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CONTRIBUTIONS TO ESTIMATION IN A CLASS OF DISCRETE DISTRIBUTIONS

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INTRODUCTION

General Background

Different approaches are possible with regard to the basic structure of the standard discrete distributions like the binomial, Poisson, negative binomial, or logarithmic series. Under plausible assumptions, these distributions may be regarded as descriptive models of populations. For instance, under the usual genetic theory, the distribution of the number x of boys in families of a fixed number of children, say n, follows the binomial law. Or again, the number of mistakes per printed page can ordinarily be assumed to have a Poisson distribution. The same is true of the distribution of accidents met with, over a period of time, by a particular individual. However, different persons may have different accident proneness as measured by the average number of accidents to the individual. If this average has, say, a Pearson's type III distribution, the distribution of the number of accidents pooled over the individuals can be shown to follow the negative binomial law. A limiting case of this is the logarithmic series distribution found useful in ecology.

Again, these distributions may arise as a result of the sampling scheme adopted. In sampling with replacement n items from a lot of manufactured items, the number of defectives follows the binomial law. If the proportion of defectives in the lot is small and the number of items sampled is moderately large, one can use the Poisson approximation for the distribution of the number of defectives. On the other hand, if one uses what is known as the inverse binomial sampling procedure (that is, one goes on sampling with replacement until he gets a fixed number such as k of defectives), the number of items sampled would then follow the negative binomial distribution.

The discrete distributions described above occur sometimes in truncated or censored forms. For instance, in human genetics, to estimate the proportion of albino children produced by couples capable of producing albinos, sampling has necessarily to be restricted to families having at least one albino child. This is because there is no way of distinguishing families incapable of producing albinos from those that are capable but by way of chance have not produced any. The number of albino children x, if sampling is restricted to families of n children, can thus take the values 1,2,...n; the value 0 being excluded. Thus, x follows what is known as the "truncated" binomial distribution--truncated on the left at 1 to be specific. We may similarly think of distributions with both extremes truncated.

Again, some types of counters can record the exact number of radio-active particles (emitted by a radio-active substance in fixed intervals of time) if the number does not exceed a certain limit such as d, otherwise it merely records that the number has exceeded d. Data obtained from such counters are samples from what are called "censored" distributions, consored on the right at d. In such cases, individual counts of all observations not exceeding d are available; only the total count of those exceeding d.

Review of Previous Work in Estimation in Discrete Distributions

The problems of estimation of parameters in various discrete distributions and their truncated or censored forms have been considered by various authors. Generally, the estimate for the case of complete distributions is neat and easy to compute, but complications come in for truncated and censored distributions.

For the truncated binomial distribution, Fisher (1936) and Haldane (1932,1938) gave the maximum likelihood procedure for estimating the parameter of the distribution. While studying the albinism in man by sampling from families with variable number of children (but having at least one albino), Haldane (1938) solved by an elaborate iterative process the complicated maximum likelihood equation based on data obtained simultaneously from different truncated binomial populations. Finney (1949) utilized the method of scores for solving the likelihood equation and provided some tables to facilitate the heavy computation. Because of computational difficulties in getting the maximum likelihood estimate, Moore (1944) suggested an alternative simple estimate which is a ratio of two suitably chosen linear functions of frequencies, whereas Rider (1955) equated the first two sample moments to the corresponding population moments and obtained another simple estimate. The sampling properties of these two simple estimates, however, have not been studied.

Estimation problems in relation to the Poisson distribution have been investigated by many authors. For truncated Poisson distributions, the case of truncation on the left has been considered by David and Johnson (1948) who provided the maximum likelihood estimate and a small numerical table for computational facility. Plackett (1953) gave a simple unbiased and highly efficient estimate which is a ratio of two linear functions of frequencies. Rider (1953) used first two moments and obtained another simple estimate, but did not study its sampling properties.

Truncation on the right has been discussed by Tippett (1932), Bliss (1948) and Moore (1952). Tippett derived the maximum likelihood solution, Bliss developed an approximation to it and Moore suggested a ratio of two linear functions of frequencies as a simple alternative estimate. For estimation in case of doubly truncated Poisson distributions, Cohen (1954) provided maximum likelihood equations, though rather unwieldy to solve; whereas Moore utilized a ratio of suitably constructed linear functions of frequencies to estimate the Poisson-parameter. They also discussed the problems of estimation for samples from censored Poisson distributions. Moore (1952) gave the simple ratio-estimate and Cohen (1954) derived maximum likelihood equations for both singly and doubly censored distributions.

For a negative binomial distribution involving two parameters, Fisher (1941) discussed the efficiency of moment-estimates and derived maximum likelihood equations for simultaneous estimation. He also gave a simple rule as to when one should proceed for getting maximum likelihood estimates. Haldane (1941) reduced the likelihood equations to a simpler form for computational facility. Sampford (1955) gave methods to obtain moment-estimates and likelihood estimates for a truncated negative binomial distribution with "zero" truncated, whereas equating the first three sample moments to the corresponding moments of the truncated negative binomial distribution, Rider obtained simple estimates for the two parameters.

Results for estimation of the parameter of a logarithmic series distribution are rather complicated, even when the distribution is complete. The estimation problems do not seem to have been thoroughly investigated. Fisher, Corbet and Williams (1943) found it useful in ecology and derived the maximum likelihood estimate of its parameter.

We thus see that previous work on estimation in these discrete distributions can be broadly classified under two heads: (1) estimation by the method of maximum likelihood, and (2) other methods of estimation, the need for the other methods arising from the fact that frequently the method of maximum likelihood leads to complicated equations for estimation.

For maximum likelihood estimation, the authors have derived the estimating equations in individual cases and have suggested the use of Fisher's iterative procedure based on "efficient scores" for the solution of the equations when these turn out to be complicated. Some numerical tables are provided here and there to help in the process of solution. The identity of the maximum likelihood and the moments method has been noticed in a few cases.

Other estimates suggested in individual cases to avoid the computational difficulties of maximum likelihood estimation are of two types. One is derived by equating the first two sample moments with the corresponding theoretical moments. The second is obtained by taking the ratio of two suitably constructed linear functions of frequencies such that the ratio of the expectation of the numerator to that of the denominator is the required parameter. These two types of estimates are easy to compute in the cases suggested, but one has to remember that they are, in general, biased and inefficient though consistent. We note that the sampling properties of these estimates have not been investigated by the previous authors.

Present Contributions to Estimation in Discrete Distributions

It is first shown that the binomial, Poisson, negative binomial and logarithmic series distributions can be regarded as special cases of a general class of discrete distributions which we refer to as generalized power series distribution (gpsd). It is then possible to examine the previous work on estimation in the case of the above discrete distributions from a general point of view. The approach in this thesis is to derive results for this general class of distributions and then apply them to the special cases of binomial, Poisson, etc.

To begin with, we present a few results which bring out some interesting properties of a gpsd. We discover an explicit functional relationship between the variance and mean of a gpsd and based on this fundamental relation, we present some characterization theorems. To mention one, we establish that the equality of variance and mean is necessary and sufficient for a gpsd to be Poisson.

Next, we investigate certain estimation problems connected with a gpsd. We show that the maximum likelihood method and the method of moments give the same estimate when the gpsd involves a single parameter. A computational method for evaluating the maximum likelihood estimate is developed which requires only a table of values of the mean of the gpsd for various values of the parameter at sufficiently close intervals. It is shown how the standard error of the estimate can be approximately evaluated by using this table. The formulae for the amount of bias in the likelihood estimate are obtained to the order of $1/N$ where N is the sample size.

Large sample methods based on maximum likelihood are then derived for testing the homogeneity of several distributions and providing the estimate for the common parameter in case the distributions are homogeneous. The likelihood equation and a method for solving it are derived for the problem of estimation in censored forms of a gpsd. Methods based on the maximum likelihood principle are given for the treatment of doubtful observations. The problem of estimation when the gpsd involves two parameters has been considered.

For the gpsd, in addition to the maximum likelihood estimate, two other simple estimates are provided. One is called the "two-moments estimate" and is derived by equating the first two sample moments to the corresponding population moments. The other estimate is called the "ratio estimate" as it is obtained by taking the ratio of two suitably constructed linear functions of frequencies such that the ratio of the expected value of the numerator to that of the denominator is equal to parameter. Expressions are derived for the bias and variance of these two estimates correct to terms of order $1/N$ where N is the sample size.

Lastly, the results obtained by general approach are applied to specific distributions - namely, the binomial, Poisson, negative binomial, and logarithmic series. In each case, exhaustive numerical tables are given to facilitate computation of the maximum likelihood estimate. The bias and efficiency of the "ratio" and "two-moments" estimates are numerically evaluated for different values of the parameter and recommendations are given for the suitability of the different methods of estimation. Illustrative examples have been worked out in detail to illustrate the methods suggested.

CHAPTER I

1.0 A CLASS OF DISCRETE DISTRIBUTIONS AND CERTAIN CHARACTERIZATION THEOREMS

1.1 Introduction

Let $g(\theta)$ be a positive function admitting a power series expansion with non-negative coefficients for non-negative values of θ smaller than the radius of convergence of the power series:

$$g(\theta) = \sum_{z=0}^{\infty} a_z \theta^z . \tag{1.1.1}$$

Noack (1950) defined a random variable Z taking non-negative integral values z with positive probabilities

$$\text{Prob}\left\{ Z=z \right\} = \frac{a_z \theta^z}{g(\theta)} \qquad (z = 0,1,2,\ldots\infty). \tag{1.1.2}$$

He called the discrete probability distribution given by (1.1.2) a power series distribution (psd) and derived some of its properties relating its moments, cumulants, etc.

To be more general, we note that the set of values of an integral-valued random variable Z need not be the entire set of non-negative integers $(0,1,2,\ldots\infty)$. For, let T be an arbitrary non-null subset of non-negative integers* and define the generating function

$$f(\theta) = \sum_{x \in T} a_x \theta^x \tag{1.1.3}$$

with $a_x \geq 0$; $\theta \geq 0$ so that $f(\theta) > 0$, is finite and differentiable.

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* In fact, one can take T to be a countable subset of real numbers; for purposes of this dissertation, however, T is chosen to be a subset of non-negative integers.

Then we can define a random variable X taking non-negative integral values in T with probabilities

$$P_x = \text{Prob}\left\{ X = x \right\} = \frac{a_x \theta^x}{f(\theta)} \qquad x \epsilon T \qquad (1.1.4)$$

and call this distribution analogously a generalized power series distribution (gpsd). It may be noted that gpsd reduces to a psd when T is the entire set of non-negative integers. We add here that we call the set of admissible values of the parameter θ of gpsd as the parameter space Θ of the gpsd. Also we refer to set T of values of random variable X defined by the gpsd, as the range T of the gpsd.

Writing the mean $\mu = E(X)$, the crude moments $m_r = E(X^r)$, the central moments $\mu_r = E(X-\mu)^r$, the moment generating function (mgf) $M(t) = E(e^{tX})$ and the cumulants $K_r = [\frac{d^r}{dt^r} \log M(t)]_{t=0}$ (in case they exist); we obtain, for a gpsd, the following relations derived on the same lines as shown by Noack (1950) for a psd:

$$\mu = \theta f'(\theta)/f(\theta) \qquad (1.1.5)$$

$$m_{r+1} = \theta m'_r + \mu m_r \qquad (1.1.6)$$

$$\mu_{r+1} = \theta \mu'_r + r\mu_2 \mu_{r-1} \qquad (1.1.7)$$

$$M(t) = f(\theta e^t) \Big/ f(\theta) \qquad (1.1.8)$$

$$m_r = \sum_{i=1}^{r} \binom{r-1}{i-1} m_{r-i} K_i \qquad (1.1.9)$$

$$K_{r+1} = \theta \sum_{i=1}^{r} \binom{r-1}{i-1} m_{r-i} K'_i - \sum_{i=2}^{r} \binom{r-1}{i-2} m_{r+1-i} K_i \qquad (1.1.10)$$

where primes denote differentiation with respect to θ.

We note further that for a gpsd

$$\mu = \theta \frac{d}{d\theta} [\log f(\theta)] \tag{1.1.11}$$

$$\mu_2 = \theta \frac{d\mu}{d\theta} \tag{1.1.12}$$

$$K_{r+1} = \theta \frac{dK_r}{d\theta} \tag{1.1.13}$$

Also, writing the factorial moment of order r as

$$\mu_{(r)} = E[(X)(X-1).....(X-r+1)]$$

we can derive for a gpsd

$$\mu_{(r)} = \frac{\theta^r}{f(\theta)} \frac{d^r}{d\theta^r} [f(\theta)] \tag{1.1.14}$$

$$\mu_{(r+1)} = (\mu-r)\mu_{(r)} + \theta \frac{d}{d\theta} [\mu_{(r)}] \tag{1.1.15}$$

The Binomial, Poisson, Negative Binomial and the Logarithmic Series distributions can be obtained as special cases of the gpsd by taking

$f(\theta) = (1+\theta)^n$, n positive integer for Binomial

$f(\theta) = e^\theta$, for Poisson

$f(\theta) = (1-\theta)^{-k}$, k positive for Negative Binomial

$f(\theta) = -\log(1-\theta)$, for Logarithmic Series.

It is interesting to note that the Poisson and Negative Binomial distributions are special cases of a psd also; however, the Binomial and Logarithmic Series are not.

The relations (1.1.5) to (1.1.15) are generalizations of corresponding results obtained by various authors (Romanovsky, Frisch, Haldane) separately for Binomial, Poisson, Negative Binomial distributions.

1.2 Functional Dependence of Variance and Mean of a gpsd

In this section, we present a few results which bring out some interesting properties of a gpsd. We discover an explicit functional relationship between the variance and the mean of a gpsd and based on this relation, present some characterization theorems.

Theorem 1: For a gpsd, Variance = Mean + $\theta^2 \frac{d^2}{d\theta^2}$ [log f(θ)].

Proof : For a gpsd, we have from (1.1.12) and (1.1.11),

$$\text{Variance } \mu_2(\theta) = \theta \frac{d}{d\theta} [\mu(\theta)] \text{ and}$$

$$\text{Mean } \mu(\theta) = \theta \frac{d}{d\theta} [\log f(\theta)].$$

Consider

$$\begin{aligned}\mu_2(\theta) &= \theta \frac{d}{d\theta} [\mu(\theta)] \\ &= \theta \frac{d}{d\theta} \left\{\theta \frac{d}{d\theta} [\log f(\theta)]\right\} \\ &= \theta \frac{d}{d\theta} [\log f(\theta)] + \theta^2 \frac{d^2}{d\theta^2} [\log f(\theta)]\end{aligned}$$

i.e.,

$$\mu_2(\theta) = \mu(\theta) + \theta^2 \frac{d^2}{d\theta^2} [\log f(\theta)]. \qquad (1.2.1)$$

Hence, the statement of the theorem.

Consider now,

Lemma 1: If the parameter space Θ of a gpsd contains zero, then the range T of the gpsd contains zero and the corresponding random variable takes the value zero with positive probabilities for all θ in the parameter space; and conversely.

Proof :

$0 \in \Theta$

$\therefore\ f(0) > 0$.

But $f(\theta) = \sum_{x \in T} a_x \theta^x$

$\therefore\ 0 \in T$ and $a_o > 0$

Converse follows by retracing the steps above.

Hence, Lemma 1.

Lemma 2: The logarithm of the generating function of a gpsd is a monotone non-decreasing function of θ.

Proof : Two cases arise: (1) $0 \notin \Theta$, i.e., Parameter space of the gpsd does not contain 0, and

(2) $0 \in \Theta$, i.e., Parameter space of the gpsd contains 0.

Case 1: Here $\theta > 0$, $\theta \in \Theta$

$$\therefore\ \mu(\theta) = \sum_{x \in T} x \frac{a_x \theta^x}{f(\theta)} > 0$$

But from (1.1.11), $\mu(\theta) = \theta \frac{d}{d\theta} [\log f(\theta)]$

$$\therefore\ \frac{d}{d\theta} [\log f(\theta)] > 0$$

Case 2: By Lemma 1, we have in this case,

$$f(\theta) = a_o + \sum_{x \in T - \{0\}} a_x \theta^x, \qquad a_o > 0$$

where $T - \{0\}$ denotes the set T without 0.

Direct computation gives, therefore:

$$\frac{d}{d\theta}[\log f(\theta)] = \frac{\frac{d}{d\theta}[f(\theta)]}{f(\theta)} = \frac{\sum_{x \in T - \{0\}} x a_x \theta^{x-1}}{f(\theta)} .$$

clearly, for $\theta > 0$, $\frac{d}{d\theta}$ [log $f(\theta)$] > 0,

for $\theta = 0$, $\frac{d}{d\theta}$ [log $f(\theta)$] $= 0$ if $1 \epsilon T$ and $a_1 = 0$ or $1 \notin T$

> 0 otherwise.

Thus, we have always $\frac{d}{d\theta}$[log $f(\theta)$] ≥ 0.

Hence, Lemma 2.

<u>Theorem 2</u>: The necessary and sufficient condition for the variance of a gpsd to equal its mean for every θ of its parameter space Θ is that the generating function be of the form

$$f(\theta) = e^{k\theta+c}$$

where $k > 0$ and c are arbitrary constants.

<u>Proof</u> : Sufficiency: obvious

Necessity : now, have for a gpsd

$\mu_2(\theta) = \mu(\theta), \qquad \theta \epsilon \Theta$

$\therefore$ By (1.2.1) of Theorem 1,

$$\theta^2 \frac{d^2}{d\theta^2}[\log f(\theta)] = 0 . \qquad (1.2.2)$$

Now, two cases arise:

<u>Case 1</u>: $0 \notin \Theta$, i.e., Parameter space of the gpsd does not contain 0. In this case, (1.2.2) reduces to

$$\frac{d^2}{d\theta^2}[\log f(\theta)] = 0$$

$\therefore$ $\frac{d}{d\theta}[\log f(\theta)] = k$, where k is some positive constant by Case 1 of Lemma 2.

$\therefore$ $\log f(\theta) = k\theta + c$, where c is arbitrary constant.

Hence, for all $\theta \epsilon \Theta$,

$$f(\theta) = e^{k\theta + c}$$

where $k > 0$ and c are arbitrary constants.

Case 2: $0 \epsilon \Theta$, i.e., Parameter space of the gpsd contains 0. For positive values of θ, Case 1 applies and we get for all $\theta \epsilon \Theta - \{0\}$,

$$f(\theta) = e^{k\theta + c}$$

where $k > 0$ and c are arbitrary constants.

To verify that this form holds for $\theta = 0$, we have by Lemma 1,

$0 \epsilon T$ and $a_0 > 0$, so that,

$$f(0) = a_0 = e^{\log a_0} = e^{k.0+c},$$

where $c = \log a_0$.

Hence, the statement of the theorem.

Theorem 3: The equality of mean and variance is necessary and sufficient for a gpsd to become Poisson. (Characterization of Poisson distribution.)

Proof: We first prove the following lemma.

Lemma: A positive constant multiple of the generating function of a gpsd does not affect it; i.e., gives rise to the same original gpsd.

Let the generating function be as given in (1.1.3) and the corresponding gpsd as given in (1.1.4).

Consider the new generating function

$h(\theta) = kf(\theta)$ where k is some positive constant

i.e., $h(\theta) = \sum_{x \in T} ka_x \theta^x$

and the gpsd corresponding to $h(\theta)$ becomes

$$\text{Prob}\left\{X=x\right\} = \frac{ka_x \theta^x}{h(\theta)} = \frac{a_x\theta^x}{f(\theta)} \quad \cdots \quad x \in T$$

which is the original gpsd given in (1.1.4).

Hence the Lemma.

Now, Theorem 3 follows immediately by applying the above lemma to Theorem 2.

Hence, the characterization of Poisson distribution as stated by Theorem 3.

Theorem 4: The necessary and sufficient condition for the variance of a gpsd to exceed its mean for every non-zero θ of its parameter space Θ is that the generating function be of the form

$$f(\theta) = e^{P(\theta)+R\theta+Q}$$

where Q and R are arbitrary constants and $P(\theta)$, along with its derivative, is a positive monotone increasing function of θ.

Proof : Necessity: Let $\mu_2(\theta) - \mu(\theta) = \theta^2 p(\theta)$, where $p(\theta)$ is a positive function of θ.

Then by Theorem 1, we have

$$\theta^2 \frac{d^2}{d\theta^2}[\log f(\theta)] = \theta^2 p(\theta).$$

i.e., $\frac{d^2}{d\Theta^2}[\log f(\Theta)] = p(\Theta)$. (1.2.3)

Integrating (1.2.3), $\frac{d}{d\Theta}[\log f(\Theta)]$

$= P'(\Theta) + R,$ (1.2.4)

where R is arbitrary constant and

$R + P'(\Theta) = \int p(\Theta)d\Theta$, a positive monotone

increasing function of Θ.

Integrating (1.2.4), we have

$\log f(\Theta) = P(\Theta) + R\Theta + Q,$ (1.2.5)

where Q is arbitrary constant and

$Q + P(\Theta) = \int P'(\Theta)d\Theta$, a positive monotone

increasing function of Θ.

Hence, from (1.2.5), we have the required form for the generating function, namely,

$f(\Theta) = e^{P(\Theta) + R\Theta + Q},$

where symbols carry the meaning as stated.

Sufficiency: Sufficiency follows from above by retracing the steps.

Theorem 5: The necessary and sufficient condition for the variance of a gpsd to be less than its mean for every non-zero Θ of its parameter space Θ is that the generating function be of the form

$$f(\Theta) = e^{A(\Theta) + B\Theta + c},$$

where B and C are arbitrary constants and $A(\Theta)$ is such that its derivative is a monotone decreasing function of Θ.

<u>Proof</u> : On the same lines as that of Theorem 4.

<u>Theorem 6</u>: The mean $\mu(\theta)$ of a gpsd is a non-negative monotone non-decreasing function of θ.

<u>Proof</u> : Consider the relation (1.1.12), which states

$$\mu_2(\theta) = \theta \frac{d}{d\theta}[\mu(\theta)] . \tag{1.2.6}$$

We know that $\mu_2(\theta) \geq 0$. Also $\theta \geq 0$.

$\therefore$ from (1.2.6) follows that

$\frac{d}{d\theta}[\mu(\theta)] \geq 0$; also $\mu(\theta) \geq 0$.

$\therefore$ $\mu(\theta)$ is a non-negative monotone non-decreasing function of θ.

<u>Theorem 7</u>: The graph of the mean of a gpsd with parameter space containing zero is convex or concave or linear in accordance with the variance of the gpsd being greater than or less than or equal to the mean and conversely.

<u>Proof</u> : Suppose $\mu_2(\theta) > \mu(\theta)$

$\therefore$ $\theta \frac{d}{d\theta}[\mu(\theta)] > \mu(\theta)$

$\therefore$ $\frac{d}{d\theta}[\mu(\theta)] > \frac{\mu(\theta)}{\theta}$ when $\theta \neq 0$.

Also, as the gpsd is taken with parameter space containing zero, we can speak of $\mu(0)$ which is clearly = 0.

Hence, follows the convexity of the graph of the mean when the variance exceeds the mean.

On similar lines the rest of the statement can be very easily established.

<u>Theorem 8</u>: The mean of a gpsd with parameter space containing zero is respectively a linear or convex or concave function of θ if and only if the generating function is respectively of the form of Theorem 2, Theorem 4, or Theorem 5.

<u>Proof</u> : The proof follows immediately from Theorem 2, Theorem 4, Theorem 5, and Theorem 7.

CHAPTER II

2.0 LIKELIHOOD ESTIMATION AND ALLIED PROBLEMS IN A CLASS OF DISCRETE DISTRIBUTIONS

We show first in this chapter that for gpsd (1.1.4), the maximum likelihood method and the method of moments give the same estimate of the gpsd parameter. The likelihood equation and a method for solving it are derived for the problem of estimation in truncated and censored forms of the gpsd (1.1.4). We mention here that we call gpsd (1.1.4) a complete gpsd as opposed to its truncated and censored forms.

Large sample methods based on maximum likelihood are then derived for testing the homogeneity of several gpsd's and providing the estimate for the common parameter in case the gpsd's are homogeneous. A treatment for doubtful observations is given. Lastly, the problem of estimation when the general distribution involves two parameters is discussed.

2.1 Estimation by Likelihood for a Complete gpsd

2.1.1 Let x_i $(i=1,2,\ldots N)$ be a random sample of size N from the gpsd (1.1.4). Then the logarithm of the likelihood function L is

$$\log L = \text{constant} + \sum_{i=1}^{N} x_i \log \theta - N \log f(\theta)$$

so that the "efficient score" for θ is

$$\psi(\theta) = \frac{d}{d\theta}[\log L]$$

$$= \sum_{i=1}^{N} x_i/\theta - Nf'(\theta)/f(\theta)$$

$$= \frac{N}{\theta}(\bar{x} - \mu), \qquad (2.1.1)$$

where

$$\bar{x} = \sum_{i=1}^{N} x_i / N \text{ is the sample mean and by (1.1.5)}$$

$\mu = \theta f'(\theta)/f(\theta)$, the mean of the gpsd (1.1.4).

The likelihood equation $\psi(\hat{\theta}) = 0$ for estimating θ thus reduces to

$$\bar{x} = \mu(\hat{\theta}) \quad (\equiv \hat{\mu}, \text{ say}) \tag{2.1.2}$$

which means equating the sample mean to the population mean. The method of maximum likelihood and the method of moments thus lead to the same estimate in the case of a gpsd.

Denoting this estimate by $\hat{\theta}$, the asymptotic variance is given by $1/I(\theta)$ where

$$\begin{aligned} I(\theta) &= -E\left(\frac{d\psi}{d\theta}\right) \\ &= -E\left[-\frac{N}{\theta}\left(\frac{d\mu}{d\theta}\right) - \frac{N}{\theta^2}(\bar{x}-\mu)\right] \\ &= \frac{N}{\theta}\left(\frac{d\mu}{d\theta}\right) \end{aligned} \tag{2.1.3}$$

also, $$= \frac{N}{\theta^2} \cdot \mu_2(\theta), \text{ because of (1.1.12).} \tag{2.1.4}$$

Thus,

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta}{N} / \left(\frac{d\mu}{d\theta}\right) \tag{2.1.5}$$

also, $$= \frac{\theta^2}{N} / \mu_2(\theta) . \tag{2.1.6}$$

2.1.2 If Equation (2.1.2) does not readily give an algebraic solution, one may use an iterative process of solution (which converges; see Rao, 1952) by starting with an approximation θ_o. An improved approximation θ_1 is then obtained from

$$\begin{aligned} \theta_1 &= \theta_o + \psi(\theta_o)/I(\theta_o) \\ &= \theta_o + [\bar{x} - \mu(\theta_o)]/\left(\frac{d\mu}{d\theta}\right)_{\theta_o} , \end{aligned} \tag{2.1.7}$$

or from the equivalent formula

$$\Theta_1 = \Theta o + \Theta o[\bar{x} - \mu(\Theta o)]/\mu_2(\Theta o)$$

$$= \Theta o[1 + \frac{\bar{x} - \mu(\Theta o)}{\mu_2(\Theta o)}] \qquad (2.1.8)$$

and the process is repeated till one gets a sufficiently accurate solution. To carry out this process by Formula (2.1.8), a table of numerical values of $\mu(\Theta)$ and $\mu_2(\Theta)$ of the gpsd under consideration for sufficiently close values of Θ could be very useful. Formula (2.1.7) would require a table of numerical values of $\mu(\Theta)$ and $\frac{d\mu}{d\Theta}$. However, it may be observed that a table for only $\mu(\Theta)$ of the gpsd under consideration for sufficiently close values of Θ would do. Because $\frac{d\mu}{d\Theta}$ can be approximated by the finite difference ratio $\frac{\Delta\mu}{\Delta\Theta}$ and this approximation is expected to be good if the tabular interval is small. Illustrative examples (4.2.5; 5.2.11; 7.2.6) given later substantiate this observation.

2.1.3 To find the amount of bias in the maximum likelihood estimate $\hat{\Theta}$, following Haldane and Smith (1956), we have the amount of bias in $\hat{\Theta}$ to order $\frac{1}{N}$ as

$$b(\hat{\Theta}) = -\frac{1}{2} \cdot \frac{1}{N} \left(\frac{B_1}{A_1^2}\right) \qquad (2.1.9)$$

where

$$A_1 = \sum_{x \in T} \left(\frac{dP_x}{d\Theta}\right)^2 / P_x \qquad (2.1.10)$$

and

$$B_1 = \sum_{x \in T} \left(\frac{dP_x}{d\Theta}\right)\left(\frac{d^2P_x}{d\Theta^2}\right)/P_x , \qquad (2.1.11)$$

in which, as usual,

$$P_x = a_x\theta^x/f(\theta) \qquad x\epsilon T$$

$$\mu = \theta\,\frac{f'(\theta)}{f(\theta)} \text{ and } \mu_2 = \theta\,\frac{d\mu}{d\theta}\ ,$$

$$\frac{dP_x}{d\theta} = \frac{P_x}{\theta}\,(x - \mu)$$

and

$$\frac{d^2P_x}{d\theta^2} = \frac{P_x}{\theta^2}[(x - \mu)^2 - \mu_2 - (x - \mu)]$$

so that

$$A_1 = \frac{\mu_2}{\theta^2}$$

and

$$B_1 = \frac{\mu_3 - \mu_2}{\theta^3}\ .$$

Therefore, from (2.1.9), we have the amount of bias in $\hat{\theta}$, to order $\frac{1}{N}$,

$$b(\hat{\theta}) = -\frac{1}{N}\cdot\frac{\theta}{2}\cdot\frac{\mu_3 - \mu_2}{\mu_2^2}\ . \qquad (2.1.12)$$

2.1.4 Next, to estimate a differentiable function of θ, such as $\omega(\theta)$, the method of maximum likelihood leads to the estimate

$$\hat{\omega} = \omega(\hat{\theta})$$

with variance

$$\text{Var}\ (\hat{\omega}) = \frac{\theta(\frac{d\omega}{d\theta})^2}{N\,\frac{d\mu}{d\theta}} = \frac{\theta^2}{N\mu_2}(\frac{d\omega}{d\theta})^2 \qquad (2.1.13)$$

where $\hat{\theta}$ is the maximum likelihood estimate of θ.

To find the amount of bias in $\hat{\omega}$, following Haldane and Smith (1956), we have that the amount of bias to order $\frac{1}{N}$ in $\hat{\omega}$ is given by:

$$b(\hat{\omega}) = -\frac{1}{2}\cdot\frac{1}{N}\left(\frac{B_2}{A_2^2}\right) \tag{2.1.14}$$

where

$$A_2 = \sum_{x\in T}\left(\frac{dP_x}{d\omega}\right)/P_x = \frac{A_1}{\left(\frac{d\omega}{d\theta}\right)^2} \tag{2.1.15}$$

and

$$B_2 = \sum_{x\in T}\left(\frac{dP_x}{d\omega}\right)\left(\frac{d^2P_x}{d\omega^2}\right)/P_x$$

$$= \frac{1}{\left(\frac{d\omega}{d\theta}\right)^3}\left[B_1 - \frac{\frac{d^2\omega}{d\theta^2}}{\frac{d\omega}{d\theta}}A_1\right] \tag{2.1.16}$$

where A_1 and B_1 are defined by (2.1.10) and (2.1.11), respectively. Then from (2.1.14),

$$b(\hat{\omega}) = -\frac{1}{N}\cdot\frac{\theta}{2}\cdot\frac{d\omega}{d\theta}\cdot\frac{\mu_3 - \mu_2\left(1 + \frac{\frac{d^2\omega}{d\theta^2}}{\frac{d\omega}{d\theta}}\cdot\theta\right)}{\mu_2^2}\,. \tag{2.1.17}$$

2.2 Estimation by Likelihood for a Truncated gpsd

Let T* be a non-null subset of the range T of gpsd (1.1.4), and consider the distribution (1.1.4) truncated to the subset T*. In this case, it can be easily verified that the (truncated) random variable X* has the probability distribution:

$$P_x^* = \text{Prob}\left\{X^* = x\right\} = \frac{a_x\theta^x}{f^*(\theta)}\quad,\qquad x\in T^*\,, \tag{2.2.1}$$

where

$$f^*(\theta) = \sum_{x \in T^*} a_x \theta^x \quad . \tag{2.2.2}$$

2.2.1 It is easy to see that the truncated gpsd (2.2.1) is in turn a gpsd in its own right with the generating function given by (2.2.2), and consequently, all the properties of (1.1.4) are valid for (2.2.1). To be explicit, distinguishing the characteristics of this truncated distribution (2.2.1) from the complete gpsd (1.1.4) by means of an asterisk(*), it immediately follows that relations analogous to those in (1.1.5) - (1.1.15) will hold for the mean μ^*, crude moments m_r^*, central moments μ_r^*, mgf $M^*(t)$, etc., such as

$$\mu^* = \theta \frac{d}{d\theta}[f^*(\theta)]$$

$$\mu_2^* = \theta \frac{d\mu^*}{d\theta} \text{ , etc.}$$

2.2.2 Similarly, the maximum likelihood estimate $\hat{\theta}$ (which in this case, is also equivalent to the moments estimate) for θ is to be computed from the likelihood equation:

$$\bar{x}^* = \mu^*(\hat{\theta}) \tag{2.2.3}$$

where $\bar{x}^*$ is the mean of a random sample of size N from the truncated gpsd (2.2.1).

