\text{Var}(\hat{\theta})$, because

$$\mu_2 = \mu(1 - \frac{1}{\theta}) - \mu).$$  \hspace{1cm} (7.2.3)

7.2.2 The expression (7.1.4) for $m_2$ gives

$$\theta = 1 - \frac{\mu}{m_2}.$$  \hspace{1cm} (7.2.4)

The two-moments estimate for $\theta$ can then be written down as

$$t = 1 - \frac{S_1}{S_2}$$  \hspace{1cm} (7.2.5)

where

$$S_i = \sum x^i n_x \quad i = 1, 2.$$
The asymptotic variance of $t$ is given by

$$\text{Var}(t) = \frac{1}{N^2} \left[ \sigma_{11} - 2(1-\theta) \sigma_{12} + (1-\theta)^2 \sigma_{22} \right]$$  \hspace{1cm} (7.2.6)$$

where

$$\sigma_{11} = m_2 - \mu^2$$

$$\sigma_{12} = m_3 - \mu m_2$$  \hspace{1cm} (7.2.7)$$

and

$$\sigma_{22} = m_4 - m_2$$

where $m_r$ denotes the $r$th moment of (7.1.2) about origin.

The efficiency of $t$ is then obtained as $\text{Eff}(t) = \frac{\text{Var}(\hat{\Theta})}{\text{Var}(t)}$

The following table gives the efficiency of $t$ for $\theta = .10, .50, .90$.

**TABLE 7.2.1**

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>.10</th>
<th>.50</th>
<th>.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Var}(\hat{\Theta})/\text{Var}(t)$</td>
<td>.228</td>
<td>.449</td>
<td>.488</td>
</tr>
</tbody>
</table>

7.2.3 Following the general approach discussed in Section 3.1, a ratio-estimate can be obtained for $\theta$ of (7.1.2). In this case, since $a_{x-1}/a_x = \frac{x}{x-1} = 1 + \frac{1}{x-1}$, the ratio-estimate for $\theta$ can be written down as

$$\theta' = (1 - \frac{n_{1}}{N}) + \frac{1}{N} \sum_{x=2}^{\infty} \frac{n_{x}}{x-1}.$$  \hspace{1cm} (7.2.8)$$
(7.2.8) provides the unique unbiased estimate of $\Theta$ linear in the frequencies. The exact variance of this estimate is

$$
\sigma^2(\Theta') = \frac{1}{N} \left[ \sum_{x=2}^{\infty} \frac{(x)}{(x-1)^2} p_x - \Theta^2 \right] \tag{7.2.9}
$$

and an unbiased estimate of $\sigma^2(\Theta')$ is

$$
[\sum_{x=2}^{\infty} \frac{(x)}{(x-1)^2} n_x - N\Theta^2]/N(N-1). \tag{7.2.10}
$$

The efficiency of $\Theta'$ is then obtained as $\text{Eff}(\Theta') = \frac{\text{Var}(\hat{\Theta})}{\sigma^2(\Theta')}$. The following table gives the efficiency of $\Theta'$ for $\Theta = .10, .50, .90$.

<table>
<thead>
<tr>
<th>$\Theta$</th>
<th>.10</th>
<th>.50</th>
<th>.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Var}(\hat{\Theta})/\sigma^2(\Theta')$</td>
<td>.895</td>
<td>.447</td>
<td>.057</td>
</tr>
</tbody>
</table>

7.2.4 One more estimate for $\Theta$ can be obtained when one notices that the expression (7.1.3) for mean $\mu = \frac{\Theta}{1-\Theta}$ can be written down as

$$
\Theta = 1 - \frac{P_1}{\mu} \tag{7.2.11}
$$

because

$$
P_1 = \Theta. \tag{7.2.12}
$$

The identity (7.2.11) suggests an estimate for $\Theta$ as

$$
\Theta'' = 1 - \frac{n_1}{S_1} \tag{7.2.13}
$$

with

$$
\text{Var}(\Theta'') = \frac{1}{N\mu^2} \left[ \sigma_{11} - 2(1-\Theta) \sigma_{12} + (1-\Theta)^2 \sigma_{22} \right] \tag{7.2.14}
$$
where

\[ \sigma_{11} = P_1 (1-P_1) \]
\[ \sigma_{12} = P_1 (1-\mu) \]

and

\[ \sigma_{22} = m_2 - \mu^2. \]