The asymptotic variance of $\hat{\theta}$ is similarly given by

$$\text{Var } (\hat{\theta}) = \frac{\theta}{N} / (\frac{d\mu^*}{d\theta}) = \frac{\theta^2}{N} / \mu_2^*(\theta) \text{ .}$$

The iterative process of solving (2.2.3) can again be put down in the form

$$\theta_1 = \theta o + [\bar{x}^* - \mu^*(\theta o)] / (\frac{d\mu^*}{d\theta})_{\theta o} \quad ,$$

or

$$\theta_1 = \theta o[1 + \frac{\bar{x}^* - \mu^*(\theta o)}{\mu_2^*(\theta o)}] .$$

The formulae for the amount of bias to order $\frac{1}{N}$ in $\hat{\theta}$ [and $\hat{\omega} = \omega(\hat{\theta})$] can be written down similarly.

2.3 Estimation by Likelihood for Censored gpsd

Let T^*, T_j $(j = 1,2,\ldots k)$ be k+1 mutually exclusive and exhaustive subsets of the range T of the gpsd (1.1.4). Suppose that in a random sample of size N from the gpsd (1.1.4), we have a record of the number n_j of observations in the subset T_j $(j = 1,2,\ldots k)$ and of the n^* observations x_i $(i = 1,2,\ldots n^*)$ in the subset T^*, so that

$$N = n^* + \sum_{j=1}^{k} n_j$$

and we write

$$\bar{x}^* = \sum_{i=1}^{n^*} x_i/n^* \quad . \tag{2.3.1}$$

2.3.1 The logarithm of the likelihood function may be written as

$$\log L = \text{constant} + \sum_{j=1}^{k} n_j \log f_j + \sum_{i=1}^{n^*} x_i \log \theta - N \log f$$

where

$$f_j = f_j(\theta) = \sum_{x\epsilon T_j} a_x\theta^x \text{ and } f = f(\theta) = \sum_{x\epsilon T} a_x\theta^x \quad . \tag{2.3.2}$$

The "efficient score" for Θ is

$$\psi(\Theta) = \frac{d}{d\Theta}[\log L]$$

$$= \sum_{j=1}^{k} n_j f_j'/f_j + \sum_{i=1}^{n^*} x_i/\Theta - Nf'/f$$

$$= \frac{1}{\Theta}[n^*\bar{x}^* - (N\mu - \sum_{j=1}^{k} n_j \nu_j)] \tag{2.3.3}$$

where

$$\nu_j = \nu_j(\Theta) = \sum_{x \in T_j} x a_x \Theta^x / f_j \tag{2.3.4}$$

is the mean of the j-th class T_j and μ is the mean of the gpsd (1.1.4).

Thus, the likelihood equation for estimating Θ is

$$n^*\bar{x}^* = N\mu - \sum_{j=1}^{k} n_j \nu_j \tag{2.3.5}$$

where

$$\mu = \mu(\Theta) \text{ and } \nu_j = \nu_j(\Theta) \quad .$$

The asymptotic variance of the estimate $\hat{\Theta}$ derived from (2.3.5) is $1/I(\Theta)$ where

$$I(\Theta) = - E(\frac{d\psi}{d\Theta}) = \frac{N}{\Theta}[\mu' - \sum_{j=1}^{k} p_j \nu_j'] \tag{2.3.6}$$

where $p_j = f_j/f$ is the probability for the j-th class T_j and primes denote the differentiation with respect to Θ.

It may be noted that when the subsets T_j $(j = 1,2,\ldots k)$ are all empty, we get our previous formula (2.1.2) for the estimate from sample for a complete gpsd.

2.4 Estimation with Doubtful Observations

Let T^* be a subset of the range T of the gpsd (1.1.4). Consider the situation where the experimenter has doubts about sample observations not in T*, and therefore, merely records the number of observations not in T*. The model considered is:

$$\text{Prob}\{X = x\} = \beta \qquad \text{for } x \notin T^*$$

$$= (1-\beta)\,\frac{a_x\theta^x}{f^*(\theta)} \qquad \text{for } x \in T^* \tag{2.4.1}$$

where

$$0 < \beta < 1 \text{ and } f^*(\theta) = \sum_{x\in T^*} a_x\theta^x \,. \tag{2.4.2}$$

2.4.1 If, in a sample of size N, the number of observations not in T* is n_1 and the records of the other $N-n_1 = n^*$ (say), are x_i $(i = 1,2,\ldots n^*)$; the logarithm of the likelihood function is then given by

$$\log L = \text{constant} + n_1 \log\beta + n^*\log(1-\beta)$$

$$+ \sum_{i=1}^{n^*} x_i \log\theta - n^*\log f^*(\theta)\,.$$

The "efficient scores" for β and θ are then

$$\psi_1 = \frac{\partial \log L}{\partial\beta} = \frac{n_1}{\beta} - \frac{n^*}{1-\beta} \tag{2.4.3}$$

$$\psi_2 = \frac{\partial \log L}{\partial\theta} = \frac{n^*}{\theta}[\bar{x}^* - \mu^*(\theta)] \tag{2.4.4}$$

where

$$\bar{x}^* = \sum_{i=1}^{n^*} x_i/n^* \tag{2.4.5}$$

and

$$\mu^*(\theta) = \sum_{x\in T^*} x a_x\theta^x/f^*(\theta) \tag{2.4.6}$$

is the mean of the subset T*.

The likelihood estimates of β and θ are thus given by

$$\hat{\beta} = n_1/N \tag{2.4.7}$$

$$\mu^*(\hat{\theta}) = \bar{x}^* \tag{2.4.8}$$

so that, the estimate of θ is derived simply by neglecting the n_1 observations not in T^* and treating the sample of n^* as one from gpsd (1.1.4) truncated to T^*.

The elements of the "information matrix" are given by:

$$I_{11} = - E(\frac{d\psi_1}{d\beta}) = \frac{N}{\beta(1-\beta)} \tag{2.4.9}$$

$$I_{12} = - E(\frac{d\psi_1}{d\theta}) = - E(\frac{d\psi_2}{d\beta}) = 0 \tag{2.4.10}$$

$$I_{22} = - E(\frac{d\psi_2}{d\theta}) = \frac{N(1-\beta)}{\theta} (\frac{d\mu^*}{d\theta}) . \tag{2.4.11}$$

2.4.2 The hypothesis H_o of interest is that the proportion of doubtful observations conforms to the gpsd (1.1.4), that is,

$$H_o: \ \beta = 1 - f^*(\theta)/f(\theta) .$$

If H_o is true, the estimate of θ is obtained by consideration of the sample as a censored one as in Section 2.3. The estimate of θ under H_o is thus obtained from:

$$n^*\bar{x}^* = N\mu - n_1\nu_1 \tag{2.4.12}$$

where

$$\nu_1 = \sum_{x \in T^*} x a_x \theta^x / [f(\theta) - f^*(\theta)] \tag{2.4.13}$$

is the mean of the subset complementary to T^*.

The estimate of θ under H_o derived from this equation will be denoted by $\hat{\theta}_o$. The estimate of β under H_o is then given by

$$\hat{\beta}_o = 1 - f^*(\hat{\theta}_o)/f(\hat{\theta}_o) . \tag{2.4.14}$$

Following the method suggested by Rao (1948) for testing H_o, we have the criterion

$$\chi_1^2 = \frac{[\psi_1(\hat{\beta}_o)]^2}{I_{11}(\hat{\beta}_o)} + \frac{[\psi_2(\hat{\theta}_o)]^2}{I_{22}(\hat{\beta}_o,\hat{\theta}_o)} \qquad (2.4.15)$$

which is asymptotically distributed as a Chisquare with one degree of freedom. In the present case, (2.4.15) takes the following form:

$$\chi_1^2 = \frac{N}{\hat{\beta}_o(1-\hat{\beta}_o)}\left(\frac{n_1}{N} - \hat{\beta}_o\right)^2 + \frac{n^{*2}}{N(1-\hat{\beta}_o)} \frac{[\bar{x}^* - \mu^*(\hat{\theta}_o)]^2}{\mu_2^*(\hat{\theta}_o)} \qquad (2.4.16)$$

where

$$\mu_2^*(\theta) = \theta \cdot \frac{d\mu^*(\theta)}{d\theta} .$$

2.5 Homogeneity and Combined Estimation

In the light of random samples from a number of gpsd's, it may be required to examine if the distributions are homogeneous in respect to the parameter θ and if so, to make a combined estimate of θ.

Let x_{ji} $(i = 1,2,\ldots N_j)$ be a random sample from the j-th gpsd characterized by the probability law:

$$a_x^{(j)}\theta_j^x/f_j(\theta_j) \qquad x \in T_j \qquad (2.5.1)$$

where

$$f_j(\theta_j) = \sum_{x \in T_j} a_x^{(j)}\theta_j^x , \qquad (2.5.2)$$

T_j is the range of the j-th gpsd, and $j = 1,2,\ldots k$.

2.5.1 The logarithm of the joint likelihood function is

$$\log L = \text{constant} + \sum_{j=1}^{k} N_j\bar{x}_j \log \theta_j - \sum_{j=1}^{k} N_j \log f_j(\theta_j) ,$$

where

$$\bar{x}_j = \sum_{i=1}^{N_j} x_{ji}/N_j$$

is the mean of the sample from the j-th gpsd. The j-th "efficient score" is then

$$\psi_j = \frac{\partial \log L}{\partial \theta_j} = \frac{N_j}{\theta_j} [\bar{x}_j - \mu_j(\theta_j)] \tag{2.5.3}$$

where $\mu_j(\theta_j)$ is the mean of the j-th gpsd.

The elements of the "information matrix" are

$$I_{jj} = \frac{N_j}{\theta_j} \cdot \frac{d\mu_j(\theta_j)}{d\theta_j} \tag{2.5.4}$$

$$= N_j \mu_{2j}(\theta_j)/\theta_j^2 \tag{2.5.5}$$

$$I_{jj'} = 0 \qquad\qquad j \neq j' \tag{2.5.6}$$

where $\mu_{2j}(\theta_j)$ is the variance of the j-th gpsd.

2.5.2 The hypothesis of homogeneity is

$$H_o: \ \theta_1 = \theta_2 = \ldots\ldots = \theta_k \ .$$

If the hypothesis H_o is true, the common value may be denoted by θ and the efficient score and the information with respect to θ are given by:

$$\psi(\theta) = \frac{N}{\theta}(\bar{x} - \sum_{j=1}^{k} N_j \mu_j(\theta)/N) \tag{2.5.7}$$

where

$$\bar{x} = \sum_{j=1}^{k} N_j \bar{x}_j/N \quad ,$$

and

$$I(\theta) = \frac{1}{\theta} \sum_{j=1}^{k} N_j \frac{d\mu_j(\theta)}{d\theta} \tag{2.5.8}$$

$$= \frac{1}{\theta^2} \sum_{j=1}^{k} N_j \mu_{2j}(\theta). \tag{2.5.9}$$

To solve the equation $\psi(\hat{\theta}) = 0$ for the maximum likelihood estimate $\hat{\theta}$, one starts with an approximation θo and derives a better approximation θ_1 from the formula:

$$\theta_1 = \theta o + N[\bar{x} - \sum_{j=1}^{i} N_j \mu_j(\theta o)/N] / \sum_{j=1}^{k} N_j \left(\frac{d\mu_j(\theta)}{d\theta}\right)_{\theta o} \tag{2.5.10}$$

or

$$\theta_1 = \theta_o[1 + N\{\bar{x} - \sum_{j=1}^{k} N_j \mu_j(\theta o)/N\} / \sum_{j=1}^{k} N_j \mu_{2j}(\theta o)]. \tag{2.5.11}$$

2.5.3 A test of the homogeneity hypothesis H_o is then given by the statistic

$$\chi^2_{k-1} = \sum_{j=1}^{k} [\psi_j(\hat{\theta})]^2 / I_{jj}(\hat{\theta}) \tag{2.5.12}$$

$$= \sum_{j=1}^{k} N \; [\bar{x}_j - \mu_j(\hat{\theta})]^2 / \hat{\theta} \left(\frac{d\mu_j(\theta)}{d\theta}\right)_{\theta=\hat{\theta}} \tag{2.5.13}$$

$$= \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\hat{\theta})]^2 / \mu_{2j}(\hat{\theta}) \tag{2.5.14}$$

which is asymptotically distributed as a Chisquare with (k-1) degrees of freedom, if H_o is true.

2.6 Estimation for a gpsd with Two Parameters

Consider a gpsd with two parameters taking the form:

$$\text{Prob}\{X = x\} = \frac{a_x(\lambda)\theta^x}{f(\theta,\lambda)} \qquad x \in T \quad , \tag{2.6.1}$$

where T is the range of the gpsd and the generating function

$$f(\theta,\lambda) = \sum_{x \in T} a_x(\lambda)\theta^x \quad , \tag{2.6.2}$$

such that $f(\theta,\lambda)$ is positive and bounded for all admissible values of the two parameters θ and λ, and the non-negative coefficients $a_x(\lambda)$ now depend on x and λ. The binomial and negative binomial distributions are special cases of (2.6.1) when they are considered to be the distributions with two parameters.

2.6.1 To estimate θ and λ on the basis of a sample x_i $(i = 1,2,\ldots N)$ of size N from (2.6.1), the logarithm of likelihood function is

$$\log L = \text{constant} + \sum_{i=1}^{N} x_i \log \theta + \sum_{i=1}^{N} \log a_{x_i}(\lambda)$$

$$- N \log f(\theta,\lambda) .$$

The "efficient score" for θ is then

$$\psi_1 = \psi_1(\theta,\lambda) = \frac{\partial}{\partial\theta}[\log L] = \frac{N}{\theta}[\bar{x} - \mu(\theta,\lambda)] \tag{2.6.3}$$

and the likelihood equation $\psi_1(\hat{\theta},\hat{\lambda}) = 0$ reduces to

$$\bar{x} = \mu(\hat{\theta},\hat{\lambda}) \tag{2.6.4}$$

which is the same as the first-moment equation.

The "efficient score" for λ is

$$\psi_2 = \psi_2(\theta,\lambda) = \frac{\partial}{\partial\lambda}[\log L]$$

$$= N[\sum_{i=1}^{N} \frac{d}{d\lambda} \log a_{x_i}(\lambda)/N$$

$$- \frac{\partial}{\partial\lambda} \log f(\theta,\lambda)] \tag{2.6.5}$$

and the estimating equation $\psi_2(\hat{\theta},\hat{\lambda}) = 0$ becomes

$$\frac{\partial}{\partial\lambda} \log f(\hat{\theta},\lambda) = \sum_{i=1}^{N} \frac{d}{d\lambda} \log a_{x_i}(\lambda)/N \quad . \tag{2.6.6}$$

This, however, is not a moment equation. The second moment equation will be

$$S^2 = \mu_2(\theta,\lambda) \tag{2.6.7}$$

where

$$S^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2/N \quad .$$

Thus, unlike gpsd's of the form (1.1.4) with single parameter, gpsd's given by (2.6.1) with two parameters do not yield identical "moment" and "maximum likelihood" estimates.

2.6.2 The elements of the "information matrix"

$$\mathcal{I} = \begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} \tag{2.6.8}$$

are given by

$$I_{11} = - E(\frac{\partial\psi_1}{\partial\theta}) = \frac{N}{\theta} (\frac{\partial\mu}{\partial\theta}) \tag{2.6.9}$$

$$I_{12} = - E(\frac{\partial\psi_1}{\partial\lambda}) = - E(\frac{\partial\psi_2}{\partial\theta})$$

$$= \frac{N}{\theta}(\frac{\partial\mu}{\partial\lambda}) \tag{2.6.10}$$

$$I_{22} = - E(\frac{\partial \psi_2}{\partial \lambda})$$

$$= N[\frac{\partial^2}{\partial \lambda^2} \log f(\theta,\lambda) - h(\lambda)] \qquad (2.6.11)$$

where

$$h(\lambda) = E[\frac{d^2}{d\lambda^2} \log a_x(\lambda)]$$

$$= \frac{1}{f(\theta,\lambda)}[\frac{\partial^2}{\partial \lambda^2} f(\theta,\lambda)] - E[\frac{d}{d\lambda} \log a_x(\lambda)]^2.$$

The asymptotic "dispersion matrix" of the estimates θ,λ obtained by solving (2.6.4) and (2.6.6) is then given by

$$\begin{pmatrix} \text{var } (\theta) & \text{cor } (\theta,\lambda) \\ \text{cor } (\theta,\lambda) & \text{var } (\lambda) \end{pmatrix} = \mathcal{I}^{-1} \qquad (2.6.12)$$

If instead of θ and λ, $\mu = \mu(\theta,\lambda)$ and λ are regarded as the parameters, the maximum likelihood estimates of μ and λ are asymptotically uncorrelated: this follows from (2.6.10).

CHAPTER III

3.0 SIMPLE METHODS OF ESTIMATION FOR A CLASS OF DISCRETE DISTRIBUTIONS

In Sections 2.1, 2.2 and 2.3, we discussed the method of maximum likelihood for estimation on the basis of samples from gpsd's which are either complete, truncated or censored. The method, though "efficient", generally involves heavy computation. Moreover, it does not yield an unbiased estimate in several cases. In this chapter, we consider some other methods of estimation and investigate their important properties. All these are easy to compute, and it also turns out that some of them provide unbiased estimates.

3.1 Estimation by the Ratio Method for a gpsd

[Range T finite and T = (c,c+1,...c+k = d) with positive probabilities]

3.1.1 Consider the gpsd (1.1.4) with range T finite and T = (c,c+1,...c+k = d) with positive probabilities, that is, the coefficients $a_x > 0$ for all $x \epsilon T$. To be explicit, the gpsd that we consider here is of the form:

$$P_x = \text{Prob}\,\{X = x\} = \frac{a_x \theta^x}{f(\theta)} \qquad (3.1.1)$$

where

$$x \epsilon T = (c, c+1, \ldots c+k = d), \text{ d finite}$$

$$f(\theta) = \sum_{x=c}^{d} a_x \theta^x \qquad (3.1.2)$$

and

$$a_x > 0 \text{ for } x \epsilon T.$$

Let

$$g_r(x) = \frac{a_{x-r}}{a_x} \qquad x \epsilon T \tag{3.1.3}$$

with r being an integer such that x-rϵT. Then

$$\sum_{x=u}^{v} g_r(x)P_x = \sum_{x=u}^{v} a_{x-r}\theta^x/f(\theta)$$

$$= \theta^r \sum_{x=u-r}^{v-r} a_x\theta^x/f(\theta)$$

$$= \theta^r \sum_{x=u-r}^{v-r} P_x \tag{3.1.4}$$

where u and v are arbitrary with $c+r \leq u \leq v \leq d$. From (3.1.4), we get the identity

$$\theta^r = \frac{\sum_{x=u}^{v} g_r(x)P_x}{\sum_{x=u-r}^{v-r} P_x} \tag{3.1.5}$$

which can be made use of in problems of estimation. In a sample of size N, if n_x is the observed frequency for x, then since $E(n_x) = NP_x$, the statistic

$$\sum_{x=u}^{v} g_r(x)n_x/\sum_{x=u-r}^{v-r} n_x \tag{3.1.6}$$

may be taken as an estimate of θ^r for admissible values of r=1,2,etc. Since u and v are arbitrary, the same method is applicable for estimation in truncated and censored gpsd's also, provided that their range contains a subset of consecutive integers. We call these estimates "ratio estimates."

It is interesting to note that methods given by Plackett (1953) and Moore (1952,1954) for estimating θ in truncated Binomial and Poisson distributions come out as special cases of the method we suggest here. The method which we call the ratio method is applicable not merely for estimating Θ, but also for its integral powers and for any gpsd of this section, truncated or censored.

The ratio estimate is not generally unbiased or efficient, but is always easy to compute. In certain cases (see Section 3.2), however, unbiased estimates can be obtained by the ratio method. In other cases, such as those in this section, the bias is generally of the order $\frac{1}{N}$ discussed below.

3.1.2 Consider the following ratio estimate of Θ for gpsd (3.1.1):

$$\Theta' = \frac{t_1}{t_2} \tag{3.1.7}$$

where

$$t_1 = \sum_{x=c+1}^{d} \left(\frac{a_{x-1}}{a_x}\right) n_x \tag{3.1.8}$$

and

$$t_2 = \sum_{x=c}^{d-1} n_x \ . \tag{3.1.9}$$

Then, writing

$$E(t_2) = N \sum_{x=c}^{d-1} P_x = N(1-P_d) = NP, \text{ say,} \tag{3.1.10}$$

where

$$P = 1 - P_d, \tag{3.1.11}$$

we have

$$E(t_1) = NP\Theta \ . \tag{3.1.12}$$

Let

$$t_1 - E(t_1) = \delta t_1 \text{ and } t_2 - E(t_2) = \delta t_2 \, . \tag{3.1.13}$$

Then

$$\Theta' = \frac{t_1}{t_2} = \Theta(1 + \frac{\delta t_1}{NP\Theta})(1 + \frac{\delta t_2}{NP})^{-1}.$$

Since the deviations δt_1, δt_2 are stochastically of order $N^{1/2}$, we get on expansion

$$\Theta' = \Theta[1 + \frac{\delta t_1}{NP\Theta} - \frac{\delta t_2}{NP} - \frac{(\delta t_1)(\delta t_2)}{N^2P^2\Theta} + \frac{(\delta t_2)^2}{N^2P^2}] \qquad (3.1.14)$$

neglecting terms of order higher than $\frac{1}{N}$. Thus, to this order of approximation,

$$E(\Theta') = \Theta[1 + \frac{E(\delta t_2)^2}{N^2P^2} - \frac{E(\delta t_1)(\delta t_2)}{N^2P^2\Theta}] . \qquad (3.1.15)$$

Now a little computation gives

$$E(\delta t_2)^2 = NP(1-P) \qquad (3.1.16)$$

and

$$E(\delta t_1)(\delta t_2 = N\Theta[P(1-P) - P_{d-1}] . \qquad (3.1.17)$$

Thus

$$E(\Theta') = \Theta + \frac{1}{N} (\frac{\Theta P_{d-1}}{P^2}) \qquad (3.1.18)$$

from which we get the magnitude of the gias in Θ', to order $\frac{1}{N}$,

$$b(\Theta') = \frac{1}{N} (\frac{\Theta P_{d-1}}{P^2})$$

$$= (\Theta P_{d-1})/N(1-P_d)^2 . \qquad (3.1.19)$$

3.1.3 The variance of Θ' correct to terms of order $\frac{1}{N}$ is

$$\text{Var } (\Theta') = \frac{1}{N^2P^2}[E(\delta t_1)^2 + \Theta^2 E(\delta t_2)^2$$

$$- 2\Theta E(\delta t_1)(\delta t_2)] . \qquad (3.1.20)$$

Now

$$E(\delta t_1)^2 = N(D - P^2\theta^2) \tag{3.1.21}$$

where

$$D = \sum_{x=c+1}^{d} \left(\frac{a_{x-1}}{a_x}\right)^2 P_x \quad . \tag{3.1.22}$$

Thus, to order $\frac{1}{N}$

$$\text{Var } (\theta') = \frac{1}{NP^2} [D - P\theta^2 + 2\theta^2 P_{d-1}] \; . \tag{3.1.23}$$

3.1.4 One simple estimate suggested by the identity

$$\theta = \frac{a_x}{a_{x+1}} \cdot \frac{P_{x+1}}{P_x} \tag{3.1.24}$$

is given by

$$m = \frac{a_x}{a_{x+1}} \cdot \frac{n_{x+1}}{n_x} \quad . \tag{3.1.25}$$

For this estimate, to terms of order $\frac{1}{N}$:

$$b(m) = \frac{1}{N} \; \frac{\theta}{P_x} \tag{3.1.26}$$

and

$$\text{Var } (m) = \frac{\theta(1 + b_x\theta)}{Nb_xP_x} \tag{3.1.27}$$

where

$$b_x = \frac{a_{x+1}}{a_x} \quad .$$

It is suggested in (3.1.26) that the order of the amount of bias for θ is only $\frac{1}{N}$. Also (3.1.26) and (3.1.27) suggest jointly that one may use the modal class for estimation with advantage.

3.2 Unbiased Estimation by the Ratio Method for a gpsd

[Range T infinite and T = $(c, c+1, \ldots \infty)$ with positive probabilities]

It is east to demonstrate that the ratio method discussed in Section 3.1 gives the unique unbiased estimate of θ, linear in frequencies, for a gpsd with range T infinite and T = $(c, c+1, \ldots \infty)$ with positive probabilities. For, consider the gpsd

$$P_x = \text{Prob}\{X = x\} = \frac{a_x\theta^x}{f(\theta)} \qquad x = c, c+1, \ldots \infty \tag{3.2.1}$$

where

$$f(\theta) = \sum_{x=c}^{\infty} a_x\theta^x \tag{3.2.2}$$

and

$$a_x > 0 \text{ for all } x = c, c+1, \ldots \infty.$$

3.2.1 Now, if in a sample of size N from gpsd (3.2.1), the frequency of x is n_x and we want an unbiased estimate for θ of the type linear in n_x, we should be able to demonstrate the existence of a function of x, t(x), such that, denoting the corresponding estimate

$$\tilde{\theta} = \sum_{x=c}^{\infty} t(x)n_x \tag{3.2.3}$$

we must have $E(\tilde{\theta}) = \theta$ for all θ in the parameter space of (3.2.1). That is

$$N \sum_{x=c}^{\infty} t(x)\, a_x\theta^x/f(\theta) = \theta$$

or

$$N \sum_{x=c}^{\infty} t(x)\, a_x\theta^x = \sum_{x=c}^{\infty} a_x\theta^{x+1} .$$

Since this is an identity in θ, equating coefficients of corresponding powers of θ, we get

$$t(x) = 0 \qquad \text{for } x = c$$
$$= \frac{1}{N}\left(\frac{a_{x-1}}{a_x}\right) \qquad \text{for } x = c+1, c+2, \ldots \infty .$$

3.2.2 Thus, the unique unbiased estimate of θ linear in the frequencies comes out to be the ratio estimate θ', because

$$\tilde{\theta} = \frac{1}{N} \sum_{x=c+1}^{\infty} \left(\frac{a_{x-1}}{a_x}\right) n_x$$

$$= \frac{\sum_{x=c+1}^{\infty} \left(\frac{a_{x-1}}{a_x}\right) n_x}{\sum_{x=c}^{\infty} n_x}$$

$$= \theta' .$$

The exact variance of this estimate is

$$\sigma^2(\theta') = \frac{1}{N}\left[\sum_{x=c+1}^{\infty} \left(\frac{a_{x-1}}{a_x}\right)^2 P_x - \theta^2 \right] . \qquad (3.2.4)$$

An unbiased estimate of $\sigma^2(\theta')$ is

$$\left[\sum_{x=c+1}^{\infty} \left(\frac{a_{x-1}}{a_x}\right)^2 n_x - N\theta'^2 \right] / N(N-1) \qquad (3.2.5)$$

the proof of which is almost immediate once one recognizes that θ' is the mean of N independent identically distributed random variables Y_i with probability distribution given by (for $i = 1,2,\ldots N$)

$$\text{Prob}\,\{ Y_i = 0 \} = P_c$$

and

$$\text{Prob}\,\left\{ Y_i = \frac{a_{x-1}}{a_x} \right\} = P_x \qquad \text{for } x = c+1, c+2, \ldots \infty .$$

One can compare $\sigma^2(\theta')$ with the asymptotic variance Var $(\hat{\theta})$ of the maximum likelihood estimate of θ and the efficiency of the ratio estimate θ' can be computed. Of course,

$$\sigma^2(\theta') > \text{Var}\ (\hat{\theta})$$

from the Cramer-Rao Information limit to the variance; but this comparison is not quite valid because the maximum likelihood estimate is not generally unbiased.

Lastly, one may establish that

$$\theta'_r = \frac{1}{N} \sum_{x=c+r}^{\infty} \left(\frac{a_{x-r}}{a_x}\right) n_x \tag{3.2.6}$$

is the only unbiased estimate of θ^r (r an integer) which is a linear function of the frequencies.

3.3 Estimation by the Two-Moments Method for a gpsd

[Range $T = (c, c+1, \ldots d)$, d finite or $T = (c, c+1, \ldots \infty)$ with positive probabilities]

Consider the gpsd (1.1.4) with finite or infinite range $T = (c, c+1, \ldots d)$ with positive probabilities; that is, consider the gpsd

$$P_x = \text{Prob}\ \{X = x\} = \frac{a_x \theta^x}{f(\theta)} \tag{3.3.1}$$

where

$$x \epsilon T = (c, c+1, \ldots d),\ d \text{ finite or infinite}$$

$$f(\theta) = \sum_{x=c}^{d} a_x \theta^x \qquad a_x > 0\ . \tag{3.3.2}$$

3.3.1 For this distribution, it is easy to see that

$$\mu = \theta G_{01} + cP_c \tag{3.3.3}$$

and

$$m_2 = \mu + \theta G_{11} + c(c-1)P_c \qquad (3.3.4)$$

where

$$G_{ij} = \sum_{x=c}^{d-1} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j P_x \ . \qquad (3.3.5)$$

Further, from (3.3.3) and (3.3.4), we have

$$\frac{m_2 - \mu - \theta G_{11}}{\mu - \theta G_{01}} = c-1 \qquad \text{when } c \neq 0 \qquad (3.3.6)$$

which when solved for θ gives the identity

$$\theta = \frac{m_2 - c\mu}{G_{11} - (c-1)G_{01}} \qquad \text{when } c \neq 0 \ . \qquad (3.3.7)$$

From (3.3.3), we have the identity

$$\theta = \frac{\mu}{G_{01}} \qquad \text{when } c = 0. \qquad (3.3.8)$$

3.3.2 The identities (3.3.7) and (3.3.8) can be made use of in estimating θ. One has only to compute

$$S_i = \sum_{x=c}^{d} x^i n_x \qquad i = 1,2 \qquad (3.3.9)$$

and

$$g_{ij} = \sum_{x=c}^{d-1} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j n_x \qquad i = 0,1;\ j = 1 \qquad (3.3.10)$$

from the sample, and then

$$t = \frac{S_2 - cS_1}{g_{11} - (c-1)g_{01}} \qquad \text{when } c \neq 0 \qquad (3.3.11)$$

or

$$t = \frac{S_1}{g_{01}} \qquad \text{when } c = 0 \ , \qquad (3.3.12)$$

can be taken as an estimate for Θ. Because we use the first two moments for the estimation of the single parameter, we call the estimate t as "two-moments estimate" and the method as "two-moments method."

3.3.3 Proceeding along the same lines as in Section 2.6, one gets to terms of order $\frac{1}{N}$,

$$b(t) = E(t) - \Theta = \frac{1}{NG^2}(\Theta\sigma_{22} - \sigma_{12}) \tag{3.3.13}$$

and

$$\mathrm{Var}\ (t) = \frac{1}{NG^2}[\sigma_{11} - 2\Theta\sigma_{12} + \Theta^2\sigma_{22}] \tag{3.3.14}$$

where (i) for $c \neq 0$,

$$G = G_{11} - (c-1)G_{01}$$
$$\sigma_{11} = (m_4 - m_2^2) + c^2(m_2 - \mu^2) - 2c(m_3 - \mu m_2)$$
$$\sigma_{12} = (G_{31} - m_2G_{11}) - c(G_{21} - \mu G_{11}) - (c-1)(G_{21} - m_2G_{01}) + c(c-1)(G_{11} - \mu G_{01})$$
$$\sigma_{22} = (G_{22} - G_{11}^2) + (c-1)^2(G_{02} - G_{01}^2) - 2(c-1)(G_{12} - G_{01}G_{11})$$

and (ii) for $c = 0$,

$$G = G_{01}$$
$$\sigma_{11} = m_2 - \mu^2$$
$$\sigma_{12} = G_{11} - \mu G_{01}$$
$$\sigma_{22} = G_{02} - G_{01}^2 \quad .$$

3.4 Estimation by the Two-Moments Method for a Truncated gpsd

Consider the gpsd (3.3.1) truncated to

$$T^* = (c^*, c^*+1, \ldots d^*), \quad d^* \neq d \text{ when } d \text{ finite}.$$

The truncated gpsd can be written as

$$P_x^* = \text{Prob}\{X^* = x\} = \frac{a_x \theta^x}{f^*(\theta)} \qquad x \in T^* \tag{3.4.1}$$

where

$$f^*(\theta) = \sum_{x=c'}^{d'} a_x \theta^x . \tag{3.4.2}$$

3.4.1 For this distribution, it is easy to see that

$$\mu^* - \theta H_{01} = c' P_{c'}^* - (d'+1) P_{d'+1}^* \tag{3.4.3}$$

and

$$m_2^* - \mu^* - \theta H_{11} = c'(c'-1) P_{c'}^* - d'(d'+1) P_{d'+1}^* \tag{3.4.4}$$

where

$$H_{ij} = \sum_{x=c'}^{d'} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j P_x^* . \tag{3.4.5}$$

3.4.2 For estimation purposes, we consider following four mutually exclusive and exhaustive cases:

Case (1) $c' = 0$ and d' Finite

Case (2) $c' = 0$ and d' Infinite

Case (3) $c' \neq 0$ and d' Infinite

Case (4) $c' \neq 0$ and d' Finite.

Case (1): $c' = 0$ and d' Finite

From (3.4.3) and (3.4.4), we have the identity

$$\theta = \frac{m_2^* - (d'+1)\mu^*}{H_{11} - d' H_{01}} \tag{3.4.6}$$

which we utilize to estimate θ. We have only to compute

$$S_i = \Sigma\, x^i n_x \qquad i = 1,2 \tag{3.4.7}$$

and

$$h_{ij} = \sum_{x=c^\cdot}^{d^\cdot} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j n_x \qquad i = 0,1;\ j = 1 \qquad (3.4.8)$$

from the sample and then

$$t^* = \frac{S_2 - (d^\cdot+1)S_1}{h_{11} - d^\cdot h_{01}} \qquad (3.4.9)$$

can be taken as an estimate for θ. The estimate t^* makes use of the (additional) information that the sample is taken from some known gpsd and truncated to the one under consideration. The estimate t of Section (2.8) does not require, and hence, does not make use of this information. The formula for the bias and variance of t^* can be written down to order $\frac{1}{N}$ as:

$$b(t^*) = \frac{1}{NH^2}(\theta\sigma^*_{22} - \sigma^*_{12}) \qquad (3.4.10)$$

and

$$\text{Var}\ (t^*) = \frac{1}{NH^2}(\sigma^*_{11} - 2\theta\sigma^*_{12} + \theta^2\sigma^*_{22}) \qquad (3.4.11)$$

where

$$H = H_{11} - d^\cdot H_{01}$$

$$\sigma^*_{11} = (m^*_4 - m^{*2}_2) + (d^\cdot+1)^2(m^*_2 - \mu^{*2}) - 2(d^\cdot+1)(m^*_3 - \mu^* m^*_2)$$

$$\sigma^*_{12} = (H_{31} - m^*_2 H_{11}) - (d^\cdot+1)(H_{21} - \mu^* H_{11})$$
$$- d^\cdot(H_{21} - m^*_2 H_{01}) + d^\cdot(d^\cdot+1)(H_{11} - \mu^* H_{01})$$

and

$$\sigma^*_{22} = (H_{22} - H^2_{11}) + d^{\cdot 2}(H_{02} - H^2_{01}) - 2d^\cdot(H_{12} - H_{01}H_{11}).$$

Case (2) and Case (3): $c' = 0$, d' Infinite and $c' \neq 0$, d' Infinite

It can be easily verified in these cases that

$$H_{ij} = G_{ij}$$

and, hence,

$$t^* = t \quad .$$

Thus, we have the same treatment as in Section 3.3.

Here we also observe that if we allow $d' = d$ -- even when d is finite -- and use $a_{d+1} = 0$ formally, we again get $H_{ij} = G_{ij}$ and $t^* = t$. This observation is specially important in the case of the binomial distribution.

Case (4): $c' \neq 0$ and d' Finite

It may be noted that the t^* estimate is not available in this case. However, the t estimate still works, and the estimate of Θ can thus be obtained by employing two moments.

3.5 An Upper Bound for Bias Per Unit Standard Error for Ratio Estimates and Two-Moments Estimates

We first establish a general result true for the bias of an estimate of a certain type. Let the probability distribution, from which a sample $x_1, x_2, \ldots x_n$ is drawn, be a general distribution of a random variable X with a single parameter Θ.

3.5.1 Let t_1 and t_2 be two statistics based on the sample such that

$$\frac{E(t_1)}{E(t_2)} = \frac{E[t_1(x_1, x_2, \ldots x_n)]}{E[t_2(x_1, x_2, \ldots x_n)]} = \Theta$$

for all Θ in the parameter space of the given distribution.

Consider the estimate $s = \frac{t_1}{t_2}$ to estimate Θ. To find the bias in s per unit standard error of s, we have

$$\begin{aligned} \text{Cov } (s,t_2) &= E(st_2) - E(s)\,E(t_2) \\ &= E(t_1) - E(s)\,E(t_2) \\ &= [\Theta - E(s)]\,E(t_2)\,. \qquad (3.5.1) \end{aligned}$$

Now

$$|\text{cov}(s,t_2)| \leq \sigma(s)\ \sigma(t_2) \qquad (3.5.2)$$

where σ denotes the standard error. Therefore, from (3.5.1) we have

$$\left| \frac{E(s) - \Theta}{\sigma(s)} \right| \leq \left| \frac{\sigma(t2)}{E(t_2)} \right| = |\ \text{c.v.}(t_2)| \qquad (3.5.3)$$

where c.v.(t_2) is the coefficient of variation of t_2. Thus, for the bias in s we have

$$\left| \frac{b(s)}{\sigma(s)} \right| \leq |\ \text{c.v.}(t_2)|\ . \qquad (3.5.4)$$

In particular, when t_2 is a constant, we have an unbiased estimate for Θ.

It may be noted that the ratio-estimates and two-moments estimates for the parameter Θ of gpsd's, which we discussed earlier, are estimates of the actual type of the estimate s that we have discussed in this section. Hence, the result in (3.5.4) also applies to them.

3.6 Estimation for a Truncated gpsd with a Finite Range of Consecutive Integers, Maximum Unknown

We have discussed methods of estimation in relation to gpsd's with known range. Sometimes, however, one has to estimate the parameter Θ even when the range is not completely known, as well as when one is interested in estimating the maximum of a finite range. The problem of estimation can be solved in such cases by using the "Ratio Method" and "Two-Moments Method" simultaneously. This is taken up in this section.

3.6.1 On the basis of a sample of size N with frequency n_x for x $(0 \leq x \leq L, \Sigma n_x = N)$ drawn from the gpsd given by the truncated gpsd (3.4.1),

$$P_x^* = \text{Prob}\{X^* = x\} = \frac{a_x\theta^x}{f^*(\theta)} \qquad x = 0,1,2,\ldots d^\cdot \qquad (3.6.1)$$

for which

$$f^*(\theta) = \sum_{x=0}^{d^\cdot} a_x\theta^x \qquad a_x > 0 \qquad (3.6.2)$$

where $d^\cdot$ is not known; to estimate θ, we choose the ratio estimate

$$\theta' = \sum_{x=0}^{L} \left(\frac{a_{x-1}}{a_x}\right)n_x \Big/ \sum_{x=0}^{L-1} n_x \quad . \qquad (3.6.3)$$

The advantage with θ' is that besides its simplicity, it does not require the knowledge of $d^\cdot$.

3.6.2 To estimate $d^\cdot$, however, the identity

$$\theta = \frac{m_2^* - (d^\cdot+1)\mu^*}{H_{11} - d^\cdot H_{01}} \qquad (3.6.4)$$

gives

$$d^\cdot = \frac{m_2^* - \mu^* - \theta H_{11}}{\mu^* - \theta H_{01}} \qquad (3.6.5)$$

where μ^* and m_2^* are the first two moments about the origin of (3.6.1) and

$$H_{ij} = \sum_{x=0}^{d} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j P_x^* \quad . \qquad (3.6.6)$$

Therefore, the estimate of d' can be obtained as

$$\tilde{d} = \frac{S_2 - S_1 - \theta' h_{11}}{S_1 - \theta' h_{01}} \qquad (3.6.7)$$

where θ' is given by (3.6.3) and

$$S_i = \sum_{x=0}^{L} x^i n_x \qquad i = 1,2 \qquad (3.6.8)$$

and

$$h_{ij} = \sum_{x=0}^{L} x^i \left[\frac{(x+1)a_{x+1}}{a_x}\right]^j n_x \; . \qquad (3.6.9)$$

CHAPTER IV

4.0 ESTIMATION PROBLEMS FOR THE BINOMIAL DISTRIBUTION

4.1 Introduction

The gpsd defined by (1.1.4) becomes

$$\text{Prob}\{X = x\} = \binom{n}{x} \theta^x / (1 + \theta)^n \qquad (4.1.1)$$

$$x = 0,1,2,\ldots n$$

when $f(\theta) = (1 + \theta)^n$. Writing $\theta = \pi/(1-\pi)$, (4.1.1) gives the probability law for X as:

$$\text{Prob}\{X = x\} = b(x, \pi, n) = \binom{n}{x} \pi^x (1-\pi)^{n-x}, \qquad (4.1.2)$$

$$x = 0,1,2,\ldots n$$

the well-known form of the binomial distribution.

The important properties of (4.1.2) can be summarily stated as follows:

$$M(t) = (1 - \pi + \pi e^t)^n. \qquad (4.1.3)$$

The first two central moments and the coefficients β_1, β_2 are of the form:

$$\begin{aligned} \mu &= n\pi \\ \mu_2 &= n\pi(1 - \pi) \\ \beta_1 &= (1 - 2\pi)^2 / n\pi(1 - \pi) \\ \beta_2 &= 3 + \{1 - 6\pi(1 - \pi)\} / n\pi(1 - \pi). \end{aligned} \qquad (4.1.4)$$

The recurrence relations reduce to:

$$\mu_{r+1} = \pi(1 - \pi)\left[\frac{d\mu_r}{d\pi} + nr\mu_{r-1}\right] \qquad (4.1.5)$$

derived first by Romanovsky (1925) and

$$K_{r+1} = \pi(1-\pi) \frac{dK_r}{d\pi} \tag{4.1.6}$$

deduced by Frisch (1925) and rediscovered by Haldane (1940).

The distribution function $B(r, \pi, n)$ defined by

$$B(r, \pi, n) = \sum_{x=o}^{r} b(x, \pi, n) \tag{4.1.7}$$

can be reduced to

$$B(r, \pi, n) = I_{1-\pi} (n-r, r+1) \tag{4.1.8}$$

where

$$I_x (m,n) = \frac{1}{B(m,n)} \int_o^x u^{m-1} (1-u)^{n-1} du$$

for which extensive tables have been edited by K. Pearson. Romig has extensively tabulated $b(x,\pi,n)$ and $B(r,\pi,n)$ for the range of arguments - = 0.01 (0.01) 0.50, n = 50(5) 100 and Applied Mathematics Series 6, 1950, gives $b(x,\pi,n)$ for n = 2(1)49. The ordnance corps tables (1952) give values of $1 - B(r,\pi,n)$ for π = 0.01 (0.01) 0.50, n = 2(1) 150. For large n and small π, one can use tables of Poisson or Normal Probabilities, because:

$$\lim_{\substack{n \to \infty,\ \pi \to o \\ n\pi = \mu}} b (x,\pi,n) = p(x,\mu) \tag{4.1.9}$$

where

$$p(x,\mu) = e^{-\mu} . \frac{\mu^x}{x!}$$

and

$$\lim_{n \to \infty} (B(r,\pi,n) = \Phi(Z) , \tag{4.1.10}$$

where

$$Z = \frac{r + 1/2 - n\pi}{\sqrt{n\pi\ (1 - \pi)}}$$

and

$$\Phi(Z) = \frac{1}{\sqrt{2\pi}} \int_{o}^{x} e^{-1/2.u^2}\ du \ .$$

On the basis of single observation on X, X = x, the maximum likelihood estimate for π is given by $\hat{\pi} = x/n$, (n known). One has $E(\hat{\pi}) = \pi$ with $Var(\hat{\pi}) = \pi(1-\pi)/n$.

On the basis of a random sample x_i (i = 1, 2, ...N) of size N from (1.2), the maximum likelihood estimate for π is given by

$$\hat{\pi} = \frac{\bar{x}}{n} \qquad (4.1.11)$$

where

$$\bar{x} = \sum_{i=1}^{N} x_i\ /N.$$

(4.1.11) provides an unbiased estimate for the parameter π with $Var(\hat{\pi}) = \pi(1-\pi)/nN$.

4.2 Estimation from a Sample for a Singly Truncated Binomial Distribution

Fisher (1936) and Haldane (1932, 1938) discussed uses of the truncated binomial distribution. For instance, in problems of human genetics, in estimating the proportion of albino children produced by couples capable of producing albinos, sampling has necessarily to be restricted to families having at least one albino child. Finney (1949) has cited some more applications. Fisher and Haldane derived the maximum likelihood procedure to estimate the parameter π. Moore

(1954) suggested a simple "ratio-estimate" based on an identity between binomial probabilities. For a slightly different problem, Rider (1955) suggested an alternative estimation procedure which uses first two moments.

We present in this section some numerical tables to facilitate the heavy computation involved in evaluating the maximum likelihood estimate of π from a sample from singly truncated binomial distribution. The estimates given by Moore and Rider have been derived from the general results discussed in Chapter III. The efficiency and the amount of bias of these estimates are investigated in certain special cases.

The probability law of the binomial distribution truncated at c on the left can be written as

$$b^* (x, \pi, n) = \left(B^* (c, \pi, n)\right)^{-1} \binom{n}{x} \pi^x (1-\pi)^{n-x}$$

$$x = c, c+1, \ldots . n. \qquad (4.2.1)$$

where

$$B^* (r + 1, \pi, n) = 1 - B (r, \pi, n) . \qquad (4.2.2)$$

The first two moments about the origin of (4.2.1), then, are

$$\mu^* = \mu^* (c, \pi, n) = n\pi . \quad B^* (c-1, \pi, N-1)/B^*(c, \pi, n) \qquad (4.2.3)$$

and

$$m_2^* = m_2^* (c, \pi, n) = \mu^* (c, \pi, n) \left\{ 1+ \mu^*(c-1, \pi, n-1)\right\} . \qquad (4.2.4)$$

The case of truncation to the right can be dealt with in a similar way by replacing π by $1 - \pi$ and the truncation point c by n - c.

4.2.1 To estimate π by likelihood on the basis of a random sample x_i (i=1, 2,...N) of size N from (4.2.1), results derived by the general approach in Section 2.2 can be written down as follows with proper substitutions in this particular case.

The likelihood equation for π is

$$\bar{x} = \hat{\mu}^* \tag{4.2.5}$$

where

$$\bar{x} = \sum_{i=1}^{N} x_i/N$$

and μ^* is defined by (4.2.3)

Denoting this estimate as $\hat{\pi}$, its asymptotic variance is given by

$$\text{Var}\ (\hat{\pi}) = \frac{\pi(1-\pi)}{N} \Big/ \left(\frac{d\mu^*}{d\pi}\right) \tag{4.2.6}$$

$$= \frac{[\pi\ (1-\pi)]^2}{N\mu_2^*} \tag{4.2.7}$$

where μ_2^* is the variance of (4.2.1).

As the equation (4.2.5) does not readily give an algebraic solution, one may use an iterative process of solution. However, (4.2.5) suggests that if tables be made available for means μ^*s for sufficiently close values of π, one can have a ready solution. The practical case of importance is c = 1 and sometimes c = 2. For the case c = 1,

$$\mu^* = n\pi/B^*\ (1,\pi,n) = \frac{n\pi}{1-(1-\pi)^n}$$

so that the likelihood equation becomes

$$\frac{\bar{x}}{n} = \frac{\pi}{1-(1-\pi)^n}\ , \tag{4.2.8}$$

and the Expression (4.2.6) for the asymptotic variance reduces to

$$\text{Var}\ (\hat{\pi}) = \frac{\pi(1-\pi)}{N} \cdot \frac{1-(1-\pi)^n}{n} \left(1 + \frac{n\pi\ (1-\pi)^{n-1}}{1 - (1-\pi)^n}\right)^{-1} \quad (4.2.9)$$

a result first derived by Fisher (1936).

Here we present in Table I the values of μ^*/n for the binomial distribution truncated on the left at $c = 1$ for values of π spaced at suitable intervals. For the case $c = 2$, we present Table II for values of π at intervals of 0.01. Suitable charts based on these tables may be also of great help in facilitating the procedure of estimation.

These tables can be used to compute Var $(\hat{\pi})$ by using either Formula (4.2.6) or Formula (4.2.7). In case (4.2.6) is used, $\frac{d\mu^*}{d\pi}$ can be approximated by the finite difference ratio $\frac{\Delta\mu^*}{\Delta\pi}$. This approximation is expected to be good since the tabular interval is small. In case Formula (4.2.7) is used, the relationship for use is

$$\mu_2^* (c, \pi, n) = \mu^* (c, \pi, n) \left[1 + \left(\mu^*(c-1, \pi, n-1) - \mu^* (c, \pi, n)\right)\right]. \quad (4.2.10)$$

4.2.2 For a slightly different problem, where, in a sample from a complete binomial distribution, the frequencies in some lowest classes are missing, Rider (1955) suggested a method of estimation, which uses first two moments of the complete binomial and leads to a linear equation.

The method of two-moments is also applicable in the usual problem of estimation from a sample from singly truncated binomial and forms a particular case of the general method discussed in

Section 3.4 . Proceeding on those lines, one gets in this case

$$\Theta = \frac{\pi}{1-\pi} = \frac{m_2^* - c\mu^*}{H_{11}-(c-1)H_{o1}} \qquad (4.2.11)$$

where μ^* and m_2^* are defined by (4.2.3) and (4.2.4) respectively, and H_{11} and H_{o1} reduce to

$$H_{11} = n\mu^* - m_2^*$$

$$H_{o1} = n - \mu^*$$

(4.2.11) gives then

$$\pi = \frac{m_2^* - c\mu^*}{(n-1)\mu^* - n(c-1)} \qquad (4.2.12)$$

so that, on the basis of a random sample of size N with n_x as the frequency of x drawn from (4.2.1), the estimate for π can be written as

$$t = \frac{S_2 - cS_1}{(n-1)S_1-n(c-1)N} \qquad (4.2.13)$$

where

$$S_1 = \Sigma\, xn_x$$

and

$$S_2 = \Sigma\, x^2 n_x \;.$$

It is obvious that (4.2.13) is quite simple and that a great deal of computational labour can be saved if (4.2.13) is used instead of (4.2.5). On the other hand, the estimate obtained from (4.2.13) is likely to be inefficient. It is important, therefore, to investigate the loss in efficiency due to the use of (4.2.13) instead of (4.2.5).

To find the asymptotic variance of the two-moments estimate t of π, one gets on some simplification,

$$\text{Var}(t) = \frac{1}{NH^2}(\sigma^*_{11} + \pi^2 \sigma^*_{22} - 2\pi\,\sigma^*_{12}) \qquad (4.2.14)$$

where

$$H = (n-1)\,\mu^* - n\,(c-1)$$

$$\sigma^*_{11} = (m^*_4 - m^{*2}_2) + c^2\,(m^*_2 - \mu^{*2}) - 2c(m^*_3 - \mu^* m^*_2)$$

$$\sigma^*_{22} = (n-1)^2\,(m^*_2 - \mu^{*2})$$

and

$$\sigma^*_{12} = (n-1)\,(m^*_3 - \mu^* m^*_2) - c(m^*_2 - \mu^{*2})$$

where m^*_r is the r-th theoretical moment of (4.2.1) about the origin. Thus,

$$\text{Var}(t) = \frac{1}{N\{(n-1)\mu^* - n(c-1)\}^2}\Big[(m^*_4 - m^{*2}_2) + \{(n-1)\pi + c\}^2 (m^*_2 - \mu^{*2}) - 2\{(n-1)\pi + c\}\,(m^*_3 - \mu^* m^*_2)\Big]. \qquad (4.2.15)$$

The asymptotic efficiency of t is then given by

$$\text{Eff}(t) = \text{Var}(\hat{\pi}) / \text{Var}(t). \qquad (4.2.16)$$

The special cases of some importance in genetics are c = 1 and π = 1/4, 1/2 or 3/4. The efficiency of the Two-Moments Estimate (TM) relative to the Maximum Likelihood Estimate (ML) in these cases is tabulated on the following page.

TABLE 4.2.1

ASYMTOTIC EFFICIENCY OF TM FOR c = 1

n	Efficiency $\pi = 1/4$	1/2	3/4
3	.925	.875	.875
4	.871	.818	.859
5	.817	.795	.870
6	.809	.789	.886
7	.781	.794	.901
8	.766	.803	.913
9	.755	.814	.923
10	.749	.823	.931

Close investigation of the above table shows that the efficiency of TM in case of $\pi = 1/2$ and $\pi = 3/4$ decreases in the beginning with n, reaches a minimum and then increases with increasing values of n. For $\pi = 1/4$, however, the efficiency decreases throughout. Let us compute, therefore, the efficiency of TM for higher values of n. The following gives the results obtained for n = 11(1)15.

Asymptotic Efficiency of TM for c = 1 and $\pi = 1/4$

n	Efficiency
11	.746
12	.744
13	.745
14	.747
15	.750

Thus, in case of $\pi = 1/4$ also, the efficiency reaches a minimum and then increases with increasing n. It is interesting to note that in all these cases the efficiency of TM has reached the minimum at $n = 3/\pi$.

4.2.3 Following the general approach discussed in Section 3.1, a simple estimate for π can be obtained in the case of singly truncated binomial distribution (4.2.1). In this case, $a_{x-1}/a_x = x/(n-x+1)$ and since $\theta = \pi/(1-\pi)$, we have the following "ratio-estimate" for π:

$$\pi' = \frac{t_1}{t_1+t_2} \tag{4.2.17}$$

where

$$t_1 = \sum_{x=c+1}^{n} \left(\frac{xn_x}{n-x+1}\right)$$

and

$$t_2 = \sum_{x=c}^{n-1} n_x \quad .$$

When c = 1, i.e. when only "zero" values are truncated, the estimate takes the form suggested by Moore (1954):

$$\pi' = \frac{t_1}{t_1+t_2} \tag{4.2.18}$$

where

$$t_1 = \sum_{x=2}^{n} \left(\frac{xn_x}{n-x+1}\right)$$

and

$$t_2 = \sum_{x=1}^{n-1} n_x \quad .$$

To investigate the efficiency of π' given by (4.2.17) its asymptotic variance can be written down as:

$$\text{Var}(\pi') = \frac{(1-\pi)^2}{NP^2}\Big[(1-\pi)^2 D - P\pi^2 + 2\pi^2 . P_{n-1}\Big] \tag{4.2.19}$$

where

$$P = \sum_{n=c}^{n-1} b^*(x,\pi,n)$$

$$D = \sum_{x=c+1}^{n} \left(\frac{x}{n-x+1}\right) b^*(x,\pi,n)$$

and

$$P_{n-1} = b^*(n-1,\pi,n).$$

Also the asymptotic variance of the maximum likelihood estimate $\hat{\pi}$ obtained from (4.2.5) is given by

$$\text{Var}(\hat{\pi}) = \frac{\pi(1-\pi)^2}{N\mu_2^*}$$

Where μ_2^* is the variance of (4.2.1)

Therefore the asymptotic efficiency of π' takes the form:

$$\text{Eff}(\pi') = \frac{P^2}{\mu_2^*}\left(\left(\frac{1-\pi}{\pi}\right)^2 D - P + 2P_{n-1}\right)^{-1}. \tag{4.2.20}$$

In particular, when c = 1

$$P = 1 - \frac{\pi^n}{1-(1-\pi)^n}$$

$$\mu_2^* = \frac{n\pi(1-\pi)}{1-(1-\pi)^n}\left(1 + \frac{n\pi(1-\pi)^{n-1}}{1-(1-\pi)^n}\right)$$

$$P_{n-1} = \frac{n\pi^{n-1}(1-\pi)}{1-(1-\pi)^n}$$

and

$$D = \sum_{x=2}^{n} \left(\frac{x}{n-x+1}\right)^2 \cdot \binom{n}{x} \pi^x (1-\pi)^{n-x} \Big/ \{1-(1-\pi)^n\}$$

reduces to

$$D = \left(\frac{\pi}{1-\pi}\right)\left[\frac{n+1}{1-(1-\pi)^n n}\left\{(1-\pi^n)\, E\left(\frac{1}{x},n,1-\pi\right) - \frac{(1-\pi)^n}{n}\right\} - P\right]$$

where

$$E\left(\frac{1}{x},n,\pi\right) = \sum_{x=1}^{n} \frac{1}{x}\binom{n}{x} \pi^x (1-\pi)^{n-x} \Big/ \{1 - (1-\pi)^n\}$$

and is tabulated by Grab and Savage (1954).

The special cases of some importance in genetics are c = 1 and π = 1/4, 1/2 or 3/4. The efficiency of the Ratio-Estimate (R) relative to the Maximum Likelihood Estimate (ML) in these cases is tabulated and shown on the following page.

TABLE 4.2.2

ASYMPTOTIC EFFICIENCY OF R FOR c = 1

n	π = 1/4	1/2	3/4
3	.924	.875	.875
4	.909	.769	.772
5	.919	.715	.664
6	.933	.694	.565
7	.947	.693	.523
8	.952	.705	.481
9	.956	.723	.435
10	.959	.776	.388

Close investigation of the above table shows that the efficiency of R in case of $\pi = 1/4$ and $\pi = 1/2$ decreases in the beginning with n, reaches a minimum and then increases with increasing values of n. For $\pi = 3/4$, however, the efficiency decreases throughout for n = 3(1)10.

4.2.4 For c = 1, we have separately discussed the Two-Moments Estimate and the Ratio-Estimate for π. To make a comparative study of these two equally simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.4 and 3.1 , one gets, to order 1/N, the amount of bias of t (TM) and π' (R) as follows:

$$b(t) = \frac{1}{N} \frac{\mu_2^{*2} + \mu^* m_2^* - m_3^*}{(n-1)\mu^{*2}} = \frac{B(t)}{N}$$

and

$$b(\pi') = \frac{(1-\pi)^2}{NP^2}\left[\pi^2 P + (\pi - 2\pi^2)P_{n-1} - (1-\pi)^2 D\right] = \frac{B(\pi')}{N}.$$

The table on the following page gives B(t), B(π') and also a relative efficiency of t over π' for c = 1 and π = 1/4, 1/2 and 3/4. The relative efficiency is given by Rel. Eff = Var (π')/Var(t).

Let us also study the amount of bias relative to the standard error of the two estimates for some sample size say 100. The following table gives the bias as a percentage of standard error (100 |b|/S.E.) for both TM and R for c = 1 and π = 1/4, 1/2 and 3/4.

TABLE 4.2.4

BIAS AS A PERCENTAGE OF STANDARD ERROR
FOR c = 1 AND N = 100

	π = 1/4		π = 1/2		π = 3/4	
n	TM	R	TM	R	TM	R
3	6.34	5.07	7.11	3.82	6.15	2.86
4	6.94	4.05	6.21	4.05	5.40	3.31
5	7.03	3.39	6.48	4.17	4.80	3.44
6	7.24	2.94	5.84	4.15	4.35	4.08
7	7.12	2.61	5.66	4.05	3.99	4.11
8	6.97	2.37	5.31	3.90	3.71	4.20
9	6.46	2.18	4.99	3.73	3.49	4.18
10	6.39	2.02	4.28	3.62	3.29	4.08

Table 4.2.3 shows that both TM and R are under-estimates of π. A closer investigation, however, brings out that the bias to order 1/N is in general considerably smaller for R. Also, Table 4.2.3 shows that whereas for π = 1/2 and π = 3/4, R is less efficient than TM, it is more efficient when π = 1/4. Thus, a closer study of the relative efficiency of the two estimates is necessary. However, Table 4.2.3 suggests for n = 3(1)10 that the Ratio-Estimate may be used to estimate

TABLE 4.2.3

COMPARISON BETWEEN TM AND R WHEN c=1

n	N(Amount of Bias to Order 1/N) TM	R	Var(R)/Var(TM)
	Case (1) π = 1/4		
3	-.2412	-.1927	1.000
4	-.2152	-.1227	.958
5	-.1896	-.0861	.889
6	-.1715	-.0648	.867
7	-.1535	-.0510	.824
8	-.1379	-.0420	.804
9	-.1187	-.0355	.790
10	-.1097	-.0301	.781
	Case (2) π = 1/2		
3	-.2717	-.1458	1.000
4	-.1940	-.1307	1.065
5	-.1748	-.1184	1.111
6	-.1398	-.1062	1.138
7	-.1230	-.0943	1.146
8	-.1063	-.0833	1.139
9	-.0934	-.0736	1.126
10	-.0748	-.0651	1.053
	Case (3) π = 3/4		
3	-.1763	-.0820	1.000
4	-.1290	-.0833	1.112
5	-.1004	-.0893	1.543
6	-.0818	-.0962	1.569
7	-.0689	-.0931	1.724
8	-.0595	-.0928	1.900
9	-.0524	-.0915	2.124
10	-.0468	-.0896	2.398

the parameter π of a binomial distribution truncated at $c = 1$, especially when π is near to 1/4 or less; whereas Two-Moments Estimate may be preferred when π is near to 1/2 or more.

4.2.5 The detailed computation procedure of evaluating the three types of estimates discussed above will be illustrated with reference to K. Pearson's data on albinism in man. The table below gives the number of families (n_x) each of five children having exactly x albino children in the family, ($x = 1, 2, 3, 4, 5$).

Number of albinos in family (x)	1	2	3	4	5
Number of families (n_x)	25	23	10	1	1

If π is the probability for a child to be an albino, we may accept the truncated binomial model:

$$\frac{\binom{n}{x} \pi^x (1-\pi)^{n-x}}{1 - (1-\pi)^n} \qquad x = 1,2,\ldots n.$$

for the probability of x albinos in a family of n. Here $n = 5$, and the problem is to estimate π on the basis of the data given in the table above.

<u>Maximum Likelihood Estimate</u>: From the table, we get

$$N = 60$$

$$S_1 = \Sigma\, xn_x = 110$$

$$\bar{x} = S_1/N = 1.83333 ,$$

so that $\bar{x}/n = 0.366667$. Referring to Table I for n = 5, we find the following:

π	μ^*/n
0.30	0.360607
0.31	0.367474

The maximum likelihood estimate is given by that value of π for which $\mu^*/n = 0.366667$. By linear interpolation, we thus get

$$\hat{\pi} = 0.30 + \frac{0.366667 - 0.360607}{0.367474 - 0.360607}(0.31 - 0.30)$$

$$= 0.3088 .$$

The variance of this estimate is estimated from the formula

$$\text{Var}(\hat{\pi}) = \frac{\pi(1-\pi)}{N\left(\frac{d\mu^*}{d\pi}\right)} .$$

$\left(\frac{d\mu^*}{d\pi}\right)$ can be obtained approximately from the tables by taking differences instead of derivatives. Thus,

$$\left(\frac{d\mu}{d\pi}\right)_{\hat{\pi}} \sim \frac{5 \times (0.367474 - 0.360607)}{0.31 - 0.30} = 3.4335 .$$

Hence,

$$\text{Var}(\hat{\pi}) \sim \frac{0.3088 \times .6912}{60 \times 3.4335} = 0.0010361 .$$

Thus the standard error of $\hat{\pi}$ is given approximately by:

$$\text{S.E.}(\hat{\pi}) = \sqrt{(0.0010361)} = 0.03219 .$$

This is as far as linear interpolation in the tables will go. If we want to carry the approximate solution of the likelihood equation further, we start with $\pi_o = 0.3088$ as the first approximation and compute, the theoretical mean

$$\mu^* = \frac{n\pi_o}{1 - (1-\pi_o)^n} = \frac{5 \times 0.3088}{1 - (0.6912)^5}$$

$$= 1.83330$$

and

$$\frac{d\mu^*}{d\pi_o} = \frac{\mu^*}{\pi_o(1-\pi_o)}\left[1 + (n-1)\,\pi_o - \mu^*\right]$$

$$= \frac{1.83330}{0.3088 \times 0.6912} \; (1 + 4 \times 0.3088 - 1.83330)$$

$$= 3.4520 .$$

It is interesting to compare the exact value of $\frac{d\mu^*}{d\pi}$ viz., 3.4520 at $\pi = 0.3088$ with the approximation obtained by differencing in the tables, viz., 3.4335. The next approximation is then given by:

$$\pi_1 = \pi_o + (\bar{x} - \mu^*) \;/\; \left(\frac{d\mu^*}{d\pi}\right)$$

$$= 0.3088 + (1.83333 - 1.83330) \;/\; 3.4520$$

$$= 0.308809$$

which does not affect the fourth place. The variance of the estimate is

$$\text{Var}\,(\hat{\pi}) = \frac{\pi(1-\pi)}{N} \;/\; \left(\frac{d\mu^*}{d\pi}\right)$$

$$= \frac{0.3088 \times 0.6912}{60} \;/\; 3.4520$$

$$= 0.0010305$$

so that the standard error is S.E. $(\hat{\pi}) = \sqrt{0.0010305} = 0.03210$.

The agreement of this with the previous estimate obtained directly from the tables is remarkable, and these latter computations are really unnecessary, especially in view of the somewhat large standard error.

It is believed that linear interpolation in the tables would be generally adequate for all practical purposes and the second cycle of approximation would not be necessary.