The asymptotic efficiency of \( \varphi'' \) is then obtained as

\[ \text{Eff} (\varphi'') = \frac{\text{Var}(\hat{\varphi})}{\text{Var}(\varphi'')} \]

The following table gives the efficiency of \( \varphi'' \) for \( \varphi = .10, .50, .90. \)

<table>
<thead>
<tr>
<th>( \varphi )</th>
<th>.10</th>
<th>.50</th>
<th>.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\text{Var}(\hat{\varphi})}{\text{Var}(\varphi'')} )</td>
<td>.983</td>
<td>.897</td>
<td>.739</td>
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</table>

7.2.5 So far, we have separately discussed the simple estimates denoted by \( t, \varphi' \) and \( \varphi'' \). To make a comparative study of these estimates, let us investigate their amount of bias and relative efficiency.

It can be easily deduced that the amounts of bias to order \( \frac{1}{N} \) of the estimates \( t \) and \( \varphi'' \) are

\[ b(t) = \frac{1}{N\mu^2} [(1-\varphi) \sigma_{22} - \sigma_{12}] = \frac{B(t)}{N} \]  \hspace{1cm} (7.2.16)

where \( \sigma_{22} \) and \( \sigma_{12} \) are defined by (7.2.7).
and
\[
b(\theta'') = \frac{1}{N\mu^2} \left[ (1-\theta) \sigma_{22} - \sigma_{12} \right] = \frac{B(\theta'')}{N} \tag{7.2.17}
\]

where \(\sigma_{22}\) and \(\sigma_{12}\) are defined by (7.2.15).

As regards \(\theta'\), it is known that it is unbiased.

The following table gives \(B(t)\), \(B(\theta'')\) and also relative efficiency of these estimates with respect to the ratio-estimate \(\theta'\) for \(\theta = .10, .50, .90\).

\begin{table}[h]  
\centering
\caption{Comparison of the Estimates}
\begin{tabular}{c|ccc}
\hline
\(\theta\) & \(N\) (amount of bias to order \(1/N\)) & \(\text{Var}(\theta')/\text{Var}(t)\) & \(\text{Var}(\theta')/\text{Var}(\theta'')\) \\
\hline
\hline
\(t\) & \(Q'\) & & & \\
.10 & 0.9128 & 0.0948 & 0.255 & 1.098 \\
.50 & 1.7329 & 0.3466 & 1.003 & 2.006 \\
.90 & 0.9279 & 0.2303 & 8.564 & 12.973 \\
\hline
\end{tabular}
\end{table}

Thus, it is easy to see that the estimate \(\theta''\) may be used with advantage to estimate the parameter \(\theta\) of the logarithmic series because of its simplicity, small bias and high efficiency.

7.2.6 The detailed computation procedure of evaluating the four types of estimates discussed above is illustrated with reference to the logarithmic series data due to Williams. The following table gives the distribution of 1534 biologists according to the number of research papers to their credit in the Review of Applied Entomology, Vol. 24, 1936.
<table>
<thead>
<tr>
<th>No. of papers per author ( (x) )</th>
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<th>4</th>
<th>5</th>
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<th>11</th>
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</thead>
<tbody>
<tr>
<td>No. of authors ( (n_x) )</td>
<td>1062</td>
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<td>0</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>

**Maximum Likelihood Estimate:** For the data we get

\[
N = 1534 \\
S_1 = 2379
\]

so that \( \bar{x} = 1.5508 \). Referring to Table VII we find the following:

\[
\begin{array}{cc}
\Theta & \mu \\
0.56 & 1.5503 \\
0.57 & 1.5706 \\
\end{array}
\]

The maximum likelihood estimate is given by that value of \( \Theta \) for which \( \mu = 1.5508 \). By linear interpolation, we get \( \hat{\Theta} = 0.5602 \). To compute the variance of \( \hat{\Theta} \), taking 0.5602 as the estimate for \( \Theta \), we require

\[
\alpha = \frac{\log_e 10}{-\log_{10}(1-\Theta)} = 1.21738
\]

\[
\mu = \frac{\Theta \mu}{1-\Theta} = 1.5507
\]

\[
m_2 = \frac{\mu}{1-\Theta} = 3.5259
\]

\[
\mu_2 = m_2 - \mu^2 = 1.1212.
\]