Two-Moments Estimate: To compute this estimate for π, we require in addition the value of $S_2 = \Sigma x^2 n_x = 248$. Then the estimate is

$$t = \frac{1}{n-1}\left[\frac{S_2}{S_1} - 1\right]$$

$$= \frac{1}{4}\left[\frac{248}{110} - 1\right] = 0.3136 .$$

To compute the variance of t we require:

$$\mu^* = \frac{n\pi}{1-(1-\pi)^n} = 1.95280$$

$$m_2^* = \mu^*\left[(n-1)\pi + 1\right] = 4.40239$$

$$m_3^* = \mu^*\left[(n-1)\pi+1 + (n-1)\pi\left\{(n-2)\pi + 2\right\}\right]$$

$$= 11.60615$$

and

$$m_4^* = \mu^*\Big[(n-1)\pi+1 + 3(n-1)\pi\left\{(n-2)\pi+2\right\}$$

$$+ (n-1)\pi.\ (n-2)\pi\left\{(n-3)\pi+3\right\}\Big]$$

all evaluated by taking 0.3136 as the estimate for π. The variance of t is estimated from the formula

$$\text{Var}(t) = \frac{1}{N\{(n-1)\mu^*\}^2}\Big[(m_4^* - m_2^{*2}) + \{(n-1)\pi+1\}^2\ (m_2^*-\mu^{*2})$$

$$- 2.\{(n-1)\ \pi+1\}\ (m_3^*-\mu^* m_2^*)\Big]$$

$$= 0.0012066$$

so that the standard error is S.E. (t) = 0.03474.

<u>Ratio Estimate</u>: The ratio estimate for π is given by

$$\pi' = \frac{t_1}{t_1+t_2}$$

where

$$t_1 = \sum_{x=2}^{n} \left(\frac{x}{n-x+1}\right) n_x$$

and

$$t_2 = \sum_{x=1}^{n-1} n_x .$$

Here $t_2 = 59$ and t_1 can be computed from the following table:

x	$\frac{x}{n-x+1}$	n_x
2	5	23
3	1	10
4	2	1
5	5	1

Thus, $t_1 = 28.50$.

The ratio estimate is obtained as $\pi' = \frac{28.50}{28.50 + 59} = 0.3257$.

To compute the variance of π', we require

$$P = 1 - \frac{\pi^n}{1-(1-\pi)^n} = 0.99574$$

$$P_{n-1} = \frac{n(1-\pi)}{\pi}(1-P) = 0.04408$$

$$\mu_2^* = \frac{n\pi}{1-(1-\pi)^n}\left[(n-1)\pi+1-\frac{n\pi}{1-(1-\pi)^n}\right]$$

$$= 0.86052 ,$$

and

$$D = \left(\frac{\pi}{1-\pi}\right)\left[\frac{n+1}{1-(1-\pi)^n}\left\{(1 - \pi^n)\ E\left(\frac{1}{X},\ n,\ 1-\pi\right) - \frac{(1-\pi)^n}{n}\right\} - P\right]$$

$$= 0.56156 .$$

The quantity $E(\frac{1}{X},n,1-\pi)$ was obtained from the table by Grab and Savage, which gives for $n = 5$,

$1 - \pi$	$E(\frac{1}{X},n,1-\pi)$
65	0.35465
70	0.32183

By interpolation, $E(\frac{1}{X},n,1-\pi) = 0.33870$ taking $\pi' = 0.3257$ as the estimate for π throughout. Then the variance of π' is estimated from the formula

$$\text{Var}(\pi') = \frac{P^2}{\mu_2^*}\left[\left(\frac{1-\pi}{\pi}\right)^2 D - P + 2P_{n-1}\right]^{-1}$$

$$= 0.0013410$$

so that the standard error of π' is S.E. $(\pi') = 0.03662$. The following table summarizes the results obtained:

Estimate	Value	Variance	Standard Error
ML	0.3088	0.0010305	0.03210
TM	0.3136	0.0012066	0.03474
R	0.3257	0.0013410	0.03662

4.3 Homogeneity and Combined Estimation for Singly Truncated Binomial Distributions

While sampling for studies in albinism, observations are available simultaneously from families with varying family-size. In such situations, one may be required to examine if the distributions (in the case of albinism families) are homogeneous in respect of the parameter π and if so, to make a combined estimate of π.

4.3.1 On the basis of a random sample of size $N = \sum_{j=1}^{k} N_j$ from k singly truncated binomial distributions characterized by the probability law:

$$\left[B^*(c, \pi, n_j)\right]^{-1} \binom{n_j}{x} \pi^x (1-\pi)^{n_j - x} \qquad x = 1, 2, \ldots n_j \tag{4.3.1}$$

$j = 1, 2, \ldots k$, following Section 2.5 .

The j-th "efficient score" is

$$\psi_j = \frac{N_j}{\pi_j(1-\pi_j)}\left[\bar{x}_j - \mu_j(\pi_j)\right] \tag{4.3.2}$$

where $\mu_j(\pi_j)$ is the mean of the j-th distribution.

The elements of the information matrix are

$$I_{jj} = \frac{N_j}{\pi_j(1-\pi_j)} \frac{d\mu_j}{d\pi_j} = \frac{N_j \, \mu_{2j}(\pi_j)}{[\pi_j(1-\pi_j)]^2}$$

$$I_{jj} = 0 \quad j \neq j' \tag{4.3.3}$$

where $\mu_{2j}(\pi_j)$ is the variance of the j-th distribution.

The hypothesis of homogeneity is H_o: $\pi_1 = \pi_2 = \ldots \pi_k$.

If the hypothesis H_o is true, the common value may be denoted by π and the efficient score and the information with respect to π are given by:

$$\psi = \frac{N}{\pi(1-\pi)} [\bar{\bar{x}} - \frac{1}{N} \cdot \sum_{j=1}^{k} N_j \mu_j(\pi)] \qquad (4.3.4)$$

where

$$\bar{\bar{x}} = \sum_{j=1}^{k} N_j \bar{x}_j / N \qquad (4.3.5)$$

and

$$I = \frac{1}{\pi(1-\pi)} \sum_{j=1}^{k} N_j \cdot \frac{d\mu_j}{d\pi} \qquad (4.3.6)$$

$$= \frac{1}{[\pi(1-\pi)]^2} \sum_{j=1}^{k} N_j \mu_{2j} \, . \qquad (4.3.7)$$

To solve the equation $\psi = 0$ for π, we may start with an approximation π_o and derive a better approximation π_1 from the formula

$$\pi_1 = \pi_o + N [\bar{\bar{x}} - \frac{1}{N} \sum N_j \mu_j(\pi_o)] / \sum N_j (\frac{d\mu_j}{d\pi})_{\pi_o} \qquad (4.3.8)$$

Or

$$\pi_1 = \pi_o + N\pi_o(1-\pi_o) [\bar{\bar{x}} - \frac{1}{N} \sum N_j \mu_j(\pi_o)] / \sum N_j \mu_{2j}(\pi_o) \qquad (4.3.9)$$

and repeat this process of iteration till sufficient accuracy is attained. This maximum likelihood estimate will be denoted by $\hat{\pi}$.

A test of the homogeneity hypothesis H_o is then given by the statistic

$$\chi^2_{k-1} = \sum_{j=1}^{k} \psi_j(\hat{\pi})^2 / I_{jj}(\hat{\pi})$$

$$= \frac{1}{\hat{\pi}(1-\hat{\pi})} \sum_{j=1}^{k} [N_j \{\bar{x}_j - \mu_j(\pi)\}^2] / [\frac{d\mu_j(\pi)}{d\pi}]_{\hat{\pi}} \qquad (4.3.10)$$

$$= \sum_{j=1}^{k} N_j [\bar{x}_j - \mu_j(\hat{\pi})]^2 / \mu_{2j}(\hat{\pi}) \, , \qquad (4.3.11)$$

which is asymptotically distributed as a Chi-square with (k-1) df if H_o is true.

4.3.2 We shall illustrate the computational technique and the use of the tables with reference to the problem of estimating the proportion of albino children from K. Pearson's (1913) data quoted below:

No. of children in family n_j	No of families N_j	Total number of albino children in the families T_j
2	40	49
3	55	76
4	50	85
5	60	110
6	53	116
7	46	103
8	27	77
9	29	73
10	20	52
11	14	50
12	8	28
13	4	19
14	4	16
15	1	10
Total	N = 411	T = 864

To get the first approximation, we compute the values of $\bar{x}_j/n_j = T_j/N_j n_j$ and referring to the tables or to the charts obtain for each household-size an estimate of π_j. This is done in columns (3) and (4) of Table 4.3.1 .

We find the estimates clustering around $\pi_o = 0.30$ which value we take as the starting point of our computations.

The next step is to read off from the tables the mean values $\mu_j^*(\pi_o)$ and the difference-ratios $\delta_j = [\Delta\mu_j^* / \Delta\pi]_{\pi_o}$ for different values of n_j's. These are shown in columns (5) and (6) of Table 4.3.1 .

TABLE 4.3.1

COMPUTATIONAL PROCEDURE FOR HOMOGENEITY AND COMBINED ESTIMATION

(1) n_j	(2) N_j	T_j	$\bar{x}_j$	(3) $\bar{x}_j/n_j$	(4) π_j (See Below*)	(5) $\mu_j^*(\pi_o)$	(6) $\left(\frac{\Delta\mu_j^*}{\Delta\pi}\right)_{\pi_o}$	$\mu_j(\pi_1)$ (See Below**)	$\left(\frac{d\mu_j}{d\pi}\right)_{\pi_1}$ (See Below**)	$\frac{N_j\{\bar{x}_j-\mu_j(\hat{\pi})\}^2}{\left(\frac{d\mu_j(\pi)}{d\pi}\right)_{\hat{\pi}}}$
2	40	49	1.22500	.611	.36	1.17647	0.699	1.18217	0.69878	0.1050
3	55	76	1.38182	.461	.31	1.36986	1.512	1.38224	1.51804	0.0000
4	50	85	1.70000	.425	.35	1.57916	2.430	1.59905	2.44156	0.2087
5	60	110	1.83333	.367	.31	1.80304	3.433	1.83115	3.44952	0.0001
6	53	116	2.18868	.351	.31	2.04001	4.500	2.07686	4.52109	0.1466
7	46	103	2.23913	.320	.29	2.28847	5.612	2.33443	5.63611	0.0741
8	27	77	2.85185	.354	.34	2.54682	6.750	2.60211	6.77694	0.2485
9	29	73	2.51724	.278	.26	2.81354	7.901	2.87826	7.92879	0.4767
10	20	52	2.60000	.260	.25	3.08721	9.052	3.16137	9.08069	0.6941
11	14	50	2.67143	.325	.32	3.36657	10.197	3.45012	10.22481	0.0201
12	8	28	3.50000	.291	.29	3.65053	11.331	3.74338	11.35621	0.0417
13	4	19	4.75000	.365	.36	3.93816	12.448	4.04018	12.47266	0.1616
14	4	16	4.00000	.286	.29	4.22867	13.552	4.33975	13.57311	0.0340
15	1	10	10.00000	.667	.67	4.52146	14.641	4.64146	14.65803	1.9589
Total	411	864								4.1701
Weighted average			2.10219			2.06462	4.57352	2.10207	4.32332	

$\pi_1 = 0.30 + \frac{2.10219 - 2.06462}{4.57352} = 0.3082$

$\pi_2 = .3082 + .00002 = 0.3082$

$\pi = 0.3082$ $\chi^2_{13} = 19.558$

$V(\hat{\pi}) = 0.000113$

$S.E.(\hat{\pi}) = 0.0106$

* From Table I

** Computed directly from the formulae $\mu_j(\pi) = \frac{n_j\pi}{1-(1-\pi)^{n_j}}$; $\frac{d\mu_j}{d\pi} = \frac{\mu_j}{\pi(1-\pi)}[1+(n_j-1)\pi-\mu_j]$

The details of the computation process are the same as on the illustrative example in Section 4.2 .

The next step is to compute the weighted averages.

$$\bar{\bar{x}} = \Sigma N_j \bar{x}_j / N = 2.10219$$

$$\bar{\mu} = \Sigma N_j \mu_j^* / N = 2.06462$$

and

$$\bar{\delta} = \Sigma N_j \delta_j / N = 4.57352 .$$

Then next approximation to the maximum likelihood estimate is thus:

$$\pi_1 = \pi_o + \frac{\bar{\bar{x}} - \bar{\mu}}{\bar{\delta}} = 0.3082$$

the same as obtained by Haldane (1938) who wrote down the likelihood equation in the form

$$\frac{T}{\pi} = \sum_j \frac{n_j N_j}{1-(1-\pi)^{n_j}}$$

and solved it directly by iteration. One single computation in our case is thus sufficient. The variance of the estimate is approximately given by:

$$V(\hat{\pi}) = \frac{\pi(1-\pi)}{N \bar{\delta}}$$

$$= \frac{0.3082 \times 0.6918}{411 \times 4.57352} = 0.000113$$

so that S.E. $(\hat{\pi}) = 0.0106$. The values for S.E. obtained by Haldane is 0.0107 which is the same for all practical purposes. To test if the proportion of albino children is the same in families of different sizes, we have to compute

$$\chi^2_{13} = \frac{1}{\hat{\pi}(1-\hat{\pi})} \Sigma \frac{N_j[\bar{x}_j - \mu^*(\hat{\pi})]^2}{\left[\frac{d\mu_j}{d\pi}\right]_\pi} .$$

But since the maximum likelihood extimate $\hat{\pi} = 0{\cdot}3082$ does not differ very much from our starting approximation $\pi_o = 0{\cdot}30$ we may use the approximation

$$\left[\frac{d\mu_j^*}{d\pi}\right]_{\hat{\pi}} \sim \left[\frac{\Delta\mu_j^*}{\Delta\pi}\right]_{\pi_o} = \delta_j$$

which are already computed. Thus

$$\chi^2_{13} = \frac{1}{0{\cdot}3082 \times 0{\cdot}6918} \sum \frac{N_j(\bar{x}_j - \mu_j^*)^2}{\delta_j}$$

$$= \frac{4.1701}{0{\cdot}3082 \times 0{\cdot}6918} = 19{\cdot}558$$

which with 14-1 = 13 degrees of freedom is not significant. The families of different sizes can thus be regarded as homogeneous in respect of the proportion of albino children and the common proportion is 0.3082 ± 0.0106.

4.4 Estimation from a Sample for a Doubly Truncated Binomial Distribution

In studying albinism, sampling may be restricted to only those families which contain at least one albino child and also at least one non-albino, Finney (1949), giving thus rise to samples from doubly truncated binomial distribution. We discuss the case of general truncation here and present some numerical tables to facilitate the heavy computation involved in evaluating the maximum likelihood estimate of π from a sample from a binomial of which only extremes are truncated. The simple "ratio-estimate" is also derived and its efficiency is investigated for this special case of practical importance.

4.4.1 The probability law of a doubly truncated binomial with truncation points, say, at c and d can be written as:

$$b^* (x, \pi, n) = [B^* (c, d, \pi, n)]^{-1} \cdot \binom{n}{x} \pi^x (1-\pi)^{n-x} \qquad (4.4.1)$$

$$x = c, c+1, \ldots d$$

where

$$B^* (c, d, \pi, n) = B (d, \pi, n) - B (c-1, \pi, n). \qquad (4.4\ 2)$$

The first two moments about the origin is (4.4.1) are

$$\mu^* = \mu^*(c, d, \pi, n) = n\pi.\ B^*(c-1, d-1, \pi, n-1) / B^*(c, d, \pi, n) \qquad (4.4.3)$$

and

$$m_2^* = m_2^*(c, d, \pi, n) = \mu^*(c, d, \pi, n). [1+\mu^*(c-1, d-1, \pi, n-1)]. \qquad (4.4.4)$$

To estimate π on the basis of a random sample of size N with frequency n_x for x drawn from (4.4.1), the likelihood equation for π can be written down as:

$$\frac{\bar{x}}{n} = \frac{\hat{\mu}^*}{n} \qquad (4.4.5)$$

and the asymptotic variance of the estimate $\hat{\pi}$ obtained from (4.4 5) is

$$\text{Var}\ (\hat{\pi}) = \frac{\pi(1-\pi)}{N} \Big/ \frac{d\mu^*}{d\pi} \qquad (4.4.6)$$

$$= \frac{[\pi\ (1-\pi)]^2}{N\mu_2^*}. \qquad (4.4.7)$$

To facilitate the solution of (4.4.5) in the special case when c = 1 and d = n-1 i.e extreme observations truncated, we present tables at suitable intervals of π for μ^*/n which reduces in this case to:

$$\mu^*/n = \mu^*(1, n-1, \pi, n) = (\pi - \pi^n)/[1-\pi^n - (1-\pi)^n]. \qquad (4.4.8)$$

An approximate value of Var ($\hat{\pi}$) may be obtained by getting $\frac{d\mu^*}{d\pi} \sim \frac{\Delta\mu^*}{\Delta\pi}$ from tables of μ^*_n or else to get the exact value, one can use

$$\mu^*_2(1,n-1,\pi,n) = \mu^*(1,n-1,\pi,n)\ [1+\mu^*\ (o,n-2,\pi,n-1) - \mu^*(1,n-1,\pi,n)] \qquad (4.4.9)$$

where values of $\mu^*(o,n-2,\pi,n-1)$ can be obtained from Table I for single truncation.

4.4.2 Unlike cases of single truncation, it may be of interest to note that the two-moments estimate is not available here. This follows from Section 3.4 , Case 4 .

4.4.3 Following the general approach discussed in Section 3.1 , the simple "ratio-estimate" for π can be written down for $c = 1$ and $d = n-1$ as:

$$\pi' = \frac{t_1}{t_1+t_2} \qquad (4.4.10)$$

where

$$t_1 = \sum_{x=2}^{n-1} \left(\frac{xn_x}{n-x+1}\right)$$

and

$$t_2 = \sum_{x=1}^{n-2} n_x \quad .$$

The asymptotic variance of π' then takes the form:

$$\text{Var}\ (\pi') = \frac{(1-\pi)^2}{NP^2}\ [(1-\pi)^2\ D - P\pi^2 + 2\pi^2.P_{n-2}]$$

where

$$P_x = \binom{n}{x}\ \pi^x(1-\pi)^{n-x}/\ [1-\pi^n - (1-\pi)^n]$$

$$P = \sum_{x=1}^{n-2} P_x \ ,$$

and

$$D = \sum_{x=2}^{n-1} \left(\frac{x}{n-x+1}\right)^2 P_x$$

which reduces, in this case, to

$$D = \left(\frac{\pi}{1-\pi}\right)\left[(n+1)\left\{\left(1-\frac{(1-\pi)^n}{1-\pi^n}\right)^{-1} E\left(\frac{1}{x}, n, 1-\pi\right) - \left(\frac{P_{n-1}}{n-1}+\frac{P_n}{n}\right)\right\} - P\right]$$

where

$$E\left(\frac{1}{x}, n, \pi\right) = \sum_{n=1}^{n} \frac{1}{x} \binom{n}{x} \pi^x (1-\pi)^{n-x} / \left\{1-(1-\pi)^n\right\} .$$

4.5 Simultaneous Estimation of Both Parameters of a Binomial Distribution

The binomial distribution has essentially two parameters π and n of which n is usually known and only π has to be estimated. However, certain cases might arise in which n is unknown, and both n and π have to be estimated. For instance, while experimenting with a radioactive substance, in addition to the mean number ($\mu = n\pi$) of disintegrating atoms, it may perhaps be of interest to know the number (n) of atoms capable of disintegration for the substance in fixed intervals of time for some specified solid angle and fit a model correspondingly.

4.5.1 To estimate π and n on the basis of a random sample of size N with observed frequency n_x for x ($\Sigma n_x = N$) drawn from (4.2.1) with n unknown, following Section 2.6, the moment-estimates are given by

$$\bar{x} = n\pi \qquad (4.5.1)$$

and

$$S^2 = n\pi(1-\pi), \qquad (4.5.2)$$

where $\bar{x} = \Sigma x n_x / N$ and $S^2 = \Sigma n_x(x - \bar{x})^2/N$.

The likelihood equations reduce to

$$\bar{x} = \hat{n}\hat{\pi} \tag{4.5.3}$$

and

$$\sum_{r \geq 0} \frac{T_{r+1}}{\hat{n}-r} + N \log (1 - \hat{\pi}) = 0 \tag{4.5.4}$$

where

$$T_{r+1} = \sum_{x>r} n_x \quad .$$

Eliminating $\hat{\pi}$ from (4.5.3) and (4.5.4) we have to solve for $\hat{n}$, the equation:

$$\sum_{r \geq 0} \frac{T_{r+1}}{\hat{n}-r} + N \log (1 - \frac{\bar{x}}{\hat{n}}) = 0 \quad . \tag{4.5.5}$$

The elements of the information matrix are:

$$\begin{aligned} I_{11} &= nN/\pi(1-\pi) \\ I_{12} &= N/(1-\pi) \\ I_{22} &= E[\Sigma\, T_{r+1}/(n-r)^2] = N \,\Sigma\, [1-B(r,\pi,n)]/(n-r)^2 . \end{aligned} \tag{4.5.6}$$

We note that n is a discrete parameter and also the range of (4.2.1) depends on n. The properties of the estimates are therefore not known, but in the (4.5.1) and (4.5.2) worked out, however, fairly accurate results are obtained.

4.5.2 To estimate π and n on the basis of a random sample of size N with observed frequency n_x for x ($\Sigma n_x = N$) drawn from a truncated binomial, say given by (4.4.1) the moment-equations are:

$$\bar{x} = \mu^* \tag{4.5.7}$$

and

$$S^2 = m_2^* \quad , \tag{4.5.8}$$

where $\bar{x} = \Sigma x n_x/N$, $S^2 = \Sigma x^2 n_x/N$ and μ^* and m_2^* are defined by (4.4.3) and (4.4.4) respectively.

The "efficient scores" for π and n reduce to:

$$\psi_1 = \frac{N}{\pi(1-\pi)} (\bar{x} - \mu^*) \tag{4.5.9}$$

and

$$\psi_2 = \sum_{r\geq 0} \frac{T_{r+1}}{n-r} + N \log (1-\pi) - N \frac{\partial B^*}{\partial n} / B^* \tag{4.5.10}$$

where B*'s are defined by (4.4.2).

The likelihood equations then become

$$\bar{x} = \hat{\mu}^* \tag{4.5.11}$$

and

$$\hat{\psi}_2 = 0 \ . \tag{4.5.12}$$

The elements of "information matrix" are

$$I_{11} = \frac{N}{\pi(1-\pi)} \cdot \frac{\partial\mu^*}{\partial\pi}$$

$$I_{12} = \frac{N}{\pi(1-\pi)} \cdot \frac{\partial\mu^*}{\partial n}$$

and

$$I_{22} = [\frac{\partial^2}{\partial n^2} B^*/B^* - (\frac{\partial}{\partial n} B^*/B^*)^2] + E\left[\Sigma \frac{T_{r+1}}{(n-r)^2}\right]$$

(4.5.11) and (4.5.12) may be solved for estimation, approximating

$$\frac{\partial}{\partial n} B(r,\pi,n) \sim \frac{\Delta B}{\Delta n}$$

and getting $\frac{\Delta B}{\Delta n}$ from binomial tables where B is defined by (4.1.7).

However, exact values of $\frac{\partial}{\partial n} B(r,\pi,n)$ and $\frac{\partial^2}{\partial n^2} B(r,\pi,n)$ which we shall call "Incomplete Dibeta and Tribeta Function" respectively, can be obtained as follows:

4.5.3 $$\frac{\partial}{\partial n} B\,(r,\pi,n) = \frac{\partial}{\partial n} I_{1-\pi}\,(n-r,\ r+1)$$

$$= I_{1-\pi}\,(n-r,\ r+1)\,[E_z(n-r,\ r+1)]_{z=1}^{z=1-\pi} \qquad (4.5.14)$$

where I's are incomplete beta functions, and

$$E_z(n,m) = \frac{\int_0^z \log u \cdot u^{n-1}\,(1-u)^{m-1}\,du}{\int_0^z u^{n-1}\,(1-u)^{m-1}\,du} \qquad (4.5.15)$$

which means the expected value of log u when u follows a beta distribution truncated on the right at z, with parameters n and m. $E_z\,(n,m)$ can be reduced to

$$E_z(n,m) = \log z - \frac{1}{I_z\,(n,m)} \sum_{r=0}^{m-1} \frac{I_z\,(n+r,\ m-r)}{n+r}\ . \qquad (4.5.16)$$

m integer

In particular,

$$E_1\,(n,m) = \sum_{r=0}^{m-1} \frac{1}{n+r} \qquad (4.5.17)$$

(4.5.16) suggests that the values of "Incomplete Dibeta Function" can be exactly obtained by using tables of "Incomplete Beta Funtion" which are extensively tabulated.

To obtain "Incomplete Tribeta Function"

$$\frac{\partial^2}{\partial n^2} B(r,\pi,n) = \frac{\partial}{\partial n}\ I_{1-\pi}(n-r,\ r+1)\ [E_z(n-r,\ r+1)]_{z=1}^{z=1-\pi}\ . \qquad (4.5.18)$$

We get after some simplification of R.H.S. of (4.5.18),

$$\frac{\partial^2}{\partial n^2} B(r,\pi,n) = I_{1-\pi}(n-r,\ r+1)\left(\left\{[E_z(n-r,\ r+1)]_1^{1-\pi}\right\}^2 + [V_z(n-r,\ r+1)]_1^{1-\pi}\right) \qquad (4.5.19)$$

where V_z (n,m) is the variance of log u when u follows a beta distribution with parameters n and m truncated on the right at z.

V_z (n,m) can be obtained from

$$V_z(n,m) = E_z^2(n,m) - [E_z(n,m)]^2 \qquad (4.5.20)$$

where E_z^2 (n,m) can be reduced to

$$E_z^2\ (n,m) = (\log z)^2 - \frac{2}{I_z\ (n,m)} \sum_{r=o}^{m-1} \frac{I_z\ (n+r,\ m-r)}{n+r} E_z\ (n+r,\ n-r). \qquad (4.5.21)$$

In particular,

$$E_1^2\ (n,m) = 2 \sum_{r=o}^{m-1} \left(\sum_{r=o}^{m-1} \frac{1}{n+r} \right)/(n+r)\ . \qquad (4.5.22)$$

4.5.4 The computation procedure for simultaneous estimation will be illustrated with reference to two examples: one on radioactive disintegrations and the second one on throwing of dice.

Example 4.5.1 The first two columns of the following table give data collected by Rutherford and others, showing the number (n_x) of intervals of time, each of 7.5 seconds, during which the number (x) of α particles omitted from a certain radioactive substance.

Data: Rutherford and Geiger: Radioactive Disintegration

No. of α particles	No. of intervals	$T_x = \sum_{s\geq x} n_s$	$\frac{Tx}{n - x + 1}$	
x	n_x	T_x	n = 77	n = 79
0	57	2608	—	—
1	203	2551	33.1299	32.2911
2	383	2348	30.8947	30.1026
3	525	1965	26.0000	25.5195
4	532	1440	19.4594	18.9474
5	408	908	12.4384	12.1067
6	273	500	6.9444	6.7568
7	139	227	3.1972	3.1096
8	45	88	1.2571	1.2222
9	27	43	0.6232	0.6056
10	10	16	0.2353	0.2286
11	4	6	0.0896	0.0870
12	2	2	0.0303	0.0294
Total	2608		134.2995	131.0065

$\sum xn_x = 10094$

$\bar{x} = 3.870$

$\sum x^2 n_x = 48650$

$s^2 = 3.676$

$n = 77$

N log n - log(n-$\bar{x}$)

134.4390   130.9693

ψ

-0.1395   +0.0372

$\hat{n} = 78$.

Moment Estimates: Following Section 4.5 , we have for this data,

$$N = 2608$$

$$\bar{x} = \frac{\sum xn_x}{N} = 3.870$$

$$s^2 = \frac{\sum(x-\bar{x})^2 n_x}{N} = 3.676$$

so that the estimate for the mean number μ of α-particles emitted per interval is

$$\mu = \bar{x} = 3.870$$

and the number (n) of particles capable of disintegration for the substance during the interval of 7.5 seconds is estimated by

$$n = \frac{\bar{x}^2}{\bar{x} - s^2} = 77.$$

Maximum likelihood estimates: The estimate for the mean number of α-particles per interval remains the same, namely $\hat{\mu} = 3.870$. To get the estimate $\hat{n}$ of n, starting with the moment estimate n = 77, we solve the equation:

$$\psi(n) = \sum_{r \geq 0} \frac{T_{r+1}}{n-r} - N\,[\log n - \log (n-\bar{x})] = 0 \,.$$

For n = 77, we have $N\,[\log n - \log (n-\bar{x})] = 134.4390$. From column 4 of the above table, we have for n = 77,

$$\sum_{r \geq 0} \frac{T_{r+1}}{n-r} = \sum_{x > 1} \frac{T_x}{n-x+1} = 134.2995$$

$$\therefore \quad \psi(77) = 134\ 2995 - 134.4390$$

$$= -\ 0.1395\,.$$

Let us try next n = 79, say. Now, $N\,[\log n - \log(n-\bar{x})] = 130.9693$ and column 5 of the above table gives for n = 79.

$$\sum_{r \geq 0} \frac{T_{r+1}}{n-r} = 131.0065$$

$$\therefore \quad \psi(79) = 0.0372$$

Thus, whereas $\psi(77)$ is negative, $\psi(79)$ is positive and therefore the likelihood estimate for n is $\hat{n} > 77$ and < 79.

$$\therefore \quad \hat{n} = 78.$$

Example 4.5.2 The first two columns of the following table give data, due to Weldon, that show the results of throwing n dice 4096 times, a throw of 4, 5 or 6 being called a success. x denotes the number of successes and n_x the frequency of x.

Successes	Frequency	$T_x = \sum_{s \geq x} n_s$	$\frac{T_x}{n - x+1}$		
x	n_x	T_x	$n = 12$	$n = 13$	$n = 11$
0	0	4096	—	—	—
1	7	4096	371.7273	315.0769	372.3636
2	60	4089	371.7273	340.7500	408.9000
3	198	4029	402.9000	366.2727	447.6666
4	430	3831	425.6667	383.1000	478.8750
5	731	3401	425.1250	377.8888	485.8571
6	948	2670	381.4285	333.7500	445.0000
7	847	1722	287.0000	245.0000	344.4000
8	536	875	175.0000	145.8333	218.7500
9	257	339	84.7500	67.8000	113.0000
10	71	82	27.3333	20.5000	41.0000
11	11	11	5.5000	3.6666	11.0000
Total	4096		2927.7641	2600.6383	3366.8123

$\sum x n_x = 25145$ $\qquad$ $N \log n - \log (n-\bar{x})$

$\bar{x} = 6.139$ $\qquad$ 2935.1030 $\quad$ 2617.7000 $\quad$ 3344.9000

$\sum x^2 n_x = 166367$ $\qquad$ ψ

$s^2 = 2.930$ $\qquad$ -7.3389 $\quad$ -17.0617 $\quad$ + 21.9123

$n = 12$. $\qquad$ $\hat{n} = 12$.

Moment Estimates: Following Section 4.5 , we have for Example (4.5.2)

$$N = 4096$$

$$\bar{x} = 6.139$$

$$s^2 = 2.930$$

so that the estimate for the number of dice thrown is given by

$$n = \frac{\bar{x}^2}{\bar{x}-s^2} = 12 ,$$

and the estimate of the proportion of successes (π) is

$$\pi = \frac{\bar{x}}{n} = \frac{6.139}{12} = 0.5116$$

Maximum likelihood estimates: To get firstly the estimate $\hat{n}$ of n, starting with the moment-estimate n = 12, we solve the equation:

$$\psi(n) = \sum_{r \geq o} \frac{T_{r+1}}{n-r} - N[\log n - \log (n-\bar{x})] = 0 \ .$$

For n = 12, we have N[log n - log $(n-\bar{x})$] = 2935.1030. From column 4 of the above table, we have for n = 12,

$$\sum_{r \geq o} \frac{T_{r+1}}{n-r} = \sum_{x > 1} \frac{T_x}{n-x+1} = 2927.7641$$

$$\therefore \quad \psi(12) = 2927.7641 - 2935.1060$$

$$= -7.3389 \ .$$

Let us try next n = 13, say, to see if $\psi(13)$ is near zero. We get by proceeding as before, $\psi(13)$ = -17.0617 which is further from zero than $\psi(12)$. Therefore, we try n = 11. We have then $\psi(11)$ = 21.9123 which indicates that $\hat{n}$ = 12 The estimate of π is then obtained by $\hat{\pi} = \frac{\bar{x}}{\hat{n}}$ = 0.5116. (Note: Weldon had thrown 12 dice).

CHAPTER V

5.0 ESTIMATION PROBLEMS FOR THE POISSON DISTRIBUTION

5.1 Introduction

The gpsd defined by (1.1.4) becomes

$$\text{Prob}\{X = x\} = \frac{1}{x!}\,\theta^x/e^\theta \qquad x = 0,1,2,\ldots\infty \qquad (5.1.1)$$

when $f(\theta) = e^\theta$. Writing $\mu = \theta$, (1.1.4) gives the probability law for X as:

$$\text{Prob}\{X = x\} = p(x,\mu) = e^{-\mu}.\,\frac{\mu^x}{x!}\,, \qquad (5.1.2)$$

$$x = 0,1,2,\ldots\infty$$

the well-known form of the Poisson distribution.