Then the variance of \( \hat{\Theta} \) is estimated from

\[
\text{Var}(\hat{\Theta}) = \frac{\Theta^2}{N_{\mu_2}} = 0.000182
\]

so that \( \text{S.E.}(\hat{\Theta}) = 0.0135 \).
Two-Moments Estimate: To estimate this estimate of $\Theta$, we require in addition the value of

$$S_2 = \sum x^2 \frac{n_x}{n} = 5439.$$ 

Then, the estimate is

$$t = 1 - \frac{S_1}{S_2} = 0.5626.$$ 

To compute the variance of $t$, taking 0.5626 as the estimate for $\Theta$, we have with usual symbols,

$$\alpha = 1.2093$$
$$\mu = 1.5555$$
$$m_2 = 3.5562$$
$$\mu_2 = 1.1366.$$ 

Also,

$$m_3 = \mu m_2 + \frac{1}{1-\Theta} \left( \mu_2 + \frac{\mu_2}{\alpha} \right)$$
$$= 12.7045.$$ 

$$m_4 = \mu m_3 + (m_3 - \mu m_2) \left( \mu + \frac{2}{1-\Theta} \right)$$
$$+ m_2 \left( m_2 + \mu_2 \right) + \frac{2\mu_2}{1-\Theta} \left( \frac{1}{\alpha} - 1 \right)$$
$$= 79.0059$$

so that

$$\sigma_{11} = m_2 - \mu^2 = 1.1366$$
$$\sigma_{12} = m_3 - \mu m_2 = 7.1728$$
$$\sigma_{22} = m_4 - m_2^2 = 66.3593.$$ 

Then, the variance of $t$ is estimated from

$$\text{Var}(t) = \frac{1}{\text{Nm}_2^2} \left[ \sigma_{11} - 2(1-\Theta) \sigma_{12} + (1-\Theta)^2 \sigma_{22} \right]$$
$$= 0.000390$$

so that $\text{S.E.}(t) = 0.0197.$
**Ratio Estimate:** The ratio estimate for $\Theta$ is given by

$$\Theta' = (1 - \frac{n_1}{N}) + \frac{1}{N} \sum_{x=2}^{\infty} \frac{n_x}{x-1}$$

$$= 0.3077 + 0.2269$$

$$= 0.5346.$$  

The variance of $\Theta'$ is estimated from the formula:

$$\text{Var}(\Theta') = \frac{1}{N} \Theta(1-\Theta) + \alpha \Theta \sum_{x=1}^{\infty} \frac{\Theta^x}{x^2}.$$  

Approximating

$$\sum_{x=1}^{\infty} \frac{\Theta^x}{x^2} \sim \sum_{x=1}^{20} \frac{\Theta^x}{x^2} = 0.6310$$

$$\text{Var}(\Theta') = 0.000450$$

so that S.E. ($\Theta''$) = 0.0212.

**Alternative Estimate:** The estimate based on $n_1$ and $S_1$ is given by

$$\Theta'' = 1 - \frac{n_1}{S_1} = 1 - \frac{1062}{2378}$$

$$= 0.5536.$$  

To compute variance of $\Theta''$, using usual symbols we have

$$\alpha = 1.23986$$

$$P_1 = 0.23986$$

$$\mu = 1.5376$$

$$m_2 = 3.4444$$

$$\mu_2 = 1.0802$$

so that

$$\sigma_{11} = P_1(1-P_1) = 0.2153$$

$$\sigma_{12} = P_1(1-\mu) = 0.013690$$

$$\sigma_{22} = m_2 - \mu^2 = 1.0802.$$
Then the variance of $\theta''$ is estimated from

$$\text{Var}(\theta'') = \frac{1}{N} \left[ \sigma_{11} - 2(1-\theta) \sigma_{12} + (1-\theta)^2 \sigma_{22} \right]$$

$$= 0.000279$$

so that $\text{S.E.}(\theta'') = 0.0167$.

The following table summarizes the results obtained.