The important properties of (5.1.2) can be summarily stated as follows:

$$M(t) = e^{\mu(e^t-1)}. \qquad (5.1.3)$$

The first two central moments and the coefficients β_1,β_2 are of the form:

$$\mu = \mu$$

$$\mu_2 = \mu$$

$$\beta_1 = 1/\mu$$

$$\beta_2 = 3 + 1/\mu\,. \qquad (5.1.4)$$

The recurrence relation connecting the central moments is

$$\mu_{r+1} = \mu\left(\frac{d\mu_r}{d\mu} + r\mu_{r-1}\right) \qquad (5.1.5)$$

and all the cumulants are equal

$$K_r = \mu \qquad r = 1,2,\ldots \qquad (5.1.6)$$

It is well-known that the equality of all cumulants is necessary and sufficient for a probability distribution to become Poisson. For any gpsd to be Poisson, however, the equality of first two cumulants only is necessary and sufficient (Chapter I, Theorem 3).

The distribution function $P(r,\mu)$ defined by

$$P(r,\mu) = \sum_{x=0}^{r} p(x,\mu) \tag{5.1.7}$$

can be reduced to

$$P(r,\mu) = I_\mu(r+1) \tag{5.1.8}$$

where

$$I_x(r) = \int_x^\infty e^{-u} \quad u^{r-1}du/\int_0^\infty e^{-u} \quad u^{r-1}du$$

is the incomplete gamma integral tabulated by K. Pearson. Molina (1947) has extensively tabulated $p(x,\mu)$ and $1-P(r,\mu)$ for the range of argument μ = 0.001 (0.0001) .010(.01).30(.1) 15(1) 100. Kitagawa (1952) has also edited "Tables of Poisson Distribution." For large μ, we have the normal approximation given by:

$$\lim_{\mu\to\infty} P(r,\mu) = \Phi(z) \tag{5.1.9}$$

where

$$z = (r + 1/2 - \mu)/\sqrt{\mu}$$

and

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-1/2 \ u^2} du.$$

On the basis of a random sample x_i $(i = 1,2,...N)$ of size N from (5.1.2), both the moments and maximum likelihood estimate for μ is given by

$$\hat{\mu} = \bar{x} \tag{5.1.10}$$

where

$\bar{x} = \sum_{i=1}^{N} x_i/N$. (5.1.10) provides an unbiased estimate of μ with Var $(\hat{\mu}) = \mu/N$.

5.2 Estimation from a Sample for Truncated Poisson Distribution

Problems of estimation in a truncated Poisson distribution with known truncation points have been discussed by various authors. The case of truncation on the left has been considered by David and Johnson (1948) who gave the maximum likelihood estimate, by Plackett (1953) who gave a simple and highly efficient ratio-estimate, and by Rider who used first two moments. Truncation on the right has been discussed by Tippett (1932), Bliss (1948), and Moore (1952). Tippett derived the maximum likelihood solution, Bliss developed an approximation to it, and Moore suggested a simple ratio estimate. Double Truncation has been studied by Moore (1954) and Cohen (1954). Moore gave ratio-estimates, while Cohen provided likelihood equations.

Neat and compact equations for estimation by the method of maximum likelihood (which has been shown to be identical with the method of moments, in general, for single-parameter gpsd's) can be derived from the general approach discussed in Chapter II. We present numerical tables and some suitable charts to facilitate the solution of these equations in certain special cases. The estimates given by Rider and Moore have been derived from the general results discussed in Chapter III. The efficiency and the amount of bias of these estimates are investigated in some cases. Problem of estimation has been also considered for single truncation with unknown truncation point.

5.2.1 The probability law of the singly truncated Poisson Distribution with truncation point on the right at d can be written as:

$$p^*(x,\mu) = [P(d,\mu)]^{-1} e^{-\mu} \frac{\mu^x}{x!} \qquad (5.2.1)$$

$$x = 0,1,2,\ldots d$$

where $P(r,\mu)$ is defined by (5.1.7).

The first four moments about the origin of (5.2.1) can be written down as:

$$\mu^* = \mu^*(d,\mu) = \mu\ P(d-1,\mu)/P(d,\mu) \tag{5.2.2}$$

$$m_2^* = m_2^*(d,\mu) = \mu^*(d,\mu)\left[1 + \mu^*(d-1,\mu)\right] \tag{5.2.3}$$

$$m_3^* = m_3^*(d,\mu) = \mu^*(d,\mu)\left[1 + 2\mu^*(d-1,\mu) + m_2^*(d-1,\mu)\right]$$

and

$$m_4^* = m_4^*(d,\mu) = \mu^*(d,\mu)\Big[1 + 3\mu^*(d-1,\mu) + 3m_2^*(d-1,\mu) + m_3^*(d-1,\mu)\Big].$$

5.2.2 To estimate μ on the basis of a random sample x_i ($i = 1,2,\ldots N$) of size N from (5.2.1), the results derived by the general approach in Chapter II can be written down as follows.

The likelihood equation for μ is

$$\bar{x} = \hat{\mu}^* \tag{5.2.4}$$

where

$$\bar{x} = \sum_{i=1}^{N} x_i/N \text{ and } \mu^* \text{ is defined by (5.2.2).}$$

Denoting this estimate as $\hat{\mu}$, the asymptotic variance is given by

$$\text{Var}\ (\hat{\mu}) = \frac{\mu}{N}\Big/\left(\frac{d\mu^*}{d\mu}\right) \tag{5.2.5}$$

$$= \frac{\mu^2}{N\mu_2^*} \tag{5.2.6}$$

where μ_2^* is the variance of (5.2.1). Equation (5.2.4) suggests that if tables be made available for means μ^*s' for sufficiently close values of μ, we can have a ready solution of (5.2.4). We present in

Table IV a numerical table for the arguments $\mu = 0.0(.1)4.9$ and $d = 4(1)10$.

This table can be used to compute Var $(\hat{\mu})$ by using Formula (5.2.5) or Formula (5.2.6). In case (5.2.5) is used, $\frac{d\mu^*}{d\mu}$ can be approximated by the finite difference ratio $\frac{\Delta\mu^*}{\Delta\mu}$. In the event Formula (5.2.6) is used, the relationship for use is

$$\mu_2^*(d,\mu) = \mu^*(d,\mu)\left[1 + \mu^*(d-1,\mu) - \mu^*(d,\mu)\right]. \qquad (5.2.7)$$

5.2.3 The method of two moments is applicable in the usual problem of estimation from a sample from a Poisson distribution singly truncated on the right, and forms a particular case of the general method discussed in Section 3.4 of Chapter III. Proceeding on those lines, one gets in this case

$$\theta = \mu = \frac{m_2^* - (d+1)\mu^*}{H_{11} - dH_{01}} \qquad (5.2.8)$$

where μ^* and m_2^* are defined by (5.2.3) and (5.2.4), respectively, and H_{11} and H_{01} reduce to

$$H_{11} = \mu^*$$

$$H_{01} = 1.$$

Then (5.2.8) gives

$$\mu = \frac{m_2^* - (d+1)\mu^*}{\mu^* - d} \qquad (5.2.9)$$

so that, on the basis of a random sample of size N with n_x as the frequency of x drawn from (5.2.1), the estimate for μ can be written as

$$t = \frac{S_2 - (d+1)S_1}{S_1 - dN} , \qquad (5.2.10)$$

where

$$S_1 = \Sigma\, x n_x \quad \text{and} \quad S_2 = \Sigma\, x^2 n_x .$$

To find the asymptotic variance of the two-moments estimate (TM) given by (5.2.10), one gets on simplification,

$$\text{Var } (t) = \frac{1}{NH^2} (\sigma^*_{11} + \mu^2 \sigma^*_{22} - 2\mu\sigma^*_{12}) \qquad (5.2.11)$$

where

$$H = \mu^* - d$$

$$\sigma^*_{11} = (m^*_4 - m^{*2}_2) + (d+1)^2 (m^*_2 - \mu^{*2})$$

$$- 2(d+1)(m^*_3 - \mu^* m^*_2) \qquad (5.2.12)$$

$$\sigma^*_{22} = m^*_2 - \mu^{*2}$$

and

$$\sigma^*_{12} = (m^*_3 - \mu^* m^*_2) - (d+1)(m^*_2 - \mu^{*2})$$

where m^*_r is the r-th theoretical moment of (5.2.1) about origin.

Thus,

$$\text{Var } (t) = \frac{1}{N(\mu^*-d)^2} [(m^*_4 - m^*_2) + (\mu+d+1)^2 (m^*_2 - \mu^{*2})$$

$$- 2(\mu+d+1)(m^*_3 - \mu^* m^*_2)] . \qquad (5.2.13)$$

The asymptotic efficiency of t is then given by

$$\text{Eff } (t) = \text{Var } (\hat{\mu})/\text{Var } (t).$$

The following table gives the asymptotic efficiency of t relative to μ for values of d = 5 with μ = .25, .5(.5) 2.5, and d = 10 with μ = .5(.5)5.

TABLE 5.2.1

EFFICIENCY OF TM

μ	.25	.50	1.00	1.50	2.00	2.50
			Case (i) d = 5			
Eff.	.978	.954	.904	.867	.850	.838

μ	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
					Case (ii) d = 10					
Eff.	.989	.988	.960	.942	.920	.897	.874	.855	.835	.815

Thus, the asymptotic efficiency of TM is never less than 82 percent in the above cases, and one may therefore use TM to estimate μ in such problems.

5.2.4 Following the general approach discussed in Section 3.1 of Chapter III, a simple ratio-estimate can be obtained for μ of (5.2.1). In this case, $a_{x-1}/a_x = x$, and since $\Theta = \mu$, the ratio-estimate for μ takes the form

$$\mu' = \sum_{x=0}^{d} xn_x / \sum_{x=0}^{d-1} n_x \tag{5.2.14}$$

as first suggested by Moore (1954).

The asymptotic variance of the "Ratio-estimate" (R) given by (5.2.14), can be obtained as

$$\text{Var}\ (\mu') = \frac{1}{NP^2} (D - p\mu^2 + 2\mu^2 P_{d-1}), \tag{5.2.15}$$

where, in this case,

$$P = \sum_{0}^{d-1} p^*(x,\mu)$$

$$D = \sum_{0}^{d} x^2 \, p^*(x,\mu) = m_2^*$$

and

$$P_{d-1} = p^*(d-1,\mu).$$

The asymptotic efficiency of μ' is then given by

$$\text{Eff } (\mu') = \text{Var } (\hat{\mu})/\text{Var } (\mu').$$

The following table gives the asymptotic efficiency of μ' relative to $\hat{\mu}$ for values of d = 5 with μ = .25, .5(.5)2.5, and d = 10 with μ = .5(.5)4.5.

TABLE 5.2.2

EFFICIENCY OF R

μ	.25	.50	1.00	1.50	2.00	2.50
			Case (i) d = 5			
Eff.	.999	.990	.979	.967	.951	.923

μ	.5	1.0	1.5	2.0	2.5	3.0	3.5	4	4.5
				Case (ii) d = 10					
Eff.	1.000	1.000	1.000	1.000	.999	.992	.981	.894	.817

Thus, R seems to be highly efficient on the whole and its efficiency always exceeds 82 percent in Table 5.2.2.

5.2.5 So far, we have separately discussed the Two Moments Estimate and the Ratio Estimate for μ. To make a comparative study of these two simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.1 and 3.3 of Chapter III, one gets to order $\frac{1}{N}$ the amount of bias of t(TM) and $\mu'(R)$ as follows:

$$b(t) = (\mu\sigma^*_{22} - \sigma^*_{12})/NH^2 = \frac{B(t)}{N} \qquad (5.2.16)$$

where H, σ^*_{22} and σ^*_{12} are defined by (5.2.12)

and

$$b(\mu') = \mu P_{d-1}/NP^2 = \frac{B(\mu')}{N} . \qquad (5.2.17)$$

The relative efficiency of R over TM is given by

$$\text{Rel.Eff.} = \text{Var } (t)/\text{Var } (\mu').$$

Table 5.2.3 gives $B(t)$, $B(\mu')$ and Rel. Eff. for values of $d = 5$ with $\mu = .25, .5(.5)2.5$ and $d = 10$ with $\mu = .5(.5)5$. Table 5.2.3 shows that both TM and R are over-estimates of μ. A closer investigation, however, brings out that the bias to order $\frac{1}{N}$ is always considerably smaller for R. Also, R is more efficient than TM. Thus, we conclude that one may prefer the Ratio Estimate to the Two-Moments Estimate for μ of the Poisson distribution singly truncated on the right because of its simplicity, small bias and high efficiency.

5.2.6 The probability law of the singly truncated Poisson distribution with truncation point on the left at c can be written as:

$$p^*(x,\mu) = [P^*(c,\mu)]^{-1} e^{-\mu} \frac{\mu^x}{x!} \qquad (5.2.18)$$

$$x = c, c+1, \ldots \infty$$

TABLE 5.2.3

COMPARISON BETWEEN TM AND R

μ	N(Amount of Bias to order 1/N) TM	N(Amount of Bias to order 1/N) R	Var(TM)/Var(R)
	Case (i) d = 5		
.25	.0526	.0003	1.022
.50	.1111	.0008	1.048
1.00	.2498	.0015	1.134
1.50	.4260	.0719	1.187
2.00	.6507	.1977	1.199
2.50	.9181	.4461	1.210
	Case (ii) d = 10		
.5	.0526	.0000	1.011
1.0	.1111	.0000	1.012
1.5	.1765	.0003	1.041
2.0	.2500	.0004	1.062
2.5	.3333	.0022	1.087
3.0	.4284	.0081	1.115
3.5	.5547	.0231	1.144
4.0	.6640	.0536	1.170
4.5	.8093	.1063	1.243
5.0	.9786	.1876	1.226

TABLE 5.2.4

BIAS AS A PERCENTAGE OF STANDARD ERROR FOR N = 100

	Bias/ S.E. X 100	
μ	TM	R
	Case (i) d = 5	
.25	1.0412	.0063
.50	1.5339	.0112
1.00	2.3566	.1550
1.50	3.9107	.5763
2.00	3.9114	1.2576
2.50	4.3275	2.3137
	Case (ii) d = 10	
.5	.7402	.0000
1.0	1.1043	.0000
1.5	1.4120	.0029
2.0	1.7150	.0057
2.5	2.0202	.0136
3.0	2.3352	.0467
3.5	2.7484	.1219
4.0	3.0121	.2607
4.5	3.3599	.4650
5.0	3.8477	.7380

where

$$P^*(c,\mu) = 1 - P(c-1,\mu). \tag{5.2.19}$$

The first two moments about origin of (5.2.1) can be written down as

$$\mu^* = \mu^*(c,\mu) = \mu\, P^*(c-1,\mu)/P^*(c,\mu) \tag{5.2.20}$$

and

$$m_2^* = m_2^*(c,\mu) = \mu^*(c,\mu)\left[1 + \mu^*(c-1,\mu)\right]. \tag{5.2.21}$$

To estimate μ on the basis of a random sample x_i $(i = 1,2,\ldots N)$ of size N from (5.2.18), results derived by the general approach in Chapter II can be written down as follows.

The likelihood equation for μ is

$$\bar{x} = \hat{\mu}^* \tag{5.2.22}$$

where

$$\bar{x} = \sum_{i=1}^{N} x_i/N \text{ and } \mu^* \text{ is defined by (5.2.20).}$$

Denoting this estimate as $\hat{\mu}$, the asymptotic variance is given by

$$\text{Var}\ (\hat{\mu}) = \frac{\mu}{N}/(\frac{d\mu^*}{d\mu}) \tag{5.2.23}$$

$$= \frac{\mu^2}{N\mu_2^*} \tag{5.2.24}$$

where μ_2^* is the variance of (5.2.18).

It is suggested by (5.2.22) that if tables be made available for means μ^*'s for sufficiently close values of μ, we can have a ready solution of (5.2.22). For a case of special significance when $c = 1$, i.e., when only zero observations are truncated, (5.2.22) becomes

$$\bar{x} = \mu/(1-e^{-\mu}) \tag{5.2.25}$$

and the asymptotic variance given by (5.2.23) reduces to

$$\text{Var}\ (\hat{\mu}) = \frac{\mu}{N}\ (1 - e^{-\mu})/(\mu+1 - \frac{\mu}{1-e^{-\mu}}). \tag{5.2.26}$$

Here, we present in a somewhat extensive table, Table V, values of $\mu^*(1,\mu)$ for the Poisson distribution truncated on the left at $c = 1$ for values of μ spaced at suitable intervals. A chart based on this table is also given to facilitate the procedure of estimation.

This table can be used to compute Var $(\hat{\mu})$ by using Formula (5.2.23) or Formula (5.2.24). In case Formula (5.2.23) is used, $\frac{d\mu^*}{d\mu}$ can be approximated by the finite difference ratio $\frac{\Delta\mu^*}{\Delta\mu}$. In case Formula (5.2.24) is used, the relationship for use is

$$\mu_2^*(1,\mu) = \mu^*(1,\mu)\left[1 + \mu - \mu^*(1,\mu)\right]. \tag{5.2.27}$$

Tables for $\mu^*(c,\mu)$ of the Poisson distribution truncated on the left have been also given for various values of c and μ.

5.2.7 For a slightly different problem, where in a sample from a complete Poisson distribution, the frequencies for some lowest "counts" are missing, Rider (1953) suggested a method of estimation which uses first two moments of the complete Poisson and leads to a linear equation.

The method of two-moments is also applicable in the usual problem of estimation from a sample from singly truncated Poisson and forms a particular case of the general method discussed in Section 3.4 of Chapter III. Proceeding on these lines, one gets the estimate for μ in this case as

$$t = \frac{S_2 - cS_1}{S_1 - (c-1)N} \tag{5.2.28}$$

where

$$S_1 = \Sigma\, xn_x \quad \text{and} \quad S_2 = \Sigma\, x^2 n_x .$$

To find the asymptotic variance of the two-moments estimate t of μ, one gets on simplification,

$$\text{Var }(t) = \frac{1}{N[\mu^* - (c-1)]^2}[(m_4^* - m_2^{*2}) + (\mu+c)^2(m_2^* - \mu^{*2})$$

$$- 2(\mu+c)(m_3^* - \mu^* m_2^*)]. \qquad (5.2.29)$$

The asymptotic efficiency of t is then given by

$$\text{Eff }(t) = \text{Var }(\hat{\mu})/\text{Var }(t).$$

The case of single truncation on the left at $c = 1$ is of practical importance. David and Johnson (1952) studied the efficiency for this particular case. The following is the table of Eff.(t) computed by them.

TABLE 5.2.5

EFFICIENCY OF TM FOR c = 1

μ	.5	1	1.5	2.0	2.5	3.0	4.0
Eff.	.87	.80	.75	.73	.71	.71	.72

Source: David and Johnson (1952)

Thus, the efficiency of TM is not less than 70 percent for $c = 1$ with $\mu = .5(.5)4.0$.

5.2.8 Following the general approach discussed in Section 3.1 of Chapter III, a simple ratio-estimate for μ can be obtained in the case of singly truncated Poisson distribution (5.2.1). In this case,

$a_{x-1}/a_x = x$ and since $\theta = \mu$, we have the following "ratio-estimate" for μ:

$$\mu' = \sum_{x=c+1}^{\infty} xn_x/N \tag{5.2.30}$$

when $c = 1$, i.e., when only "zero" counts are truncated, the estimate takes the form suggested by Plackett (1953):

$$\mu' = \sum_{x=2}^{\infty} xn_x/N\,. \tag{5.2.31}$$

The unique unbiased estimate of μ linear in the frequencies (ibid., Section 3.2 of Chapter III), is provided in (5.2.30). The exact variance of this estimate is

$$\sigma^2(\mu') = \frac{1}{N}\Big[\sum_{x=c+1}^{\infty} x^2 P_x - \mu^2\Big] \tag{5.2.32}$$

and an unbiased estimate of $\sigma^2(t)$ is

$$\Big\{\sum_{x=c+1}^{\infty} x^2 n_x - N\mu'^2\Big\}/N(N-1) \tag{5.2.33}$$

when $c = 1$, (5.2.32) reduces to

$$\sigma^2(\mu') = \frac{1}{N}[\mu + \mu^2/(e^{\mu} - 1)] \tag{5.2.34}$$

first derived by Plackett (1953). Plackett computed also the efficiency of μ' in this special case. The following table gives the efficiencies of μ' relative to $\hat{\mu}$.

It can be shown that the efficiency of μ' never falls below 0.9536, the minimum value being attained when $\mu = 1.355$ (Plackett).

TABLE 5.2.6

EFFICIENCY OF R FOR c = 1

μ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
Eff.	.9693	9559	9539	9586	9662	9743	9815	9872

Source: Plackett (1953)

5.2.9 So far, we have separately discussed the Two Moments Estimate and the Ratio Estimate for μ. To make a comparative study of these two simple estimates, let us investigate their amount of bias and relative efficiency.

Following Sections 3.1 and 3.3 one gets to order 1/N the amount of bias of t (TM) as follows:

$$b(t) = (\mu\sigma^*_{22} - \sigma^*_{12})/NH^2 = \frac{B(t)}{N} \qquad (5.2.35)$$

where $H = \mu^* - (c-1)$

$\sigma^*_{22} = m^*_2 - \mu^{*2}$

and $\sigma^*_{12} = (m^*_3 - \mu^* m^*_2) - c\,(m^*_2 - \mu^{*2})$.

For μ', however, one has

$$b(\mu') = 0. \qquad (5.2.36)$$

The relative efficiency of R over TM is given by

$$\text{Rel. Eff} = \text{Var}(t)/\text{Var}(\mu').$$

The following table gives bias and relative efficiency of TM and R for $\mu = .5(.5)4$. Thus, we conclude that one may prefer Ratio Estimate to estimate μ of the Poisson distribution singly truncated on the left at c = 1 because of its simplicity, unbiasedness and high efficiency.

TABLE 5.2.7

BIAS AND RELATIVE EFFICIENCY OF TM AND R

for c = 1

μ	N(Amount of Bias to order 1/N) TM	R	$\frac{Var(TM)}{Var(R)}$
.5	- .3935	.0000	1.11
1.0	- .6321	.0000	1.19
1.5	- .6373	.0000	1.27
2.0	- .8647	.0000	1.31
2.5	- .9179	.0000	1.36
3.0	- .9502	.0000	1.37
3.5	- .9698	.0000	1.37
4.0	- .9817	.0000	1.37

5.2.10 Some cases are likely to arise in which one is aware of the type of truncation, but does not know the point at which truncation occurs. For instance, when lots of manufactured items come for acceptance to a consumer from the producer who has earlier censored items having more than, say, d defects, consumer has to draw a sample or samples from a singly truncated Poisson population with unknown truncation point on the right. Estimation of μ and d thus becomes essential before setting up any acceptance sampling plan.

On the basis of a sample x_i $(i = 1, 2, \ldots N)$ of size N with observed frequency n_x for x $(0 < x < R, \Sigma n_x = N)$ drawn from

$$p^* (x, \mu, d) = (P (d,\mu))^{-1} e^{-\mu} \cdot \frac{\mu^x}{x!} \qquad (5.2.37)$$

$$x = 0, 1, 2, \ldots d$$

to estimate μ, we choose the Ratio-estimate

$$\mu' = \sum_{x=0}^{R} xn_x / \sum_{x=0}^{R-1} n_x . \qquad (5.2.38)$$

The advantage with μ' is that besides its simplicity, it does not need the knowledge of the truncation point.

To estimate d, the identity

$$\mu = \frac{m_2^* - (d+1)\mu^*}{\mu^* - d}$$

gives

$$d = \frac{m_2^* - (\mu+1)\mu^*}{\mu^* - \mu} .$$

Therefore the estimate of d can be obtained as:

$$d' = \frac{S_2 - (\mu'+1)S_1}{S_1 - \mu' N} \qquad (5.2.39)$$

where μ' is given by (5.2.38), $S_1 = \Sigma x n_x$ and $S_2 = \Sigma x^2 n_x$.

5.2.11 The detailed computation procedure of evaluating the three types of estimates discussed above is illustrated with reference to data collected by Varley (1949) to study population balance in the Knapweed Gall-fly. The table below gives the number of flower-heads (n_x) each having exactly x gall-cells (x = 1,2, ...)

Number of gall-cells in a flower-head (x)	1	2	3	4	5	6	7	8	9	10
Number of flower-heads (n_x)	287	272	196	79	29	20	2	0	1	0

Assuming the truncated Poisson model:

$$\frac{\mu^x}{x!\,(e^{\mu}-1)} \qquad x = 1, 2, \ldots$$

for the probability of x gall-cells in a flower-head, the problem is to estimate μ on the basis of the given data.

Maximum Likelihood Estimate: For the data, we get

$$N = 886$$

$$S_1 = 2023$$

so that $\bar{x} = 2.2833$. Referring to Table V for c = 1, we find the following:

μ	μ^*
1.9	2.2342
2.0	2.3130

The maximum likelihood estimate is given by that value of μ for which $\mu^* = 2.2833$. By linear interpolation we thus get

$$\hat{\mu} = 1.9623 .$$

The variance of this estimate is estimated from the formula:

$$\text{Var}(\hat{\mu}) = \frac{\mu^2}{N\mu_2^*}$$

where $\mu_2^* = \mu^*(1 + \mu - \mu^*)$.

On computation, $\mu_2^* = 2.2833\ (1 + 1.9623 - 2.2833)$

$= 1.5504$

and so,

$$\text{Var}(\hat{\mu}) = \frac{(1.9623)^2}{886 \times 1.5504} = 0.002803 .$$

Thus, the standard error of μ is given by

$$S.E.\ (\hat{\mu}) = \sqrt{0.002803} = 0.0529 \quad .$$

<u>Two Moments Estimate</u>: To compute this estimate of μ, we require in addition the value of

$$S_2 = \Sigma\, x^2 n_x = 6027$$

Then the estimate is

$$t = (\frac{S_2}{S_1} - 1) = \frac{6027}{2023} - 1$$

$$= 1.9792.$$

To compute the variance of t, taking 1.9792 as the estimate for μ, we have

$$\mu^* = \frac{\mu\ e^{\mu}}{e^{\mu}-1} = 2.2965$$

$$m_2^* = \mu^*(1+\mu) = 6.8417$$

$$\mu_2^* = m_2^* - \mu^{*2} = 1.5678$$

$$m_3^* = \mu(\mu^*+m_2^*) + (1+\mu)\mu_2^*$$

$$= 22.7571$$

and $m_4^* = \mu[\mu^* + m_2^* + 2\mu_2^* + m_3^* - 2\mu(1 + \mu)] + (m_3^* - \mu m_2^*)(1 + 2\mu)$

$= 91.6896.$

The variance of m is then estimated from the formula:

$$Var(t) = \frac{1}{N\mu^{*2}}[(m_4^* - m_2^{*2}) + (1 + \mu)^2 \cdot \mu_2^* - 2(1 + \mu)(m_3^* - \mu^* m_2^*)]$$

$$= 0.003600\ ,$$

so that the standard error is

$$S.E.(t) = 0.0546.$$

The following table summarizes the results obtained:

Estimate	Value	Variance	Standard Error
ML	1.9623	0.002803	0.0529
TM	1.9792	0.003600	0.0600
R	1.9594	0.002982	0.0546

5.3 Estimation from a Sample for a Censored Poisson Distribution

Moore (1952) and Cohen (1954) discussed the problem of estimation of μ from a censored sample of the Poisson distribution. Moore gave a simple ratio-estimate and Cohen derived likelihood equations for both singly and doubly censored samples. In this section, we provide a neat and compact likelihood equation for estimation. The amount of bias involved in estimating μ by the sample mean after pooling observations of higher counts has been investigated. A suitable chart is provided to suggest as to when one should resort to a finer method of estimation.

5.3.1. Suppose that in a random sample of size N from (5.1.2), we have a record of the number n_1 of observations in the right tail defined by ($\geq c$) and of the n^* observations $x_i (i = 1, 2, \ldots n^*)$, $x_i < c$) so that $N = n^* + n_1$. Results for estimation derived by the general approach in Section 2.3 can be written down in this case as follows:

The efficient score for μ is

$$\psi(\mu) = \frac{1}{\mu} \{n^* \bar{x}^* - (N\mu - n_1 \nu_1)\} \tag{5.3.1}$$

where $\bar{x} = \sum_{i=1}^{n^*} x_i/n^*$ and ν_1 is the mean of the Poisson distribution truncated on the left at c.

Thus, the likelihood equation for estimating μ is

$$n^* \bar{x}^* = N\mu - n_1 \nu_1 \qquad (5.3.2)$$

The asymptotic variance of the estimate μ derived from (5.3.2) is $1/I(\mu)$ where

$$I(\mu) = \frac{N}{\mu}\left(1 - P \frac{d\nu_1}{d\mu}\right) \qquad (5.3.3)$$

where P is the probability of the right tail.

Equation (5.3.2) does not readily give an algebraic solution and the iterative process of solution has to be resorted to. To facilitate the process of estimation we use Table V . to obtain values of means for values of μ and c spaced at suitable intervals.

5.3.2 A simple estimate of μ from a censored sample under consideration can be obtained as

$$m = \frac{\sum_{i=1}^{n^*} x_i + cn^*}{N} \qquad (5.3.4)$$

which, though always an underestimate, may at times, when the magnitude of the bias is small may be useful.

The relative bias is

$$b = \frac{E(m) - \mu}{\mu} = - \frac{P(\mu^* - c)}{\mu} \qquad (5.3.5)$$

where μ^* is the mean of the left tail.

Here we present in Table 5.3.1 the values of the relative bias expressed as a percentage for various values of μ and c spaced at suitable intervals. Chart 4 based on this table may be also of help to suggest as to when one should resort to a finer method of estimation. Also, before

TABLE 5.3.1

$$- b = \frac{P(\mu^* - c)}{\mu}$$

μ \ c	4	5	6	7	8	9	10
.5	.000372	.000326	.000000	.000000	.000000	.000000	.000000
1.0	.004348	.000688	.000096	.000000	.000000	.000000	.000000
1.5	.016109	.003723	.000752	.000013	.000000	.000000	.000000
2.0	.037570	.011244	.002962	.000695	.000147	.000028	.000005
2.5	.068305	.024781	.007971	.002296	.000597	.000141	.000030
3.0	.106454	.044873	.016901	.005731	.001762	.000494	.000127
3.5	.149588	.071141	.030462	.011810	.004169	.001348	.000400
4.0	.195368	.102571	.048856	.021189	.008405	.003063	.001030
4.5	.241801	.137821	.071805	.034255	.015016	.006068	.002270
5.0	.287375	.175461	.098660	.051093	.024418	.010798	.004431
5.5	.331034	.214227	.128168	.071544	.036846	.017637	.007858
6.0	.372169	.253006	.160623	.094999	.052330	.026868	.012878
6.5	.392222	.290968	.193899	.121057	.070709	.038639	.019773
7.0	.445659	.327507	.227610	.148995	.091659	.052955	.028736
7.5	.477929	.362219	.261078	.178163	.114771	.069693	.039877
8.0	.507431	.394879	.293784	.207953	.139563	.088626	.053193
8.5	.534296	.425390	.325334	.237822	.165527	.109411	.068572
9.0	.558749	.453743	.355472	.267357	.192186	.131684	.100000
9.5	.581036	.480000	.384046	.296118	.219109	.155060	.104685

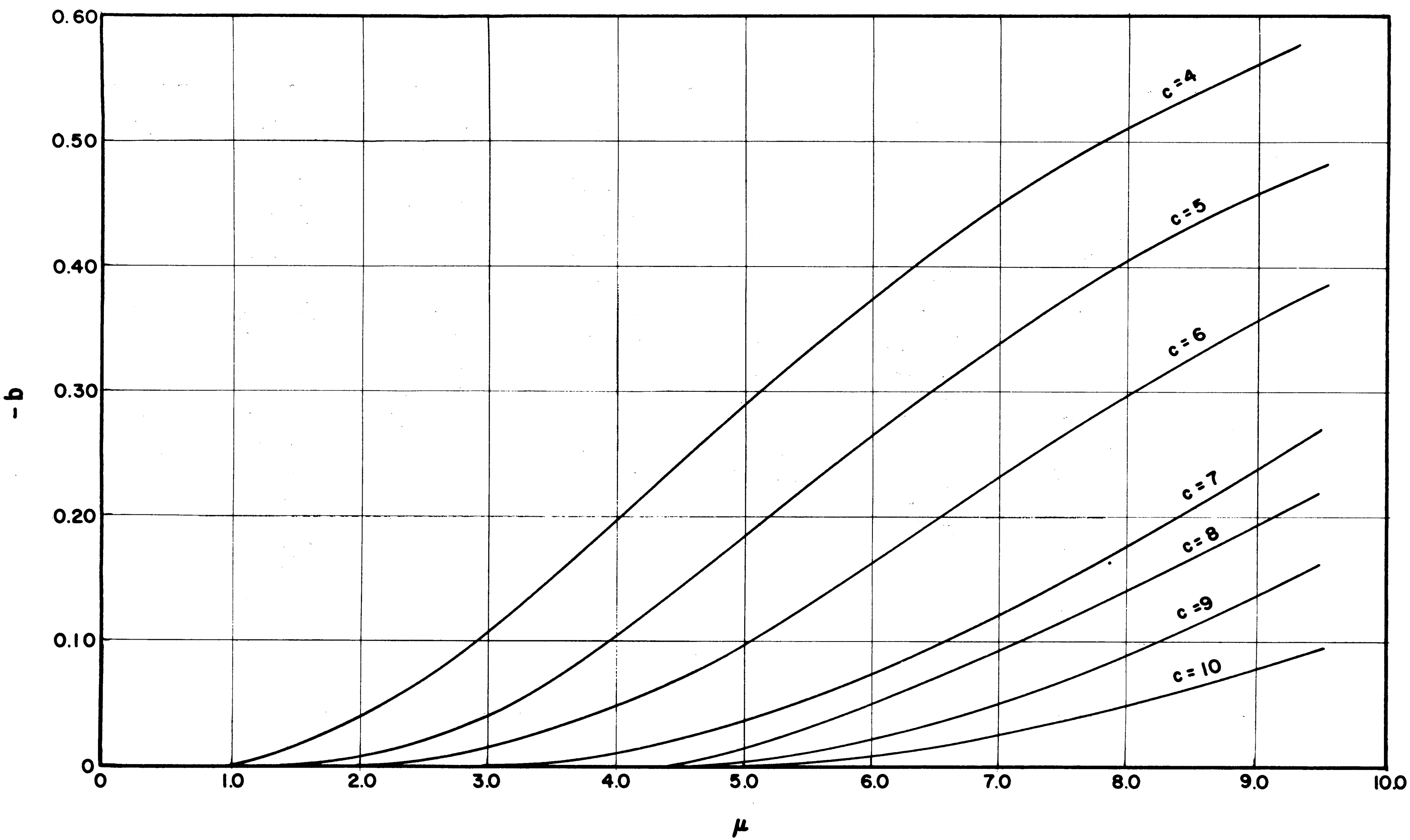

Chart 5.3.1 Chart for the Relative Bias of the "Pooled Estimate" for c = 4(1)10

starting an experiment on "Poisson-counts", Table 5.3.1 and Chart 5.3.1 may be of great help to an experimentor in deciding the value of the count beyond which he need only record not the individual counts, but only the total frequency.