<table>
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<tr>
<th>Estimate</th>
<th>Value</th>
<th>Variance</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
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<tr>
<td>TM</td>
<td>0.5626</td>
<td>0.000390</td>
<td>0.0197</td>
</tr>
<tr>
<td>R</td>
<td>0.5346</td>
<td>0.000450</td>
<td>0.0212</td>
</tr>
<tr>
<td>$\theta''$</td>
<td>0.5536</td>
<td>0.000279</td>
<td>0.0167</td>
</tr>
</tbody>
</table>

7.3 Estimation from a Sample for Truncated Logarithmic Series Distribution

Following the general approach discussed in Chapters I, II, and III, the results for the estimation of the parameter $\theta$ can be written down on the basis of a sample drawn from a truncated logarithmic series distribution.
TABLES I – VII
TABLE I

\[ n^* \] OF SINGLY TRUNCATED BINOMIAL DISTRIBUTION ON THE LEFT AT \( c = 1 \)

(Zero-Observations Truncated)

\[ n = 3(1.15); n = .00(.01).99 \]

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### Table II

Table: Approximations of singly truncated binomial distribution on the left at \( e^{-2} \)

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| \( n = 4 \) |
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| 20   | 53928 | 54221 | 54529 | 54845 | 55166 | 55494 | 55839 | 56193 | 56552 | 56920 |
| 30   | 56891 | 56878 | 67173 | 57474 | 57781 | 58099 | 58413 | 58738 | 59071 | 59410 |
| 40   | 59756 | 60102 | 60465 | 60837 | 61218 | 61590 | 61973 | 62367 | 62772 | 63191 |
| 50   | 63636 | 64070 | 64512 | 64962 | 65428 | 65902 | 66384 | 66871 | 67368 | 67875 |
| 60   | 68420 | 68976 | 69507 | 70056 | 70637 | 71223 | 71816 | 72426 | 73046 | 73683 |
| 70   | 74331 | 74993 | 75668 | 76357 | 77056 | 77777 | 78502 | 79254 | 80031 | 80848 |
| 80   | 81581 | 82326 | 83006 | 84031 | 84877 | 85740 | 86614 | 87502 | 88404 | 89317 |
| 90   | 90243 | 91181 | 92130 | 93081 | 94057 | 95038 | 96017 | 97006 | 98002 | 99003 |

\( \pi = \frac{k}{n} \)

| \( n = 5 \) |
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# Table V

μ* of Singly Truncated Poisson Distribution on the Left at c

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c = 1(1)10; \mu = 0.0(0.1)9.9
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| 1.0 | 2.3922 | 2.4382 | 2.4866 | 2.5342 | 2.5842 | 2.6342 | 2.6888 | 2.7419 | 2.7970 | 2.8535 |
| 5.0 | 5.1755 | 5.2647 | 5.3544 | 5.4448 | 5.5356 | 5.6270 | 5.7189 | 5.8112 | 5.9040 | 5.9972 |
| 7.0 | 7.0450 | 7.1419 | 7.2389 | 7.3362 | 7.4336 | 7.5313 | 7.6290 | 7.7270 | 7.8250 | 7.9233 |

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| 4.0 | 4.7653 | 4.8158 | 4.8664 | 4.9170 | 4.9687 | 5.0215 | 5.0743 | 5.1282 | 5.1820 | 5.2354 |
| 5.0 | 5.4011 | 5.5577 | 5.6332 | 5.7136 | 5.7928 | 5.8729 | 5.9539 | 6.0356 | 6.1181 | 6.2014 |</p>
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TABLE VI

\( \Phi^* \) OF SIMPLY TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION ON THE LEFT AT \( c=1 \)

(Zero-Observations Truncated)

\( k = 1(1)3; \ \Phi = .01(.01).99 \)

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### TABLE VII

μ of LOGARITHMIC SERIES DISTRIBUTION

θ = .01(.01).99

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Chart 1. Estimation of $x$ of Singly Truncated Binomial Distribution at $c = 1$ for $n = 3(1)6$
Chart 3. Estimation of \( \pi \) of Singly Truncated Binomial Distribution at \( c = 1 \) for \( n = 11(1)15 \)
Chart 4. Estimation of $\mu$ of Singly Truncated Poisson Distribution at $c = 1$
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<td>The cumulants and moments of the binomial distribution and the cumulants of chi-square for a(nx2)-fold table.</td>
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