Equation of estimation in other types of censored samples may be written down on the same lines as above.

5.4 Estimation With Doubtful Observations

While sampling from a Poisson distribution (5.1.2), cases arise in which one is doubtful if the observations in the "zero-class" really come from the Poisson population. For instance, while recording data on counts of minute particles in an experiment, one is doubtful if the "zero-counts" sometimes occur because of failure in the working of the counter. Before actually estimating the parameter, therefore, one has to test if the zero-counts conform to the Poisson distribution under consideration, which, incidentally also tests if the counter works throughout or not.

5.4.1 We take the model for the distribution as

$$\text{Prob}\ \{X = x\} = \begin{cases} \beta & \text{for } x = 0 \\ \dfrac{1-\beta}{e^{\mu}-1} \cdot \dfrac{\mu^x}{x!} & \text{for } x = 1,2,\ldots\infty \end{cases} \qquad (5.4.1)$$

On the basis of a sample of size N with n_1 zero-counts and n^* non-zero counts x_i ($i = 1, 2, \ldots n^*$, $n^* = N-n_1$) one gets the efficient scores for β and μ as:

$$\psi_1 = \frac{n_1}{\beta} - \frac{n^*}{1 - \beta} \qquad (5.4.2)$$

and

$$\psi_2 = \frac{n^*}{\mu} (\bar{x}^* - \mu^*) \tag{5.4.3}$$

where $\bar{x}^* = \frac{\sum_{i=1}^{n^*} x_i}{n^*}$ and $\mu^* = \mu/(1-e^{-\mu})$

The estimates therefore are given by

$$\hat{\beta} = \frac{n_1}{N} \tag{5.4.4}$$

and

$$\hat{\mu}^* = \bar{x}^* \tag{5.4.5}$$

Now, the hypothesis H_o of interest is

$$\beta = e^{-\mu} \tag{5.4.6}$$

obviously the estimate of μ under H_o is given by

$$\hat{\mu}_o = \bar{x}$$

where

$$\bar{x} = \sum_{i=1}^{n^*} x_i/N \tag{5.4.7}$$

Therefore, the estimate of β under H_o becomes

$$\hat{\beta}_o = e^{-\hat{\mu}_o} \tag{5.4.8}$$

and the Chi-square criterion with 1 df given by (2.4.16) reduces to

$$\chi^2 = \chi_o^2 \left(1 + \frac{1}{V_o}\right) \tag{5.4.9}$$

where

$$\chi_o^2 = \frac{N(\frac{n_1}{N} - \hat{\beta}_o)^2}{\hat{\beta}_o (1 - \hat{\beta}_o)}$$

and $$V_o = \frac{1 - \hat{\beta}_o}{\hat{\beta}_o \ \hat{\mu}_o} - 1$$

Illustrative Example: Let us illustrate the method of estimation with doubtful observations with reference to data given by Scrase. The problem is connected with the number of dust nuclei in the air and the data give the frequency distribution of the number of drops in a small volume of air that fall on to a stage in a chamber containing moisture and filter. The data are as follows:

Number of dust nuclei (x)	0	1	2	3	4	5	6	7	8
Frequency (n_x)	23	56	88	95	73	40	17	5	3

For the full distribution, we have

$$N = 400$$

$$S_1 = 1170$$

and

$$\overline{x} = 2.9250$$

Scrase is of the opinion that this mean (and hence the estimate of the Poisson parameter given by $\mu = \overline{x}$) is slightly high in that a number of zero counts were wrongly rejected as being due to the apparatus not working. The zero-counts are doubtful and so before writing down the estimate, we shall have to test if these zero-counts conform to the full Poisson-data.

Under the null hypothesis H_o that zero-counts conform to the full distribution, we have, following the results derived in Section 5.4,

$$\hat{\mu}_o = \overline{x} = 2.9250.$$

Therefore under H_o the estimate of β, the proportion of zero-counts to the total number of counts, is given by

$$\hat{\beta}_o = e^{-\hat{\mu}_o} = e^{-2.9250}$$

$$= 0.53665$$

To compute the Chi-square criterion given by (5.4.9) we have

$$X_o^2 = \frac{N\left(\frac{n_1}{N} - \hat{\beta}_o\right)^2}{\hat{\beta}_o(1 - \hat{\beta}_o)}$$

$$= \frac{400\left(\frac{23}{400} - 0.053665\right)^2}{0.053665 \times 0.946335}$$

$$= 0.115839$$

and $$V_o = \frac{1 - \hat{\beta}_o}{\hat{\beta}_o \hat{\mu}_o} - 1 = 5.028763$$

so that, the Chi-square criterion with 1 degree of freedom to test H_o comes out to be

$$X^2 = X_o^2 \left(1 + \frac{1}{V_o}\right)$$

$$= 0.138874$$

which is not significant, showing thereby that the zero-counts conform to the full Poisson distribution and the estimate of μ (obtained to be $\hat{\mu}_o = 2.925$) computed from the full data is statistically quite legitimate. This conclusion brings out that the rejections of zero-counts were rightly judged by the experimenter, rightly because the counter had really failed to work then.

CHAPTER VI

6.0 ESTIMATION PROBLEMS FOR THE NEGATIVE BINOMIAL DISTRIBUTION

6.1 Introduction

The gpsd with two parameters defined by (2.6.1) becomes

$$\text{Prob}\,\{X = x\} = \binom{\lambda + x - 1}{x}\,\theta^{x}/(1-\theta)^{-\lambda} \qquad (6.1.1)$$

$$x = 0,1,2,\ldots \infty$$

when

$$f(\theta,\lambda) = (1 - \theta)^{-\lambda}, \quad \lambda > 0.$$

Writing $k = \lambda$ and $\frac{\mu}{\mu + k} = \theta$, (6.1.1) gives the probability for X as:

$$\text{Prob}\,\{X = x\} = y(x,\mu,k) = \binom{k+x-1}{x}\left(\frac{k}{\mu+k}\right)^{k}\left(\frac{\mu}{\mu+k}\right)^{x}$$

$$x = 0,1,2,\ldots\infty \qquad (6.1.2)$$

a well-known form of the negative binomial distribution.

The important properties of (6.1.2) can be summarily stated as follows:

$$M(t) = [1 - \frac{\mu}{k}(e^{t} - 1)]^{-k}. \qquad (6.1.3)$$

The first two central moments are given by

$$\mu = \mu$$

$$\mu_2 = \frac{\mu}{k}(\mu + k) \qquad (6.1.4)$$

and the coefficients β_1, β_2 take the form:

$$\beta_1 = \frac{1}{\mu} + \frac{1}{k}\left(3 + \frac{\mu}{\mu+k}\right)$$

$$\beta_2 = 3 + \frac{1}{\mu_2} + \frac{6}{k}. \qquad (6.1.5)$$

The recurrence relation connecting central moments

$$\mu_{r+1} = \mu_2 \left(\frac{d\mu_r}{d\mu} + r\,\mu_{r-1}\right) \qquad (6.1.6)$$

does not seem to have been noticed before, this follows immediately from (1.1.7).

The recurrence relation connecting cumulants is

$$K_{r+1} = \mu_2 \frac{dK_r}{d\mu} \qquad (6\ 1.7)$$

which is derived in a slightly different form by Guldberg (1935) and Wishart (1949).

The distribution function Y (r, μ, k) defined by

$$Y(r, \mu, k) = \sum_{x=o}^{r} y(x, \mu, k) \qquad (6.1\ 8)$$

can be reduced to

$$Y(r, \mu, k) = I_{\phi}(k, r + 1)$$

where

$$\phi = \frac{k}{\mu + k} \qquad (6.1.9)$$

and

$$I_x(m,n) = \frac{1}{B(m,n)} \int_o^x u^{m-1}(1-u)^{n-1} du$$

for which extensive tables have been edited by K. Pearson. When k is positive integer, one can use

$$Y(r,\mu,k) = \sum_{x=k}^{r+k} \binom{r+k}{x} \left(\frac{k}{\mu+k}\right)^x \left(\frac{\mu}{\mu+k}\right)^{r+k-x}$$

$$= 1 - B(k - 1, \phi, r+k) \qquad (6.1.10)$$

where ϕ is given by (6.1.9) and $B(r,\pi,n)$ is the cumulative binomial probability defined in (4.1.7). For large k, one can use tables of Poisson probabilities, because

$$\lim_{k \to \infty} y(x, \mu, k) = p(x, \mu) \qquad (6.1.11)$$

where

$$p(x, \mu) = e^{-\mu} \cdot \frac{\mu^x}{x!} .$$

For large μ and k, however, we have the normal approximation given by

$$\lim_{\mu \to \infty,\ k \to \infty} Y(r, \mu, k) = \Phi(z) \qquad (6.1.12)$$

where $z = (r + 1/2 - \mu)/\sqrt{\mu_2}$

and

$$\Phi(z) = \frac{1}{2\pi} \int_{-\infty}^{z} e^{-1/2 \cdot u^2} du .$$

6.2 Estimation of Parameters of Complete Negative Binomial Distribution

The negative binomial distribution can be viewed as a compound Poisson distribution, Greenwood and Yule (1920), when the mean of the Poisson distribution follows the gamma distribution. One finds, therefore, its applications in all fields wherever the data under consideration are too heterogeneous to be fitted by a Poisson distribution. For instance, Greenwood and Yule (1920) applied it to accident-data, Fisher (1941) and Bliss (1953) to biological data, Sichel (1951) to psychological data, whereas Wise (1946) found its use in an industrial sampling problem.

6.2.1 To estimate μ and k on the basis of a random sample of size N with observed frequency n_x for x ($\Sigma n_x = N$) drawn from (6.1.2), following Section 2.6, the moment-estimates are given by

$$\bar{x} = \mu \tag{6.2.1}$$

and

$$s^2 = \frac{\mu}{k}(\mu + k) \tag{6.2.2}$$

where

$$\bar{x} = \Sigma x n_x / N \text{ and } s^2 = \Sigma n_x (x-\bar{x})^2/N.$$

The likelihood equations reduce to

$$\bar{x} = \hat{\mu} \tag{6.2.3}$$

and

$$\sum_{r \geq o} \frac{T_{r+1}}{\hat{k}+r} = N \log \left(\frac{\hat{\mu}+\hat{k}}{\hat{k}}\right) \tag{6.2.4}$$

where $T_{r+1} = \sum_{x > r} n_x$ which was first derived by Haldane (1941).

Eliminating $\hat{\mu}$ from (6.2.3) and (6.2.4), we have to solve for $\hat{k}$ the equation

$$\sum_{r \geq o} \frac{T_{r+1}}{\hat{k}+r} = N \log \left(\frac{\bar{x} + \hat{k}}{\hat{k}}\right) \tag{6.2.5}$$

The elements of the information-matrix are:

$$I_{11} = \frac{N}{\mu_2}$$

$$I_{12} = 0 \tag{6.2.6}$$

and

$$I_{22} = \sum_{r \geq o} \left[\frac{1 - Y(r, \mu, k)}{(k + r)^2}\right] + \frac{N\mu}{k(\mu+k)}$$

6.2.2 It is easy to see that the moment-estimate is identical with the likelihood-estimate for the mean μ and is given by the sample mean $\bar{x}$. The estimates for k are however different. The moment estimate for k, though inefficient, is easy to compute, whereas the likelihood estimate, though efficient, is rather difficult to obtain. It is important to know, therefore, as to when one should finally proceed to obtain likelihood-estimate for k. For large μ, Sichel (1951) showed the efficiency of the moment-estimate for k to be never less than 0.80. He also showed that the efficiency is minimum for large μ when $k = 5.5$ and recommended that for values of $k > 5.5$, it is not necessary to estimate k by the arduous likelihood-estimate. Fisher (1941) examined the efficiency of the moment-estimates and concluded that if μ is less than $k/9$ for any value of k or if k exceeds 18 for any value of μ, high efficiency is assured, for intermediate values, however, if the product $(1 + \frac{k}{\mu})(k + 2)$ exceeds 20, the efficiency is satisfactorily high.

6.3 Estimation of Parameters of a Truncated Negative Binomial Distribution

Sampford (1955) discussed an application of truncated negative binomial distribution. Taking k and $\omega = \frac{k}{\mu+k}$ to be parameters, he gave methods to obtain "moment-estimates" and "likelihood-estimates" for ω and k when zero-observations are truncated and also investigated the efficiency of the moment estimates so obtained. Rider (1955) used three moments and provided simple estimates for k and $p = \frac{\mu}{k}$. Following the general approach discussed in Section 2.6 , neat and compact likelihood equations can be made available.

The probability law of the negative binomial distribution of which zero-observations are truncated can be written as

$$y^*(x, \mu, k) = [1 - (\frac{k}{\mu+k})^k]^{-1} . \binom{k+x-1}{x} (\frac{\mu}{\mu+k})^x (\frac{k}{\mu+k})^k$$

$$x = 1,2,3,\ldots\infty \qquad (6.3.1)$$

The first two moments about the origin of (6.3.1) then, are

$$\mu^* = \frac{\mu}{1 - (\frac{k}{\mu+k})^k} \qquad (6.3.2)$$

and

$$m_2^* = \mu^* (1 + \mu + \frac{\mu}{k}). \qquad (6.3.3)$$

6.3.1 To estimate μ and k on the basis of a random sample of size N with frequency n_x for $x(\Sigma n_x = N)$ drawn from (6.3.1) the moment equations can be written down as

$$\bar{x} = \mu^* = \frac{\mu}{1 - (\frac{k}{\mu+k})^k} \qquad (6.3.4)$$

and

$$S^2 = m_2^* = \mu^*(1 + \mu + \frac{\mu}{k}) \qquad (6.3.5)$$

where

$$\bar{x} = \Sigma x n_x/N \text{ and } S^2 = \Sigma x^2 n_x/N .$$

Eliminating k from (6.3.4) and (6.3.5), one gets

$$\mu = \bar{x} \left[1 - \left(\frac{\bar{x}}{S^2 - \bar{x}\mu} \right)^{\frac{\bar{x}\mu}{S^2 - \bar{x}(\mu+1)}} \right]. \qquad (6.3.6)$$

An estimate of μ can be obtained by solving this single equation (6.3.6) by using an iterative method. It is easy to see that the estimate of k is also available during the process of iteration, because one has

$$k = \frac{\bar{x}\mu}{S^2 - \bar{x}(\mu+1)} \tag{6.3.7}$$

6.3.2 Following the general approach discussed in Section 2.6 , the likelihood equations for μ and k in this case can be written down as:

$$\bar{x} = \hat{\mu}^* \tag{6.3.8}$$

and

$$\sum_{r \geq o} \frac{T_{r+1}}{k+r} = \frac{N\hat{\mu}^*}{\hat{\mu}} \log \left(\frac{\hat{\mu}+\hat{k}}{\hat{k}}\right) \tag{6.3.9}$$

where μ^* is defined by (6.3.2) and $T_{r+1} = \sum_{x > r} n_x$. The elements of the information matrix are:

$$I_{11} = \frac{N}{\mu_2} \cdot \frac{\partial \mu^*}{\partial \mu} = \frac{N\mu_2^{*2}}{\mu^2} \tag{6.3.10}$$

$$I_{12} = \frac{N}{\mu_2} \cdot \frac{\partial \mu^*}{\partial k} = -\frac{N\mu_2^*}{\mu_2} \cdot \frac{1}{\mu+k} \tag{6.3.11}$$

and

$$I_{22} = \sum_{r \geq o} \left[\frac{1 - Y(r,\mu,k)}{(k+r)^2}\right] + \frac{N}{\mu(\mu+k)} \left[\mu_2^* \log \frac{k}{\mu+k} + \mu \left(\frac{\mu_2^*}{\mu+k} - \frac{\mu}{k}\right)\right] \tag{6.3.12}$$

where μ_2 and μ_2^* are the variances of (6.1.2) and (6.3.1) respectively.

6.3.3 Equating the first three sample-moments to the corresponding theoretical moments about origin of (6.3.1) Rider (1955) gets simple estimates for μ and k as follows:

$$\mu' = \frac{2S_2^2 - S_1(S_2 + S_3)}{S_1(S_2 - S_1)} \tag{6.3.13}$$

and

$$k' = \frac{2S_2^2 - S_1(S_2 + S_3)}{S_1(S_1 + S_3)+S_2(S_1+S_2)} \tag{6.3.14}$$

where

$$S_i = \Sigma\, x^i n_x/N \qquad i = 1,2,3.$$

6.4 Estimation when k is Known

The negative binomial distribution is essentially a distribution with two parameters. Cases, however, arise in which the parameter k is known and only the other parameter has to be estimated. This is taken up in this section.

When k is a positive integer, writing $\phi = \frac{k}{\mu+k}$, the negative binomial law given by (6.1.2) becomes

$$y(x, \phi, k) = \binom{k + x - 1}{x} \phi^k (1-\phi)^x \tag{6.4.1}$$

$$x = 0,1,2,\ldots\infty$$

(6.4.1) can be looked upon as the probability law for the number $X = k + x$ of independent successive trials required to get k successes when ϕ is the probability of success at each trial and can be used in sampling problems in which sampling is done until a certain number k of "character-bearers" is available in the sample. It has been used by Haldane (1945) in biology and by Craig (1953) among others in industrial problems.

6.4.1 On the basis of single observation on X, $X = k+x$, the likelihood estimate for ϕ can be obtained as

$$\hat{\phi} = \frac{k}{k+x} = \frac{k}{X}. \tag{6.4.2}$$

It can be seen that $\hat{\phi}$ is biased for ϕ. Following Section 3.2 , however, a unique unbiased ratio-estimate for ϕ can be obtained. In this case, $a_{x-1}/a_x = x|(k+x-1)$ and since $\Theta = 1 - \phi$, we have the ratio-estimate for ϕ as

$$\phi' = \frac{k - 1}{k+x-1} = \frac{k - 1}{X - 1} \qquad k > 1 \tag{6.4.3}$$

provided first by Haldane (1945) and shown to be the only unbiased estimate for ϕ by Girshick, Mosteller and Savage (1946).

To estimate ϕ on the basis of a random sample $X = k + x$ of size N with frequency n_x for x drawn from (6.4.1), the following estimates are available:

<u>Maximum Likelihood Estimate</u>: Following Section 2.1 , the likelihood equation (which is the same as moment equation) can be written down as $\bar{x} = \hat{\mu}$ where $\bar{x} = \Sigma x n_x/N$ and μ is the mean of (6.4.1) given by

$$\mu = k(1-\phi)/\phi \tag{6.4.4}$$

one gets the estimate for ϕ, therefore, as

$$\hat{\phi} = \frac{k}{k+\bar{x}} . \tag{6.4.5}$$

The asymptotic variance and the amount of bias to order $\frac{1}{N}$ of $\hat{\phi}$ can be easily obtained as

$$\text{Var}(\hat{\phi}) = \frac{\phi^2 (1 - \phi)}{Nk} \tag{6.4.6}$$

and

$$b(\hat{\phi}) = \frac{\phi(1 - \phi)}{Nk} . \tag{6.4.7}$$

The following table gives the amount of bias and standard error of $\hat{\phi}$ (ML) for k = 1, 2, 3 with ϕ = .01, .05, .10, .25, 50, .75.

TABLE 6.4.1

Bias and Standard Error of ML for N = 100

k	ϕ	Amount of Bias to order $\frac{1}{N}$	Standard Error	100 X $\frac{\text{Bias}}{\text{S.E.}}$
	.01	.000099	.000990	10.00
	.05	.000475	.004873	9.75
	.10	.000100	.009487	9.49
1	.25	.001875	.021660	8.66
	.50	.002500	.035355	7.07
	.75	.001875	.037500	5.00
	.01	.000050	.000704	7.04
	.05	.000238	.003446	6.89
	.10	.000450	.006708	6.70
2	.25	.000938	.015300	6.13
	.50	.000125	.025000	5.00
	.75	.000938	.026510	3.54
	.01	.000033	.000574	5.75
	.05	.000158	.002813	5.63
	.10	.000625	.005477	5.48
3	.25	.000625	.012490	5.00
	.50	.000833	.020411	4.08
	.75	.000625	.021660	2.89

<u>Ratio Estimate</u>: Following Section 3.1 , the unique unbiased linear estimate for ϕ can be written down as

$$\phi' = 1 - \frac{1}{N} \sum_{x=1}^{\infty} \left(\frac{x}{k+x-1}\right) n_x . \qquad (6.4.8)$$

The exact variance of this ratio-estimate ϕ' is

$$\sigma^2(\phi') = \frac{1}{N} \left[\sum_{x=1}^{\infty} \left(\frac{x}{k+x-1}\right)^2 y(x,\phi,k) - (1-\phi)^2 \right] \qquad (6.4.9)$$

and its unbiased estimate is given by

$$[\sum_{x=1}^{\infty} (\frac{x}{k+x-1})^2 n_x - N(1-\phi')^2]/N(N-1).$$

The efficiency of ϕ' is then obtained as Eff.(ϕ') = Var $(\phi')/\sigma^2(\phi')$

The following table gives the efficiency of ϕ' for k = 1,2 with ϕ = .01, .05, .10, .25, .50, .75.

TABLE 6.4.2

Efficiency of R

k \ ϕ	.01	.05	.10	.25	.50	.75
1	.010	.050	.100	.250	.500	.750
2	.136	.221	.289	.442	.647	.830

Also, if one notices that $S_1 = \Sigma\, xn_x$ follows the negative binomial law with parameters ϕ and Nk, one more unbiased ratio-estimate can also be written down for ϕ on the lines of the estimate given for single observation by (6.4.3) as,

$$\phi'' = \frac{Nk - 1}{Nk + S_1 - 1}$$

$$= \frac{k - 1/N}{k + \bar{x} - 1/N} . \qquad (6.4.10)$$

The exact variance of ϕ'' is then given by

$$\sigma^2(\phi'') = \sum_{S_1=o}^{\infty} (\frac{Nk - 1}{Nk+S_1-1})^2 y(S_1, \phi, Nk) - \phi^2 . \qquad (6.4.11)$$

Also the asymptotic variance of ϕ'' can be written down as

$$\text{Var}(\phi'') = \frac{\phi^2(1-\phi)}{Nk} \cdot \frac{1}{(1 - \frac{1}{Nk})^2}$$

$$= \frac{\text{Var}(\hat{\phi})}{(1 - \frac{1}{Nk})^2} \qquad (6.4.12)$$

which indicates that ϕ'' is highly efficient. It is also unbiased and easy to compute. Hence it may be used with advantage to estimate ϕ because the maximum likelihood estimate ϕ given by (6.4.5), though efficient, has some bias whereas the ratio-estimate ϕ' given by (6.4.8), though unbiased, involves serious loss of efficiency.

6.4.2 Cases can arise in which one has to estimate ϕ on the basis of a sample drawn from truncated negative binomial distribution.

The negative binomial distribution (6.4.1) truncated say, on the left at c can be written down as:

$$y^*(x,\phi,k) = [Y^*(c,\phi,k)]^{-1} \binom{k+x-1}{x} \phi^k(1-\phi)^x \qquad (6.4.13)$$

$$x = c, c+1, \ldots \infty$$

where $Y^*(r+1,\phi,k) = 1 - Y(r,\phi,k)$. The first two moments about origin of (6.4.13) are

$$\mu^* = \mu^*(c, \phi, k) = \frac{k(1-\phi)}{\phi} \cdot \frac{Y^*(c-1, \phi, k+1)}{Y^*(c, \phi, k)} \qquad (6.4.14)$$

and

$$m_2^* = m_2^*(c, \phi, k) = \mu^*(c, \phi, k)\, [1+\mu^*(c-1,\phi,k+1)]. \qquad (6.4.15)$$

To estimate ϕ on the basis of a random sample of size N with frequency n_x for x drawn from (6.4.13), results derived by the general approach in Section 2.2 can be written down as follows with proper substitutions in this particular case.

Maximum Likelihood Estimate: The likelihood equation for ϕ is

$$\bar{x}^* = \mu^* \tag{6.4.16}$$

where $\bar{x}^* = \sum x n_x / N$ and μ^* is defined by (6.4.14). Denoting this estimate as $\hat{\phi}$, its asymptotic variance is given by

$$\mathrm{Var}(\hat{\phi}) = - \frac{(1-\phi)}{N} / \left(\frac{d\mu^*}{d\phi}\right) \tag{6.4.17}$$

$$= \frac{(1-\phi)^2}{N\mu_2^*} \tag{6.4.18}$$

where μ_2^* is the variance of (6.4.13).

As the equation (6.4.16) does not readily give an algebraic solution, one may use an iterative process of solution. However, one can have a ready solution of (6.4.16), if tables be made available for μ's for sufficiently close values ϕ. Here we present in Table VI values of μ^* for the special case of c=1 with k=1, 2, 3 at suitable intervals of ϕ. When c=1,(6.4.16) reduces to

$$\mu^* = \frac{k(1-\phi)}{\phi(1-\phi^k)}. \tag{6.4.19}$$

This table can be used to compute $\mathrm{Var}(\hat{\phi})$ by using either (6.4.17) or (6.4.18). In case (6.4.17) is used, $\frac{d\mu^*}{d\phi}$ can be approximated by the finite difference ratio $\frac{\Delta\mu^*}{\Delta\phi}$. In case Formula (6.4.18) is used, the relationship for use is

$$\mu_2^* (c,\phi,k) = \mu^* (c,\phi,k) \, [1+ \mu^* (c-1,\phi,k+1)-\mu^* (c,\phi,k)].$$

Ratio Estimate: The simple ratio estimate for ϕ of (6.4.13) can be written down as

$$\phi' = 1 - \frac{1}{N} \sum_{x=c+1}^{\infty} \left(\frac{x}{k+x-1}\right) n_x . \tag{6.4.20}$$

(6.4.20) provides the unique unbiased linear function of frequencies for estimating ϕ and its exact variance is given by

$$\sigma^2(\phi') = \frac{1}{N} \left[\sum_{x=c+1}^{\infty} \left(\frac{x}{k+x-1}\right)^2 y\,(x,\phi,k) - (1-\phi)^2 \right]. \tag{6.4.21}$$

We have already seen, however, for the complete negative binomial distribution that this type of ratio-estimate, though unbiased, involve serious loss of efficiency. In the case of truncated distribution also, therefore, it is expected to have a similar pattern of low efficiencies.

Two-Moments Estimate: Following Section 3.4 , we get for the distribution (6.4.1)

$$\Theta = 1 - \phi = \frac{m_2^* - c\mu^*}{H_{11} - (c-1)H_{o1}} \tag{6.4.22}$$

where μ^* and m_2^* are defined by (6 4.14) and (6.4.15) respectively and H_{o1} and H_{11} reduce to

$$H_{o1} = k + \mu^*$$

$$H_{11} = k\mu^* + m_2^*$$

(6.4.22) gives then

$$\phi = \frac{(k+1)\mu^* - (c-1)k}{m_2^* + (k-c+1)\mu^* - (c-1)k} \tag{6.4.23}$$

so that a simple estimate for ϕ can be written down as

$$t = \frac{(k+1)\ S_1 - (c-1)kN}{S_2+(k-c+1)S_1-(c-1)kN} \tag{6.4.24}$$

where $S_i = \Sigma\ x^i n_x \qquad i = 1,2.$

When c = 1, (6.4.24) reduces to

$$t = \frac{(k+1)\ S_1}{S_2+\ kS_1}. \tag{6.4.25}$$

The asymptotic variance of t given by (6.4.25) reduces to

$$\text{Var}(t) = \frac{1}{NH^2}\ (\sigma^*_{11} + \phi^2\ \sigma^*_{22} - 2\phi\ \sigma^*_{12}) \tag{6.4.26}$$

where

$$H = m^*_2 + k\mu^*$$

$$\sigma^*_{11} = (k+1)^2\ \mu^*_2$$

$$\sigma^*_{22} = (m^*_4 - m^{*2}_2) + k^2\ \mu^*_2 + 2k\ (m^*_3 - \mu^* m^*_2)$$

and

$$\sigma^*_{12} = (k+1)[(m^*_3 - \mu^* m^*_2) + (k\ \mu^*_2)]$$

where m^*_r is the r-th theoretical moment and μ^* and μ^*_2 are the mean and variance respectively of (6.4.13) with c = 1.

Also one gets to order $\frac{1}{N}$ the amount of bias of t for c = 1 as

$$b(t) = (\phi\ \sigma^*_{22} - \sigma^*_{12})/NH^2$$

where H, σ^*_{22}, σ^*_{12} are defined above.

6.5 Homogeneity and Combined Estimation, k Known

Consider a situation in which m different machines in a section of a factory are producing items of some kind. One has to examine if these different machines are homogeneous in respect of the quality (ϕ) of the product as judged by the proportion defectives, and if so, to make a combined estimation of ϕ.

On the basis of inverse binomial sampling applied to lots of fairly large number of items from a machine corresponding to each lot from every machine, one will be having an observation of number of items (k+x) that had to be inspected to get k defectives. Let N_j be the number of lots inspected from j-th machine, so that $N = \sum_{j=1}^{m} N_j$ denotes the total number of lots inspected.

6.5.1 Thus, on the basis of a random sample of size $N = \sum_{j=1}^{m} N_j$ from m distributions characterized by the probability law:

$$\binom{k+x-1}{k-1} \phi_j^k (1 - \phi_j)^x \tag{6.5.1}$$

j = 1, 2,...m where ϕ_j denotes the quality of product from j-th machine. Following Section 2.5, we have the j-th "efficient-score"

$$\psi_j = \frac{N_j}{1-\phi_j} (\bar{x}_j - \mu_j(\phi_j)) \tag{6.5.2}$$

where $\mu_j(\phi_j)$ is the mean of the j-th distribution.

The elements of the information matrix are

$$I_{jj} = - \frac{N_j}{1-\phi_j} \frac{d\mu_j}{d\phi_j} = \frac{N_j \mu_{2j} (\phi_j)}{(1 - \phi_j)^2} \tag{6.5.3}$$

$$I_{jj'} = 0 \qquad (j \neq j')$$

where $\mu_{2_j} (\phi_j)$ is the variance of the j-th distribution.

The hypothesis of homogeneity is $H_0: \phi_1 = \phi_2 = \dots \phi_m$.

If the hypothesis H_o is true, the value may be denoted by ϕ and the efficient score and the information with respect to ϕ are given by

$$\psi = \frac{N}{1-\phi}(\bar{x} - \mu) \tag{6.5.4}$$

where

$$\bar{x} = \sum_{j=1}^{m} N_j \bar{x}_j / N$$

and

$$\mu = \sum_{j=1}^{m} N_j \mu_j(\phi)/N = \frac{k}{\phi}(1-\phi)$$

$$I = -\frac{1}{1-\phi} \sum_{j=1}^{m} N_j \cdot \frac{d\mu_j}{d\phi}$$

$$= \frac{1}{(1-\phi)^2} \sum_{j=1}^{m} N_j \mu_{2j}(\phi)$$

$$= \frac{NK}{\phi^2(1-\phi)} \tag{6.5.5}$$

Solving $\hat{\psi} = 0$, we have $\hat{\phi} = \frac{k}{k+\bar{x}}$ (6.5.6)

A test of the homogeneity hypothesis H_o is then given by the statistic

$$\chi^2 = \sum_{j=1}^{m} [\psi_j(\hat{\phi})]^2 / I_{jj}(\hat{\phi}) \tag{6.5.7}$$

$$= \sum_{j=1}^{m} N_j [\bar{x}_j - \mu_j(\hat{\phi})]^2 / \mu_{2j}(\hat{\phi}) \tag{6.5.8}$$

which is asymptotically distributed as a chisquare with (m-1) degrees of freedom if H_o is true.

6.5.2 We will illustrate the computation procedure with reference to data in the table below, which gives, for 10 (m=10) pairs of pages from Tippett's random sampling numbers, the observed distribution of the number (x) of even integers between two consecutive zeros (k=1) characterized by the model:

$$\phi(1-\phi)^x \qquad x = 0$$

For the distribution of x, we have to test if the pairs of pages are homogeneous in respect of ϕ and if so, to get a combined estimate of ϕ.

FREQUENCY FOR PAGES

No. of even integers (x)	j=1	2	3	4	5	6	7	8	9	10	Total
Pair: j / x	2 & 3	4 & 5	6 & 7	8 & 9	10 & 11	12 & 13	14 & 15	16 & 17	18 & 19	20 & 21	
0	67	71	53	65	79	71	59	70	60	66	661
1	44	58	50	47	59	59	51	54	52	47	521
2	50	32	35	43	26	44	46	44	48	44	412
3	42	25	23	33	27	37	29	36	25	21	298
4	26	23	28	26	27	30	25	29	23	30	267
5	19	28	21	19	24	21	24	15	22	20	213
6	16	17	18	17	18	15	17	5	16	17	157
7	14	20	12	14	20	10	6	8	21	16	141
8	7	9	11	13	13	8	15	9	11	11	107
9	5	5	9	9	11	7	9	5	5	8	73
10	9	4	11	5	10	5	3	7	3	9	66
11	6	9	7	7	4	9	2	6	6	7	63
12	3	3	1	4	2	1	3	2	5	3	27
13	1	2	2	4	5	3	3	2	3	3	28
14	0	2	4	3	5	2	2	4	6	4	32
15	4	3	2	1	2	2	6	2	3	1	26
16	4	2	1	3	1	3	1	3	0	0	18
17	1	2	2	2	3	2	0	0	1	2	15
18	1	2	0	0	2	1	1	1	1	0	9
19	2	0	2	1	2	0	2	2	1	1	13
20	0	1	2	0	0	1	2	0	1	1	8
21	0	0	1	1	0	0	0	0	0	0	2
22	0	0	0	0	1	0	1	0	0	1	3
23	0	0	0	0	0	2	1	3	0	1	7
24	1	0	0	0	0	0	0	1	2	0	4
25	0	0	1	0	0	0	0	1	0	0	2
26	2	1	0	0	0	0	0	0	0	0	3
27	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	1	0	0	0	1
Total N_j	324	319	296	317	342	333	309	309	315	313	3177
S_{1j}	1242	1210	1252	1222	1378	1194	1225	1138	1247	1223	
$\bar{x}_j$	3.8333	3.7931	4.2297	3.8549	4.0146	3.5856	3.9644	3.6828	3.9587	3.9073	

For the above data, we have the following:

j	N_j	$\bar{x}_j$	j	N_j	$\bar{x}_j$
1	324	3.8333	6	333	3.5856
2	319	3.7931	7	309	3.9644
3	296	4.2247	8	309	3.6828
4	317	3.8549	9	315	3.9587
5	342	4.0146	10	313	3.9073

$$N = \sum_{j=1}^{10} N_j = 3177$$

$$S_1 = \sum x n_x = 12326$$

so that

$$\bar{x} = \frac{S_1}{N} = 3.8798.$$

The maximum likelihood estimate of ϕ under the hypothesis of homogeneity is then given by

$$\hat{\phi} = \frac{1}{1 + \bar{x}} = \frac{1}{4.8798} = 0.2049.$$

To test if ϕ is the same for different pairs of pages, we have to compute

$$\sum N_j[\bar{x}_j - \mu_j(\hat{\phi})]^2 = \sum N_j(\bar{x}_j - \bar{x})^2 = 90.974$$

and

$$\mu_2(\hat{\phi}) = \frac{(1-\hat{\phi})}{\hat{\phi}^2} = \frac{\bar{x}}{\hat{\phi}} = 18.935$$

so that

$$\chi^2 = \frac{\sum N_j\,(\bar{x}_j - \bar{x})^2}{\mu_2(\hat{\phi})} = \frac{90.974}{18.935} = 4.8045$$

which with 10-1 = 9 degrees of freedom is not significant. The pairs of pages of Tippett's random sampling numbers can thus be regarded as homogeneous in respect of the distribution of even integers between two consecutive zeros, as they should be expected.

CHAPTER VII

7.0 ESTIMATION PROBLEMS FOR THE LOGARITHMIC SERIES DISTRIBUTION

7.1 Introduction

The gpsd defined by (1.1.4) becomes

$$\text{Prob}\left\{X = x\right\} = \frac{\theta^x}{-x \log(1-\theta)} \qquad x = 1,2,3,\ldots\infty \tag{7.1.1}$$

when $f(\theta) = -\log(1-\theta)$.

Writing $\alpha = \frac{1}{-\log(1-\theta)}$, (7.1.1) gives the probability law for X as:

$$\text{Prob}\left\{X = x\right\} = p(x, \theta) = \frac{\alpha\, \theta^x}{x} \qquad x = 1,2,3,\ldots\infty \tag{7.1.2}$$

a well-known form of the Logarithmic series distribution.

The first two moments about origin of (7.1.2) can be obtained as:

$$\mu = \frac{\alpha\theta}{1-\theta} \tag{7.1.3}$$

and

$$m_2 = \frac{\mu}{1-\theta}. \tag{7.1.4}$$

7.2 Estimation from a Sample for Complete Logarithmic Series Distribution

Applications of logarithmic series distribution have been discussed among others by Fisher (1943), Williams (1943, 1944), Harrison (1945) and Kendall (1948). Problems of estimation, however, do not seem to have been thoroughly investigated. Following the general approach discussed in Chapters II and III, we provide in this section different estimates for the parameter θ of the logarithmic series and investigate their efficiency and the amount of bias in certain special cases. A table of mean-values of the logarithmic series is provided to obtain the maximum likelihood estimate with facility.

7.2.1 To estimate Θ by likelihood on the basis of a random sample of size N with frequency n_x for $x(1 \leq x \leq \infty, \sum n_x = N)$ drawn from (7.1.2), results derived by the general approach in Section 2.1 can be written down as follows:

The likelihood equation for Θ is

$$\bar{x} = \hat{\mu} \tag{7.2.1}$$

where $\bar{x} = \sum xn_x/N$ and μ is defined by (7.1.3).

Denoting this estimate as $\hat{\Theta}$, its asymptotic variance is given by

$$\text{Var}(\hat{\Theta}) = \frac{\Theta^2}{N\mu_2} \tag{7.2.2}$$

where μ_2 is the variance of (7.1.2). (7.2.1) suggests that if a table be made available for means μ's for sufficiently close values of Θ, we can have a ready solution of (7.2.1). Here, we present a numerical Table VII for the argument $\Theta = .01\ (.01).99$. This table can be used to compute $\text{Var}(\hat{\Theta})$, because

$$\mu_2 = \mu\left(\frac{1}{1-\Theta} - \mu\right). \tag{7.2.3}$$

7.2.2 The expression (7.1.4) for m_2 gives

$$\Theta = 1 - \frac{\mu}{m_2}. \tag{7.2.4}$$

The two-moments estimate for Θ can then be written down as

$$t = 1 - \frac{S_1}{S_2} \tag{7.2.5}$$

where

$$S_i = \sum x^i n_x \qquad i = 1,2.$$

The asymptotic variance of t is given by

$$Var(t) = \frac{1}{Nm_2^2}[\sigma_{11} - 2(1-\theta)\,\sigma_{12} + (1-\theta)^2\,\sigma_{22}] \qquad (7.2.6)$$

where

$$\sigma_{11} = m_2 - \mu^2$$

$$\sigma_{12} = m_3 - \mu m_2 \qquad (7.2.7)$$

and

$$\sigma_{22} = m_4 - m_2^2$$

where m_r denotes the rth moment of (7.1.2) about origin.

The efficiency of t is then obtained as $Eff(t) = \frac{Var(\hat{\theta})}{Var(t)}$

The following table gives the efficiency of t for θ = .10, .50, .90.

TABLE 7.2.1

Efficiency of TM

θ	.10	.50	.90
$\frac{Var(\hat{\theta})}{Var(t)}$	.228	.449	.488

7.2.3 Following the general approach discussed in Section 3.1 , a ratio-estimate can be obtained for θ of (7.1.2). In this case, since $a_{x-1}/a_x = \frac{x}{x-1} = 1 + \frac{1}{x-1}$, the ratio-estimate for θ can be written down as

$$\theta' = (1-\frac{n_1}{N}) + \frac{1}{N}\sum_{x=2}^{\infty}\frac{n_x}{x-1}. \qquad (7.2.8)$$

(7.2.8) provides the unique unbiased estimate of Θ linear in the frequencies. The exact variance of this estimate is

$$\sigma^2(\Theta') = \frac{1}{N}\left[\sum_{x=2}^{\infty}\left(\frac{x}{x-1}\right)^2 P_x - \Theta^2\right] \qquad (7.2.9)$$

and an unbiased estimate of $\sigma^2(\Theta')$ is

$$\left[\sum_{x=2}^{\infty}\left(\frac{x}{x-1}\right)^2 n_x - N\Theta'^2\right]/N(N-1). \qquad (7.2.10)$$

The efficiency of Θ' is then obtained as $\text{Eff}(\Theta') = \frac{\text{Var}(\hat{\Theta})}{\sigma^2(\Theta')}$.

The following table gives the efficiency of Θ' for Θ = .10, .50, .90.

Efficiency of R

Θ	.10	.50	.90
$\frac{\text{Var}(\hat{\Theta})}{\sigma^2(\Theta')}$	.895	.447	.057

7.2.4 One more estimate for Θ can be obtained when one notices that the expression (7.1.3) for mean $\mu = \frac{\alpha\Theta}{1-\Theta}$ can be written down as

$$\Theta = 1 - \frac{P_1}{\mu} \qquad (7.2.11)$$

because

$$P_1 = \alpha\Theta. \qquad (7.2.12)$$

The identity (7.2.11) suggests an estimate for Θ as

$$\Theta'' = 1 - \frac{n_1}{S_1} \qquad (7.2.13)$$

with

$$\text{Var}(\Theta'') = \frac{1}{N\mu^2}\left[\sigma_{11} - 2(1-\Theta)\,\sigma_{12} + (1-\Theta)^2\,\sigma_{22}\right] \qquad (7.2.14)$$

where

$$\sigma_{11} = P_1(1-P_1)$$

$$\sigma_{12} = P_1(1-\mu) \qquad (7.2.15)$$

and

$$\sigma_{22} = m_2 - \mu^2.$$

The asyptotic efficiency of Θ'' is then obtained as

$$\text{Eff}\ (\Theta'') = \frac{\text{Var}(\hat{\Theta})}{\text{Var}(\Theta'')}.$$

The following table gives the efficiency of Θ'' for Θ = .10, .50, .90.

Efficiency of Θ''

Θ	.10	.50	.90
$\frac{\text{Var}(\Theta)}{\text{Var}(\Theta'')}$	.983	.897	.739

7.2.5 So far, we have separately discussed the simple estimates denoted by t, Θ' and Θ''. To make a comparative study of these estimates, let us investigate their amount of bias and relative efficiency.

It can be easily deduced that the amounts of bias to order $\frac{1}{N}$ of the estimates t and Θ'' are

$$b(t) = \frac{1}{N\mu^2}\,[(1-\Theta)\,\sigma_{22} - \sigma_{12}] = \frac{B(t)}{N} \qquad (7.2.16)$$

where σ_{22} and σ_{12} are defined by (7.2.7).

and

$$b(\Theta'') = \frac{1}{N\mu^2}\left[(1-\Theta)\,\sigma_{22} - \sigma_{12}\right] = \frac{B(\Theta'')}{N} \qquad (7.2.17)$$

where σ_{22} and σ_{12} are defined by (7.2.15).

As regards Θ', it is known that it is unbiased.

The following table gives $B(t)$, $B(\Theta'')$ and also relative efficiency of these estimates with respect to the ratio-estimate Θ' for $\Theta = .10, .50, .90$.

TABLE 7.2.2

Comparison of the Estimates

Θ	N(amount of bias to order $\frac{1}{N}$)		$\frac{Var(\Theta')}{Var(t)}$	$\frac{Var(\Theta')}{Var(\Theta'')}$
	t	Θ''		
.10	0.9128	0.0948	0.255	1.098
.50	1.7329	0.3466	1.003	2.006
.90	0.9279	0.2303	8.564	12.973

Thus, it is easy to see that the estimate Θ'' may be used with advantage to estimate the parameter Θ of the logarithmic series because of its simplicity, small bias and high efficiency.

7.2.6 The detailed computation procedure of evaluating the four types of estimates discussed above is illustrated with reference to the logarithmic series data due to Williams. The following table gives the distribution of 1534 biologists according to the number of research papers to their credit in the Review of Applied Entomology, Vol. 24, 1936.

No. of papers per author (x)	1	2	3	4	5	6	7	8	9	10	11
No. of authors (n_x)	1062	263	120	50	22	7	6	2	0	1	1

<u>Maximum Likelihood Estimate</u>: For the data we get

$$N = 1534$$

$$S_1 = 2379$$

so that $\bar{x} = 1.5508$. Referring to Table VII we find the following:

Θ	μ
.56	1.5503
.57	1.5706.

The maximum likelihood estimate is given by that value of Θ for which $\mu = 1.5508$. By linear interpolation, we get $\hat{\Theta} = 0.5602$. To compute the variance of $\hat{\Theta}$, taking 0.5602 as the estimate for Θ, we require

$$\alpha = \frac{\log_{10} e}{-\log_{10}(1-\Theta)} = 1.21738$$

$$\mu = \frac{\alpha\Theta}{1-\Theta} = 1.5507$$

$$m_2 = \frac{\mu}{1-\Theta} = 3.5259$$

$$\mu_2 = m_2 - \mu^2 = 1.1212.$$

Then the variance of $\hat{\Theta}$ is estimated from $\mathrm{Var}(\hat{\Theta}) = \dfrac{\Theta^2}{N\mu_2} = 0.000182$

so that S.E. $(\hat{\Theta}) = 0.0135$.

Two-Moments Estimate: To estimate this estimate of θ, we require in addition the value of

$$S_2 = \Sigma x^2 n_x = 5439.$$

Then, the estimate is

$$t = 1 - \frac{S_1}{S_2} = 0.5626.$$

To compute the variance of t, taking 0.5626 as the estimate for θ, we have with usual symbols,

$$\alpha = 1.2093$$

$$\mu = 1.5555$$

$$m_2 = 3.5562$$

$$\mu_2 = 1.1366.$$

Also,

$$m_3 = \mu m_2 + \frac{1}{1-\theta}\left(\mu_2 + \frac{\mu_2}{\alpha}\right)$$

$$= 12.7045.$$

$$m_4 = \mu m_3 + (m_3 - \mu m_2)\ (\mu + \frac{2}{1-\theta})$$

$$+ m_2\ (m_2 + \mu_2) + \frac{2\mu\mu_2}{1-\theta}\ (\frac{1}{\alpha} - 1)$$

$$= 79.0059$$

so that

$$\sigma_{11} = m_2 - \mu^2 = 1.1366$$

$$\sigma_{12} = m_3 - \mu m_2 = 7.1728$$

$$\sigma_{22} = m_4 - m_2^2 = 66.3593.$$

Then, the variance of t is estimated from

$$\text{Var}(t) = \frac{1}{Nm_2^2}\ [\sigma_{11} - 2(1-\theta)\ \sigma_{12} + (1-\theta)^2\ \sigma_{22}]$$

$$= 0.000390$$

so that S.E.(t) = 0.0197.

<u>Ratio Estimate</u>: The ratio estimate for Θ is given by

$$\Theta' = (1- \frac{n_1}{N}) + \frac{1}{N} \sum_{x=2}^{\infty} \frac{n_x}{x-1}$$

$$= 0.3077 + 0.2269$$

$$= 0.5346.$$

The variance of Θ' is estimated from the formula:

$$Var(\Theta') = \frac{1}{N} \Theta(1-\Theta) + \alpha \Theta \sum_{x=1}^{\infty} \frac{\Theta^x}{x^2}.$$

Approximating

$$\sum_{x=1}^{\infty} \frac{\Theta^x}{x^2} \sim \sum_{x=1}^{20} \frac{\Theta^x}{x^2} = 0.6310$$

$$Var(\Theta'') = 0.000450$$

so that S.E. $(\Theta'') = 0.0212$.

<u>Alternative Estimate</u>: The estimate based on n_1 and S_1 is given by

$$\Theta'' = 1 - \frac{n_1}{S_1} = 1 - \frac{1062}{2378}$$

$$= 0.5536.$$

To compute variance of Θ'', using usual symbols we have

$$\alpha = 1.23986$$

$$P_1 = 0.23986$$

$$\mu = 1.5376$$

$$m_2 = 3.4444$$

$$\mu_2 = 1.0802$$

so that

$$\sigma_{11} = P_1(1-P_1) = 0.2153$$

$$\sigma_{12} = P_1(1 - \mu) = -0.13690$$

$$\sigma_{22} = m_2 - \mu^2 = 1.0802.$$

Then the variance of θ'' is estimated from

$$\mathrm{Var}(\theta'') = \frac{1}{N\mu^2}[\sigma_{11} - 2(1-\theta)\,\sigma_{12} + (1-\theta)^2\,\sigma_{22}]$$

$$= 0.000279$$

so that $\mathrm{S.E.}(\theta'') = 0.0167$.

The following table summarizes the results obtained.

Estimate	Value	Variance	Standard Error
ML	0.5602	0.000182	0.0135
TM	0.5626	0.000390	0.0197
R	0.5346	0.000450	0.0212
θ''	0.5536	0.000279	0.0167

7.3 Estimation from a Sample for Truncated Logarithmic Series Distribution

Following the general approach discussed in Chapters I, II, and III, the results for the estimation of the parameter θ can be written down on the basis of a sample drawn from a truncated logarithmic series distribution.

TABLES I - VII

TABLE I

$\frac{\mu^*}{n}$ OF SINGLY TRUNCATED BINOMIAL DISTRIBUTION ON THE LEFT AT c=1

(Zero-Observations Truncated)

n = 3(1)15; π = .00(.01).99

π	00	01	02	03	04	05	06	07	08	09
					n = 3					
00	000000	336689	340098	343536	347029	350570	354158	357795	361481	365217
10	369004	372842	376733	380677	384675	388727	392835	396999	401220	405499
20	409836	414233	418690	423209	427789	432432	437139	441911	446748	451651
30	456621	461659	466766	471943	477190	482509	487900	493364	498902	504515
40	510204	515969	521812	527732	533732	539811	545971	552212	558534	564940
50	571428	578001	584658	591401	598229	605144	612145	619233	626409	633673
60	641026	648466	655996	663613	671321	679117	687002	694975	703037	711187
70	719424	727749	736160	744657	753239	761905	770653	779484	788395	797384
80	806451	815594	824810	834098	843455	852878	862366	871915	881523	891186
90	900901	910664	920471	930319	940203	950119	960061	970026	980008	990001
					n = 4					
00	000000	253781	257626	261841	265510	269551	273659	277835	282080	286396
10	290782	295241	299772	304377	309056	313812	318643	323553	328540	333607
20	338753	343981	349290	354681	360156	365714	371357	377086	382900	388801
30	394789	400864	407028	413281	419622	426053	432574	439185	445885	452677
40	459559	466531	473594	480748	487991	495325	502749	510262	517864	525555
50	533333	541199	549151	557189	565311	573518	581807	590177	598627	607157
60	615763	624446	633203	642033	650933	659903	668939	678041	687206	696432
70	705716	715057	724453	733900	743397	752941	762530	772161	781831	791539
80	801282	811057	820862	830694	840551	850430	860330	870248	880182	890130
90	900090	910060	920037	930022	940012	950006	960002	970001	980000	990000
					n = 5					
00	000000	204040	208162	212365	216652	221025	225483	230027	234660	239382
10	244194	249098	254098	259181	264364	269641	275014	280484	286050	291715
20	297477	303339	309300	315362	321523	327785	334148	340611	347176	353841
30	360607	367474	374441	381508	388675	395941	403304	410766	418324	425978
40	433727	441569	449503	457529	465644	473848	482138	490513	498971	507510
50	516129	524825	533596	542440	551356	560340	569390	578504	587680	596916
60	606207	615554	624952	634399	643893	653432	663012	672632	682289	691981
70	701705	711459	721241	731049	740880	750733	760605	770496	780402	790323
80	800256	810200	820155	830118	840088	850064	860046	870032	880022	890014
90	900009	910005	920003	930001	940001	950000	960000	970000	980000	990000

TABLE I (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 6					
00	000000	170882	175196	179611	184126	188745	193467	198295	203229	208270
10	213420	218680	224049	229530	235123	240828	246646	252567	258623	264782
20	271056	277443	283944	290559	297287	304128	311082	318146	325322	332607
30	340001	347502	355109	372820	370634	378550	386564	394676	402884	411185
40	419576	428056	436622	445271	454002	462811	471696	480653	489681	498777
50	507936	517158	526439	535775	545165	554605	564093	573626	583201	592816
60	602468	612154	621872	631620	641396	651197	661021	670866	680731	690613
70	700511	710422	720327	730283	740229	750183	760145	770114	780088	790068
80	800051	810038	820028	830020	840014	850010	860006	870004	880002	898001
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 7					
00	000000	147200	151659	156236	160932	165748	170686	175747	180933	186243
10	191680	197243	202934	208753	214700	220776	226979	233312	239772	246359
20	253073	259914	266879	273968	281179	288512	295963	393532	311217	319015
30	326924	334941	343064	351291	359618	368042	376561	385172	393871	402654
40	411520	420464	429483	438574	447723	456957	466243	475587	484986	494437
50	503937	513482	523071	532699	542364	552063	561794	571553	581340	591151
60	600985	610838	620710	630599	640502	650418	660347	670286	680234	690190
70	700153	710122	720097	730076	740059	750046	760035	770026	780019	790014
80	800010	810007	820005	830003	840002	850001	860001	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 8					
00	000000	129441	134015	138724	143570	148553	153676	158940	164345	169892
10	175582	181416	187393	193513	199771	206183	212731	219420	226248	233215
20	240319	247557	254928	262429	270058	277813	285689	293684	301796	310020
30	318352	326790	335330	343967	352699	361520	370427	379415	388482	397623
40	406833	416110	425448	434845	444297	453800	463350	472944	482580	492253
50	501961	511700	521469	531265	541085	550926	560788	570667	580562	590471
60	600393	610327	620270	630221	640180	650146	660118	670094	680075	690059
70	700046	710035	720027	730020	740015	750011	760008	770006	780004	790003
80	800002	810001	820001	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE I (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 9					
00	000000	115630	120299	125120	130119	135226	140513	145958	151561	157323
10	163244	169324	175561	181957	188508	195215	202075	209087	216247	223554
20	231005	238596	246324	254186	262178	270295	278534	286890	295358	303945
30	312615	321394	330267	339229	348276	357402	366604	375876	385215	394615
40	404072	413583	423143	432748	442396	452082	461803	471556	481338	491146
50	500978	510832	520704	530594	540498	550416	560346	570287	580236	590193
60	600157	610127	620102	630082	640065	650051	660040	670031	680024	690018
70	700014	710010	720007	730005	740004	750003	760002	770001	780001	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 10					
00	000000	104583	109333	114253	119343	124607	130043	135654	141440	147400
10	153534	159841	166320	172970	179787	186770	193916	201222	208683	216297
20	224058	231963	240006	248184	256489	264981	273465	282125	290891	299758
30	308721	317773	326911	336127	345417	354776	364199	373680	383216	392802
40	402433	412106	421817	431562	441339	451143	460972	470823	480695	490584
50	500489	510407	520338	530279	540229	550187	560152	570123	580099	590079
60	600063	610050	620039	630030	640023	650018	660013	670010	680008	690006
70	700004	710003	720002	730001	740001	750001	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 11					
00	000000	095546	100367	105375	110570	115955	121531	127297	133253	139398
10	145732	152207	158957	165843	172908	180146	187555	195129	202864	210754
20	218794	226978	235299	233751	252329	261024	269832	278745	287757	296861
30	306052	315322	324667	334080	343556	353090	362676	372310	381988	391704
40	401456	411240	421052	430889	440749	450628	460524	470436	480361	490298
50	500244	510199	520162	530131	540105	550084	560067	570053	580041	590013
60	600025	610019	620015	630011	640008	650006	660004	670003	680002	690002
70	700001	710001	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE I (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 12					
00	000000	088016	092901	097989	103282	108781	114486	120398	126515	132837
10	139359	146080	152997	160105	167399	174874	182525	190346	198330	206469
20	214758	223159	231736	240445	249256	258178	267205	276328	285542	294838
30	304211	313653	323159	332723	342339	352002	361708	371452	381230	391038
40	400873	410731	420609	430506	440419	450345	460283	470231	480188	490152
50	500122	510098	520078	530061	540048	550038	560029	570023	580017	590005
60	600010	610007	620005	630004	640003	650002	660001	670001	680001	690000
70	700000	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 13					
00	000000	081647	086588	091751	097135	102742	108571	114622	120892	127380
10	134082	140993	148110	155486	162935	170630	178505	186550	194943	203124
20	211635	220283	229061	237959	246970	256084	265293	274591	283968	293419
30	302935	312511	322141	331819	341540	351299	361091	370913	380762	390632
40	400523	410431	420353	430289	440234	450190	460153	470122	480097	490077
50	500061	510048	520037	530029	540022	550017	560013	570010	580007	590002
60	600004	610003	620002	630001	640001	650001	660000	670000	680000	690000
70	700000	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 14					
00	000000	076188	081183	086415	091885	097594	103542	109725	116143	122790
10	129663	136755	144060	151572	159282	167181	175261	183513	191927	200493
20	209201	218041	227004	236080	245260	254535	263896	273336	282846	292419
30	302048	311728	321453	331217	341015	350843	360698	370575	380472	390385
40	400314	410254	420205	430164	440131	450104	460082	470065	480051	490039
50	500030	510023	520018	530013	540010	550008	560006	570004	580003	590001
60	600001	610001	620001	630000	640000	650000	660000	670000	680000	690000
70	700000	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE I (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 15					
00	000000	071458	076502	081800	087353	093160	099221	105533	112091	118892
10	125927	133191	140676	148371	156269	164357	172627	181066	189665	198411
20	207291	216302	225425	234654	243977	253386	262872	272427	282043	291713
30	301431	311191	320986	330814	340669	350548	360446	370362	380292	390235
40	400188	410150	420119	430094	440073	450057	460044	470034	480026	490020
50	500015	510011	520008	530006	540004	550003	560002	570002	580001	590001
60	600001	610000	620000	630000	640000	650000	660000	670000	680000	690000
70	700000	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE II

$\frac{\mu^*}{n}$ OF SINGLY TRUNCATED BINOMIAL DISTRIBUTION ON THE LEFT AT c=2

n = 3(1)15; π = .00(.01).99

π	00	01	02	03	04	05	06	07	08	09
					n = 3					
00	000000	667771	668916	670066	671232	672413	673611	674825	676056	677305
10	678571	679856	681159	682482	683823	685185	686567	687970	689394	690840
20	692308	693798	695312	696850	698413	700000	701613	703252	704918	706612
30	708333	710084	711864	713675	715517	717391	719298	721239	723214	725225
40	727273	729358	731481	733643	735849	738095	740385	742718	745098	747525
50	750000	752525	755102	757732	760417	763158	765957	768817	771739	774725
60	777778	780899	784091	787356	790698	794118	797619	801205	804878	808642
70	812500	816456	820513	824675	828947	833333	837838	842466	847222	852113
80	857143	862319	867647	873134	878788	884615	890625	896825	903226	909836
90	916667	923729	931034	938596	946428	954545	962963	971698	980769	990196
					n = 4					
00	000000	501674	503388	505127	506895	508693	510523	512384	514277	516204
10	518164	520160	522191	524258	526363	528505	530687	522909	535171	537476
20	539823	542214	544650	547132	549663	552239	554866	557543	560272	563055
30	565891	568784	671734	574742	577810	580939	584131	587388	590711	594101
40	597561	601092	604695	608373	612128	615960	619873	623867	627946	632111
50	636364	640707	645142	649671	654298	659023	663849	668771	673814	678957
60	684210	689576	695057	700656	706373	712213	718176	724266	730485	736834
70	743314	749932	756684	763575	770606	777778	785092	792549	800151	807898
80	815789	823826	832006	840331	848797	857404	866149	875028	884040	893178
90	902439	911816	921302	930891	940573	950338	960177	970076	980023	990003
					n = 5					
00	000000	402014	404080	406184	408330	410520	412756	415037	417366	419743
10	422170	424648	427178	429761	432399	435093	437845	440655	443525	446457
20	449452	452511	455637	458830	462092	465426	468831	472311	475867	479500
30	483212	487006	490883	494844	498891	503027	507253	511571	515983	520491
40	525096	529801	534608	539517	544531	549653	554883	560223	565675	571242
50	576923	582721	588638	594674	600832	607112	613515	620043	626695	633473
60	640378	647410	654569	661856	669269	676809	684475	692266	700182	708220
70	716380	724660	733057	741569	750194	758928	767769	776713	785756	794894
80	804124	813439	822836	832310	841855	851466	861138	870866	880643	890464
90	900324	910217	920138	930083	940046	950022	960009	970003	980000	990000

TABLE II (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 6					
00	000000	335577	337882	340239	342652	345121	347650	350239	352891	355606
10	358386	361233	364149	367135	370194	373326	376534	379819	383183	386629
20	390158	393772	397472	401262	405142	409115	413183	417347	421609	425973
30	430438	435009	439685	444470	449364	454370	459490	464725	470076	475546
40	481135	486846	492679	498635	504716	510922	517255	523715	530302	537017
50	543860	550830	557929	565154	572507	579985	587588	595314	603163	611132
60	619219	627423	635740	644169	652706	661348	670093	678936	687874	696903
70	706019	715218	724496	733849	743271	752759	762308	771914	781572	791277
80	801025	810813	820636	830489	840370	850274	860199	870140	880096	890064
90	900040	910024	920014	930007	940003	950001	960000	970000	980000	990000
					n = 7					
00	000000	288125	290605	293151	295765	298451	301209	304042	306952	309941
10	313010	316163	319401	322727	326142	329648	333241	336946	340741	344636
20	348635	352738	356948	361268	365699	370244	374904	379682	384579	389598
30	394739	400005	405398	410918	416568	422348	428251	434303	440479	446790
40	453324	459813	466526	473374	480355	487469	494716	502094	509602	517237
50	525000	532887	540896	549024	557270	565629	574099	582677	591158	600140
60	609017	617987	627044	636186	645406	654702	664069	673501	682995	692547
70	702151	711805	721503	731241	741017	750825	760663	770527	780414	790321
80	800246	810185	820127	830100	840071	850049	860033	870022	880014	890008
90	900005	910002	920001	930000	940000	950000	960000	970000	980000	990000
					n = 8					
00	000000	252536	255152	257848	260624	263485	266433	269470	272599	275822
10	279142	282561	286082	289708	293440	297282	301237	305306	309492	313717
20	318225	322776	327455	332262	337199	342270	347474	352815	358294	363912
30	369670	375569	381611	387795	394122	400593	407207	413964	420863	427904
40	435085	442405	449862	457454	465180	473036	481020	489128	497359	505707
50	514170	522743	531423	530205	549085	558058	567121	576267	585493	594793
60	604164	613600	623097	632651	642256	651909	661606	671342	681114	690918
70	700751	710609	720490	730390	704308	750240	760185	770141	780106	790078
80	800057	810041	820029	830020	840013	850008	860005	870003	880002	890001
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE II (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 9					
00	000000	224857	227584	230401	233314	236324	239435	242650	245972	249403
10	252947	256607	260385	264285	268309	272461	276743	281158	285708	290396
20	295225	300196	305311	310574	315984	321544	327255	333118	339133	345303
30	351625	358101	364731	371512	378446	385529	392761	400140	407663	415327
40	423131	431070	439141	447340	455664	464108	472668	481338	490115	498994
50	507968	517034	526186	535418	544727	554106	563552	573058	582620	592234
60	601895	611598	621340	631117	640926	650762	660623	670505	680407	690325
70	700257	710202	720157	730120	740091	750068	760051	770037	780027	790019
80	800013	810009	820006	830004	840002	850001	860001	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 10					
00	000000	202715	205533	208455	211485	214626	217882	221256	224753	228374
10	232125	236007	240025	244182	248480	252924	257515	262257	267152	272202
20	277410	282777	288305	293996	299850	305868	312051	318399	324910	331586
30	333823	345422	352579	359894	367362	374982	382750	390661	398712	406899
40	415217	423661	432226	440907	449698	458593	467588	476675	485851	495108
50	504442	513847	533317	532848	542433	552069	561751	571474	581234	591028
60	600851	610700	620572	630464	640374	650300	660238	670187	680146	690113
70	700087	710066	720049	730036	740027	750019	760014	770009	780006	790004
80	800003	810002	820001	830001	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 11					
00	000000	184599	187496	190509	193642	196901	200289	203810	207468	211267
10	215211	219304	223549	227949	232509	237230	242117	247171	252394	257790
20	263359	269103	275022	281117	287389	293836	300458	307253	314221	321358
30	328661	336128	343756	351539	359474	367555	375779	384138	392628	401243
40	409976	418822	427773	436825	445970	455202	464514	573902	483359	492878
50	502456	512085	521763	531482	541240	551032	560854	570703	580575	590468
60	600378	610303	620241	630191	640150	650116	660090	670069	680052	690039
70	700029	710021	720015	730011	740008	750005	760003	770002	780001	790001
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE II (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 12					
00	000000	169504	172469	175562	178790	182156	185666	189324	193134	197102
10	201230	205523	209985	214620	219430	224419	229590	234944	240483	246209
20	252123	258225	264515	270991	277654	284501	291529	298737	306120	313674
30	321395	329277	337317	345507	353841	362314	370919	379648	388494	397452
40	406513	415670	424917	434247	443653	453129	462668	472265	481914	491610
50	501347	511121	520929	530765	540627	550510	560413	570332	580265	590211
60	600166	610130	620101	630078	640059	650045	660003	670025	680018	690013
70	700009	710007	720005	730003	740002	750001	760001	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 13					
00	000000	156733	159757	162924	166238	169704	173329	177116	181072	185200
10	189505	193992	198664	203525	208578	213825	219270	224913	230756	236799
20	243042	249484	256124	262959	269987	277204	284607	292189	299947	307185
30	315965	324214	332608	341145	349818	358616	367533	376561	385693	394919
40	404233	413627	423095	432629	442223	451871	461567	471306	481083	490894
50	500734	510599	520486	530392	540314	550250	560198	570156	580122	590094
60	600072	610055	620042	630031	640023	650017	660012	670009	680006	690004
70	700003	710002	720001	730001	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000
					n = 14					
00	000000	145786	148865	152098	155493	159054	162787	166698	170793	175077
10	179554	184228	189104	194185	199473	204972	210682	216604	222738	229085
20	235641	242406	249375	256545	263912	271471	279214	287136	295230	303487
30	311901	320462	329162	337993	346946	356011	365181	374447	383800	393233
40	402739	412309	421938	431619	441345	451112	460915	470749	480610	490493
50	500397	510317	520252	530199	540156	550122	560094	570072	580055	590042
60	600031	610023	620017	630012	640009	650006	660004	670003	680002	690001
70	700001	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE II (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 15					
00	000000	136300	139428	142724	146194	149844	153682	157713	161943	166377
10	171021	175878	180953	186248	181766	197508	203474	209666	216081	222717
20	229571	236640	243918	251400	259080	266950	275002	283228	291618	300165
30	308858	317688	326644	335718	344900	354181	363551	373002	382527	292117
40	401764	411464	421208	430992	440810	450658	460532	470427	480341	490271
50	500214	510167	520130	530101	540077	550059	560045	570033	580025	590018
60	600013	610010	620007	630005	640003	650002	660002	670001	680001	690000
70	700000	710000	720000	730000	740000	750000	760000	770000	780000	790000
80	800000	810000	820000	830000	840000	850000	860000	870000	880000	890000
90	900000	910000	920000	930000	940000	950000	960000	970000	980000	990000

TABLE III

$\frac{\mu^*}{n}$ OF DOUBLE TRUNCATED BINOMIAL DISTRIBUTION AT c=1 AND d=n

$n = 3(1)15;\ \pi = .00(.01).99$

π	00	01	02	03	04	05	06	07	08	09
					n = 3					
00	000000	336667	340000	343333	346667	350000	353333	356667	360000	363333
10	366667	370000	373333	376667	380000	383333	386667	390000	393333	396667
20	400000	403333	406667	410000	413333	416667	420000	423333	426667	430000
30	433333	436667	440000	443333	446667	450000	453333	456667	460000	463333
40	466667	470000	473333	476667	480000	483333	486667	490000	493333	496667
50	500000	503333	506667	510000	513333	516667	520000	523333	526667	530000
60	533333	536667	540000	543333	546667	550000	553333	556667	560000	563333
70	566667	570000	573333	576667	580000	583333	586667	590000	593333	596667
80	600000	603333	606667	610000	613333	616667	620000	623333	626667	630000
90	633333	636667	640000	643333	646667	650000	653333	656667	660000	663333
					n = 4					
00	000000	253781	247625	261530	265498	269526	273616	277766	281977	286247
10	290576	294963	299409	303911	308470	313084	317753	322476	327251	332078
20	336956	341884	346861	351884	356954	362061	367227	372428	377669	382950
30	388268	393623	399013	404435	409890	415374	420886	426425	431988	437574
40	443182	448808	454452	460112	465785	471469	477164	482866	488574	494286
50	500000	505714	511426	517134	522836	528531	534215	539888	545548	551192
60	556818	562426	568012	573575	579114	584626	590110	595565	600987	606377
70	611732	617050	622331	627572	632773	637931	643046	648116	653139	658116
80	663043	667922	672749	677524	682247	686916	691530	696089	700591	705036
90	709424	713753	718023	722234	726384	730474	734502	738470	742375	746219
					n = 5					
00	000000	204040	208162	212365	216652	221024	225480	230023	234653	239370
10	244176	249070	254054	259127	264289	269542	274884	280315	285836	291445
20	297143	302928	308800	314757	320798	326923	333129	339415	345780	352220
30	358734	365320	371975	378697	385483	392330	399235	406194	413206	420265
40	427368	434513	441694	448909	456153	463422	470712	478019	485339	492667
50	500000	507333	514661	521981	529288	536578	543847	551091	558306	565487
60	572632	579735	586794	593805	600765	607670	614517	621303	628025	634680
70	641266	647780	654220	660585	666871	673077	679202	685243	691200	697072
80	702857	708555	714164	719685	725116	730458	735771	740873	745946	750930
90	755824	760630	765347	769977	774520	778976	783348	787635	791838	795960

TABLE III (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 6					
00	000000	170882	175196	179611	184126	188745	193467	198294	203228	208269
10	213419	218677	224045	229524	235113	240814	246627	252551	258587	264734
20	270992	277361	283839	290426	297121	303922	310827	317835	324944	332151
30	339455	346852	354339	361914	369573	377312	385127	393015	400972	408992
40	417071	425205	433388	441616	449883	458184	466513	474864	483233	491614
50	500000	508386	516766	525135	533487	541816	550117	558384	566612	574795
60	582929	591008	599028	606985	614873	622688	630427	638086	645661	653148
70	660545	675056	667849	682165	689173	696078	702879	709574	716161	722639
80	729008	735266	741413	747449	753373	759186	764886	770476	775955	781323
90	786581	791731	796772	801706	806533	811255	815874	820389	824804	829118
					n = 7					
00	000000	147200	151659	156236	160932	165748	170686	175747	180933	186243
10	191680	197243	202934	208752	214699	220774	226976	233307	239765	246350
20	253061	259897	266857	273938	281141	288461	295899	303450	311114	318885
30	326763	334743	342822	350996	359262	367614	376050	384564	393151	401807
40	410526	419304	428135	437013	445933	454888	463874	472884	481913	490953
50	500000	509046	518087	527115	536125	545111	554067	562987	571865	580696
60	589474	598193	606849	615436	623950	632386	640738	649004	657178	665257
70	673237	681114	688886	696549	704101	711538	718859	726062	733143	740103
80	746939	753650	760235	766693	773023	779226	785301	791248	797066	802757
90	808320	813757	819067	824253	829314	834252	839068	843764	848341	852800
					n = 8					
00	000000	129441	134015	138724	143570	148553	153676	158940	164345	169892
10	175582	181416	187393	193513	199777	206183	212730	219419	226247	233214
20	240316	247537	254923	262423	270049	277800	285673	293663	301767	309983
30	318305	326730	335253	343871	352579	361371	370244	379192	388210	397294
40	406438	415636	424884	434177	443508	452872	462264	471679	481110	490552
50	500000	509448	518890	528321	537736	547128	556492	565823	575116	584364
60	593562	602706	611790	620808	629756	638629	647421	656129	664746	673270
70	681695	690017	698233	706337	714327	722200	729951	737577	745077	752446
80	759683	766786	773753	780581	878269	793817	800223	806487	812607	818584
90	824418	830108	835655	841060	846324	851447	856430	861276	865985	870559

TABLE III (CONT'D)

n = 9

π	00	01	02	03	04	05	06	07	08	09
00	000000	115630	120299	125120	130096	135226	140513	145958	151561	157323
10	163244	169324	175561	181957	188508	195215	202075	209087	216247	223554
20	231004	238595	246323	254185	262176	270292	278513	286884	295350	203924
30	312601	321375	330243	339198	348235	357351	366539	375794	385112	394487
40	403914	413389	422340	432461	442049	451665	461304	470962	480633	490314
50	500000	509685	519366	529038	538696	548335	557951	567538	577093	586611
60	596086	605513	614888	624206	633461	642649	651765	660802	669757	678624
70	687399	696076	704650	713116	721470	729708	737824	745815	753677	761405
80	768995	776446	783753	790913	797925	804785	811492	818043	824439	830676
90	836756	842677	848439	854042	859486	864774	869904	874880	879701	884370

n = 10

π	00	01	02	03	04	05	06	07	08	09
00	000000	104583	109333	114253	119343	124607	130043	135654	141440	147400
10	153534	159841	166320	172970	179787	186770	193916	201222	208693	216297
20	224058	231963	240006	248183	256489	264918	273464	282183	290888	299755
30	308716	317768	326903	336117	345404	354758	364175	373650	383177	392752
40	402370	412027	421718	431439	441186	450955	460743	470544	480357	490177
50	500000	509823	519643	527901	539257	549045	558814	568561	578288	587973
60	597630	607248	616823	626350	635825	645242	654596	663883	673097	682232
70	691284	700225	709111	717877	726536	735082	743511	751817	759994	768037
80	775942	783703	791317	798778	806084	813230	820213	827030	833679	840159
90	846466	852600	858560	864346	869957	875393	880657	885747	890667	895417

n = 11

π	00	01	02	03	04	05	06	07	08	09
00	000000	095546	100367	105375	110570	115955	121531	127297	133253	139398
10	145732	152253	158957	165843	172908	180146	187555	195129	202864	210754
20	218794	226978	235299	243751	252329	261024	269832	278745	287756	296860
30	306050	315315	324664	334077	343551	353084	362668	372299	381973	391685
40	401431	411208	421010	430836	440681	450543	460419	470305	480199	490098
50	500000	509902	519801	529695	539581	549456	559318	569164	578990	588792
60	598569	608315	618027	627701	637332	646916	656448	665923	675335	684679
70	693950	712244	703140	721255	730168	738976	747671	756249	764701	773022
80	781206	789245	797136	805980	812445	819854	827092	834156	841042	847747
90	854268	860602	866747	872703	878469	884045	889430	894625	899633	904454

TABLE III (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 12					
00	000000	088016	092901	097989	103282	108781	114486	120398	126515	132837
10	139359	146080	152997	160105	167399	174874	182525	190346	198330	206469
20	214758	223189	231753	240445	249256	258178	267205	276328	285542	294838
30	304210	313652	323158	332721	342337	351994	361705	371448	381224	391030
40	400862	410717	420592	430484	440389	450307	460234	470169	480110	490054
50	500000	509946	519890	529395	539765	549693	559611	569516	579408	589283
60	599137	608970	618776	628552	638295	648000	657663	667279	676842	686348
70	695790	714458	705162	723672	732795	741822	750744	759555	768247	776811
80	785242	793531	801670	809654	817474	825125	832601	839895	847003	853919
90	860641	867163	873485	879602	885514	891219	896718	902011	907099	911984
					n = 13					
00	000000	086588	081647	091751	097135	102742	108571	114622	120892	127380
10	134082	140993	148110	155426	162935	170630	178505	186550	194760	203124
20	211635	220283	229061	237951	246970	256084	265293	274591	283968	293418
30	302935	312511	322141	331819	341539	351298	361090	370912	380759	390629
40	400519	410425	420346	430279	440221	450173	460130	470093	480060	490029
50	500000	509970	519940	529906	539869	549827	559778	569721	579654	589575
60	599481	609370	619240	629088	638910	648702	658461	668181	677859	687489
70	697065	706581	716032	725409	734707	743916	753030	762041	770939	779717
80	788365	796876	805240	793509	821495	829370	837065	844574	851890	859007
90	865918	872620	879108	885378	891429	897258	902865	908249	913412	918353
					n = 14					
00	000000	076188	081183	086415	091885	097594	103542	109725	116143	122790
10	129663	136755	144060	151572	159282	167181	175261	183513	191927	200493
20	209201	218041	227004	236080	245260	254535	263896	273336	282846	292419
30	302048	311728	321453	331216	341015	350843	360697	370574	380471	390384
40	400312	410252	420202	430160	440125	450097	460072	470051	480033	490016
50	500000	509984	519967	529949	539928	549903	559874	569840	579798	589748
60	599688	609616	619529	629426	639303	649158	658985	668783	678547	688272
70	697951	707581	717154	726664	736103	745465	754740	763920	772996	781959
80	790799	799507	808073	816487	824738	832819	840718	848428	855940	862245
90	870337	877210	883857	890275	850137	902406	908115	913585	918817	923812

TABLE III (CONT'D)

π	00	01	02	03	04	05	06	07	08	09
					n = 15					
00	000000	071458	076502	081800	087353	093160	099221	105533	112091	118892
10	125927	133191	140676	148371	156269	164357	172627	181066	189665	198411
20	207293	216302	225425	234654	243977	253386	262872	272427	282043	291713
30	301431	311191	320986	330814	340669	350547	360446	370362	380292	390235
40	400187	410149	420117	430092	440071	450054	460040	470028	480018	490008
50	500000	509991	519982	529972	539960	549946	559929	569908	579882	589851
60	599812	609765	619708	629638	639554	649452	658317	669186	679013	688809
70	698569	708287	717957	727573	737128	746614	756023	765346	774575	783698
80	792706	801589	810335	818933	827373	835643	843731	851629	859324	866809
90	874073	881108	887909	894467	900779	906840	912647	918200	923498	928542

TABLE IV

μ* OF SINGLY TRUNCATED POISSON DISTRIBUTION ON THE RIGHT AT d

d = 4(1)10; μ = 0.0(0.1)4.9

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					d = 4					
0.0	0.0000	0.1000	0.1200	0.2999	0.3997	0.4992	0.5982	0.6965	0.7939	0.8900
1.0	0.9846	1.0775	1.1685	1.2574	1.3439	1.4281	1.5097	1.5886	1.6649	1.7386
2.0	1.8095	1.8778	1.9435	2.0065	2.0671	2.1252	2.1809	2.2344	2.2856	2.3346
3.0	2.3817	2.4267	2.4699	2.5113	2.5510	2.5891	2.6255	2.6605	2.6941.	2.7263.
4.0	2.7573	2.7870	2.8156	2.8430	2.8694	2.8948	2.9192	2.9428	2.9654	2.9872
					d = 5					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.4999	0.5998	0.6995	0.7990	0.8982
1.0	0.9969	1.0951	1.1925	1.2890	1.3845	1.4787	1.5716	1.6630	1.7527	1.8406
2.0	1.9266	2.0107	2.0926	2.1725	2.2502	2.3502	2.3989	2.4700	2.5388	2.6054
3.0	2.6698	2.7321	2.7923	2.8504	2.9065	2.9606	3.0128	3.0632	3.1118	3.1586
4.0	3.2037	3.2473	3.2892	3.3297	3.3688	3.4064	3.4427	3.4778	3.5116	3.5442
					d = 6					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.7999	0.8997
1.0	0.9995	0.0991	1.1985	1.2976	1.3964	1.4947	1.5925	1.6896	1.7859	1.8814
2.0	1.9758	2.0692	2.1613	2.2521	2.3415	2.4294	2.5157	2.6003	2.6832	2.7643
3.0	2.8435	2.9211	2.9964	3.0700	3.1416	3.2113	3.2791	3.3450	3.4090	3.4711
4.0	3.5314	3.5898	3.6465	3.7014	3.7547	3.8062	3.8562	3.9046	3.9515	3.9968
					d = 7					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000
1.0	0.9999	1.0999	1.1997	1.2996	1.3993	1.4989	1.5983	1.6975	1.7964	1.8950
2.0	1.9931	2.0908	2.1879	2.2844	2.3801	2.4750	2.5691	2.6621	2.7540	2.8448
3.0	2.9344	3.0227	3.1096	3.1950	3.2790	3.3614	3.4422	3.5214	5.5990	3.6748
4.0	3.7490	3.8215	3.8922	3.9613	4.0286	4.0942	4.1582	4.2204	4.2811	4.3400

TABLE IV (CONT'D)

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					d = 8					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000
1.0	1.0000	1.1000	1.2000	1.2999	1.3999	1.4998	1.5997	1.6995	1.7992	1.8988
2.0	1.9983	2.0976	2.1967	2.2955	2.3940	2.4922	2.5900	2.6873	2.7841	2.8801
3.0	2.9756	3.0703	3.1642	3.2572	3.3493	3.4404	3.5304	3.6192	3.7068	3.7932
4.0	3.8783	3.9621	4.0444	4.1253	4.2048	4.2828	4.3592	4.4342	4.5076	4.5795
					d = 9					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000
1.0	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.5999	1.6999	1.7998	1.8997
2.0	1.9996	2.0994	2.1992	2.2989	2.3984	2.4978	2.5979	2.6962	2.7950	2.8936
3.0	2.9919	3.0898	3.1873	3.2844	3.3810	3.4770	3.5724	3.6671	3.7611	3.8543
4.0	3.9464	4.0381	4.1286	4.2181	4.3066	4.3940	4.4802	4.5652	4.6490	4.7485
					d = 10					
0.0	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000
1.0	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	1.6000	1.7000	1.8000	1.8999
2.0	1.9999	2.0999	2.1998	2.2997	2.3996	2.4995	2.5992	2.6990	2.7986	2.8981
3.0	2.9976	3.0968	3.1960	3.2949	3.3935	3.4920	3.5900	3.6879	3.7853	3.8822
4.0	3.9788	4.0748	4.1702	4.2651	4.3593	4.4528	4.5455	4.6375	4.7286	4.8188

TABLE V

μ* OF SINGLY TRUNCATED POISSON DISTRIBUTION ON THE LEFT AT c

c = 1(1)10; μ = 0.0(0.1)9.9

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					c = 1					
0.0	0.0000	1.0508	1.1033	1.1575	1.2133	1.2708	1.3298	1.3905	1.4528	1.5166
1.0	1.5820	1.6489	1.7172	1.7870	1.8582	1.9308	2.0048	2.0800	2.1565	2.2342
2.0	2.3130	2.3930	2.4741	2.5563	2.6394	2.7236	2.8086	2.8945	2.9813	3.0689
3.0	3.1572	3.2462	3.3360	3.4264	3.5174	3.6090	3.7011	3.7938	3.8870	3.9806
4.0	4.0746	4.1691	4.2639	4.3591	4.4547	4.5506	4.6467	4.7431	4.8398	4.9368
5.0	5.0339	5.1313	5.2288	5.3266	5.4245	5.5226	5.6208	5.7191	5.8176	5.9162
6.0	6.0149	6.1137	6.2126	6.3116	6.4107	6.5098	6.6090	6.7083	6.8076	6.9070
7.0	7.0064	7.1059	7.2054	7.3049	7.4045	7.5041	7.6038	7.7035	7.8032	7.9029
8.0	8.0027	8.1025	8.2023	8.3021	8.4019	8.5017	8.6016	8.7014	8.8013	8.9012
9.0	9.0011	9.1010	9.2009	9.3008	9.4008	9.5007	9.6006	9.7006	9.8005	9.9005
					c = 2					
0.0	0.0000	2.0339	2.0689	2.1051	2.1424	2.1810	2.2208	2.2617	2.3039	2.3475
1.0	2.3922	2.4382	2.4856	2.5342	2.5842	2.6354	2.6880	2.7419	2.7970	2.8535
2.0	2.9114	2.9705	3.0309	3.0926	3.1556	3.2198	3.2853	3.3521	3.4200	3.4892
3.0	3.5595	3.6310	3.7036	3.7774	3.8522	3.9281	4.0050	4.0830	4.1611	4.2418
4.0	4.3226	4.4043	4.4869	4.5703	4.6546	4.7396	4.8254	4.9119	4.9991	5.0870
5.0	5.1755	5.2647	5.3544	5.4448	5.5356	5.6270	5.7189	5.8112	5.9040	5.9972
6.0	6.0908	6.1848	6.2792	6.3739	6.4689	6.5642	6.6599	6.7558	6.8519	6.9484
7.0	7.0450	7.1419	7.2389	7.3362	7.4336	7.5313	7.6290	7.7270	7.8250	7.9233
8.0	8.0215	8.1199	8.2185	8.3172	8.4159	8.5147	8.6136	8.7126	8.8117	8.9108
9.0	9.0100	9.1093	9.2086	9.3079	9.4073	9.5068	9.6062	9.7058	9.8053	9.9049
					c = 3					
0.0	0.0000	3.0244	3.0515	3.0785	3.1062	3.1347	3.1642	3.1944	3.2256	3.2576
1.0	3.2906	3.3245	3.3594	3.3952	3.4320	3.4698	3.5086	3.5484	3.5893	3.6313
2.0	3.6743	3.7184	3.7636	3.8099	3.8572	3.9058	3.9554	4.0062	4.0580	4.1111
3.0	4.1652	4.2206	4.2770	4.3346	4.3933	4.4532	4.5142	4.5763	4.6395	4.7038
4.0	4.7693	4.8358	4.9034	4.9720	5.0417	5.1125	5.1842	5.2570	5.3307	5.4054
5.0	5.4811	5.5577	5.6352	5.7136	5.7928	5.8729	5.9539	6.0356	6.1181	6.2014
6.0	6.2854	6.3711	6.4555	6.5416	6.6284	6.7157	6.8037	6.8922	6.9813	7.0710
7.0	7.1612	7.2518	7.3430	7.4346	7.5266	7.6191	7.7119	7.8052	7.8988	7.9928
8.0	8.0871	8.1817	8.2766	8.3718	8.4673	8.5631	8.6590	8.7553	8.8517	8.9484
9.0	9.0452	9.1423	9.2395	9.3369	9.4345	9.5322	9.6301	9.7280	9.8262	9.9244

TABLE V (CONT'D)

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					c = 4					
0.0	0.0000	4.0710	4.0440	4.0626	4.0842	4.1062	4.1301	4.1542	4.1782	4.2032
1.0	4.2290	4.2554	4.2824	4.3103	4.3388	4.3681	4.3981	4.4289	4.4604	4.4927
2.0	4.5259	4.5599	4.5947	4.6304	4.6670	4.7044	4.7428	4.7802	4.8222	4.8633
3.0	4.9053	4.9483	4.9922	5.0371	5.0830	5.1299	5.1778	5.2267	5.2766	6.3275
4.0	5.3794	5.4323	5.4863	5.5413	5.5973	5.6544	5.7124	5.7714	5.8317	5.8928
5.0	5.9550	6.0181	6.0823	6.1475	6.2136	6.2807	6.3489	6.4179	6.4879	6.5589
6.0	6.6308	6.7034	6.7773	6.8518	6.9274	7.0036	7.0807	7.1587	7.2375	7.3170
7.0	7.3974	7.4785	7.5603	7.6428	7.7261	7.8099	7.8946	7.9798	8.0656	8.1521
8.0	8.2391	8.3267	8.4149	8.5036	8.5928	8.6825	8.7727	8.8633	8.9544	9.0459
9.0	9.1378	9.2302	9.3229	9.4159	9.5093	9.6031	9.6972	9.7915	9.8862	9.9812
					c = 5					
0.0	0.0000	—	4.9391	5.0468	5.0738	5.0930	5.1137	5.1057	5.1481	5.1677
1.0	5.1880	5.2100	5.2314	5.2540	5.2771	5.3006	5.3248	5.3494	5.3746	5.4006
2.0	5.4271	5.4542	5.4820	5.5104	5.5395	5.5693	5.5997	5.6309	5.6628	5.6954
3.0	5.7287	5.7628	5.7977	5.8333	5.8697	5.9069	5.9450	5.9838	6.0235	6.0641
4.0	6.1054	6.1477	6.1908	6.2348	6.2797	6.3255	6.3721	6.4197	6.4682	6.5177
5.0	6.5680	6.6193	6.6716	6.7247	6.7788	6.8339	6.8899	6.9468	7.0047	7.0635
6.0	7.1233	7.1840	7.2457	7.3083	7.3718	7.4362	7.5016	7.5679	7.6351	7.7031
7.0	7.7721	7.8420	7.9127	7.9843	8.0567	8.1300	8.2042	8.2791	8.3548	8.4313
8.0	8.5086	8.5867	8.6655	8.7451	8.8253	8.9063	8.9880	9.0703	9.1533	9.2369
9.0	9.3212	9.4061	9.4915	9.5776	9.6642	9.7513	9.8390	9.9272	10.0159	10.1050
					c = 6					
1.0	6.1616	6.1761	6.1968	6.2134	6.2337	6.2532	6.2734	6.2940	6.3146	6.3359
2.0	6.3577	6.3700	6.4028	6.4261	6.4499	6.4742	6.4991	6.5245	6.5505	6.5771
3.0	6.6042	6.6319	6.6602	6.6891	6.7186	6.7488	6.7796	6.8119	6.8432	6.8760
4.0	6.9095	6.9437	6.9786	7.0142	7.0506	7.0877	7.1255	7.1641	7.2034	7.2435
5.0	7.2845	7.3262	7.3687	7.4120	7.4561	7.4964	7.5469	7.5936	7.6410	7.6894
6.0	7.7386	7.7886	7.8395	7.8913	7.9440	7.9975	8.0519	8.1072	8.1634	8.2204
7.0	8.2784	8.3372	8.3969	8.4575	8.5190	8.5813	8.6445	8.7086	8.7735	8.8393
8.0	8.9060	8.9735	9.0418	9.1110	9.1810	9.2518	9.3235	9.3959	9.4691	9.5431
9.0	9.6178	9.6934	9.7696	9.8466	9.9243	10.0028	10.0819	10.1617	10.2422	10.3234

TABLE V (CONT'D)

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					c = 7					
1.0	—	7.1463	7.1713	7.1790	7.2048	7.2181	7.2335	7.2524	7.2715	7.2885
2.0	7.3067	7.3255	7.3446	7.3642	7.3842	7.4046	7.4254	7.4467	7.4684	7.4905
3.0	7.5131	7.5361	7.5596	7.5836	7.6081	7.6331	7.6586	7.6846	7.7111	7.7382
4.0	7.7658	7.7940	7.8227	7.8520	7.8819	7.9124	7.9435	7.9573	8.0076	8.0406
5.0	8.0742	8.1085	8.1434	8.1790	8.2153	8.2533	8.2900	8.3284	8.3675	8.4073
6.0	8.4478	8.4891	8.5312	8.5740	8.6175	8.6619	8.7070	8.7529	8.7995	8.8470
7.0	8.8953	8.9443	8.9942	9.0449	9.0964	9.1488	9.2019	9.2559	9.3107	9.3664
8.0	9.4229	9.4802	9.5383	9.5973	9.6571	9.7177	9.7791	9.8414	9.9045	9.9684
9.0	10.0331	10.0987	10.1651	10.2322	10.3001	10.3688	10.4384	10.5087	10.5797	10.6516
					c = 8					
2.0	8.2677	8.2838	8.3002	8.3170	8.3344	8.3517	8.3695	8.3876	8.4061	8.4249
3.0	8.4441	8.4636	8.4836	8.5039	8.5246	8.5457	8.5672	8.5891	8.6115	8.6342
4.0	8.6575	8.6811	8.7052	8.7298	8.7548	8.7804	8.8063	8.8328	8.8598	8.8873
5.0	8.9154	8.9440	8.9731	9.0027	9.0330	9.0637	9.0951	9.1270	9.1596	9.1927
6.0	9.2264	9.2607	9.2957	9.3313	9.3675	9.4045	9.4420	9.4802	9.5191	9.5587
7.0	9.5989	9.6399	9.6815	9.7239	9.7670	9.8108	9.8554	9.9097	9.9467	9.9935
8.0	10.0410	10.0892	10.1383	10.1881	10.2387	10.2900	10.3421	10.3949	10.4486	10.5031
9.0	10.5583	10.6144	10.6712	10.7288	10.7871	10.8462	10.9062	10.9670	11.0284	11.0901
					c = 9					
2.0	9.2365	9.2499	9.2643	9.2790	9.2947	9.3095	9.3252	9.3410	9.3571	9.3734
3.0	9.3900	9.4061	9.4242	9.4417	9.4595	9.4777	9.4962	9.5150	9.5342	9.5537
4.0	9.5736	9.5937	9.6143	9.6352	9.6566	9.6783	9.7004	9.7229	9.7458	9.7691
5.0	9.7929	9.8171	9.8417	9.8667	9.8923	9.9182	9.9447	9.9716	9.9990	10.0269
6.0	10.0553	10.0841	10.1136	10.1435	10.1740	10.2050	10.2365	10.2687	10.3013	10.3345
7.0	10.3683	10.4027	10.4377	10.4733	10.5095	10.5463	10.5838	10.6219	10.6606	10.7000
8.0	10.7401	10.7807	10.8220	10.8640	10.9067	10.9501	10.9941	11.0388	11.0841	11.1303
9.0	11.1772	11.2249	11.2732	11.3221	11.3718	11.4221	11.4733	11.5252	11.5778	11.6313

TABLE V (CONT'D)

μ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
					c = 10					
2.0	10.2062	10.2147	10.2277	10.2423	10.2565	10.2713	10.2863	10.3006	10.3155	10.3303
3.0	10.3450	10.3596	10.3754	10.3908	10.4063	10.4226	10.4389	10.4552	10.4722	10.4892
4.0	10.5065	10.5240	10.5417	10.5599	10.5785	10.5975	10.6151	10.6360	10.6556	10.6756
5.0	10.6961	10.7170	10.7380	10.7596	10.7815	10.8037	10.8264	10.8494	10.8728	10.8965
6.0	10.9207	10.9453	10.9703	10.9958	11.0217	11.0482	11.0749	11.1022	11.1299	11.1581
7.0	11.1867	11.2159	11.2455	11.2756	11.3063	11.3376	11.3694	11.4016	11.4343	11.4677
8.0	11.5017	11.5360	11.5710	11.6066	11.6428	11.6796	11.7168	11.7546	11.7930	11.8322
9.0	11.8720	11.9125	11.9536	11.9950	12.0371	12.0798	12.1234	12.1677	12.2123	12.2581

TABLE VI

μ* OF SINGLY TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION ON THE LEFT AT c=1

(Zero-Observations Truncated)

$k = 1(1)3;\ \phi = .01(.01).99$

ϕ	00	01	02	03	04	05	06	07	08	09
					k = 1					
00	—	100.0000	50.0000	33.3333	25.0000	20.0000	16.6667	14.2857	12.5000	11.1111
10	10.0000	9.0909	8.3333	7.6923	7.1429	6.6667	6.2500	5.8824	5.5556	5.2632
20	5.0000	4.7619	4.5455	4.3478	4.1667	4.0000	3.8462	3.7037	3.5714	3.4483
30	3.3333	3.2258	3.1250	3.0303	2.9412	2.8571	2.7778	2.7027	2.6316	2.5641
40	2.5000	2.4390	2.3809	2.3256	2.2727	2.2222	2.1739	2.1277	2.0833	2.0408
50	2.0000	1.9608	1.9231	1.8868	1.8519	1.8182	1.7857	1.7544	1.7241	1.6949
60	1.6667	1.6393	1.6129	1.5873	1.5623	1.5385	1.5152	1.4925	1.4706	1.4493
70	1.4286	2.4085	1.3889	1.3699	1.3514	1.3333	1.3158	1.2987	1.2821	1.2658
80	1.2500	1.2346	1.2195	1.2048	1.1905	1.1765	1.1628	1.1494	1.1364	1.1236
90	1.1111	1.0989	1.0870	1.0753	1.0638	1.0526	1.0417	1.0309	1.0204	1.0101
					k = 2					
00	—	99.0099	49.0196	32.3624	24.0384	19.0476	15.7233	13.3511	11.5741	10.1937
10	9.0909	8.1900	7.4404	6.8073	6.2657	5.7971	5.3879	5.0277	4.7081	4.4229
20	4.1667	3.9355	3.7258	3.5348	3.3602	3.2000	3.0525	2.9163	2.7902	2.6731
30	2.5641	2.4624	2.3674	2.2784	2.1949	2.1164	2.0425	1.9728	1.9070	1.8447
40	1.7857	1.7298	1.6767	1.6263	1.5783	1.5326	1.4890	1.4474	1.4076	1.3697
50	1.3333	1.2985	1.2652	1.2332	1.2025	1.1730	1.1447	1.1175	1.0912	1.0660
60	1.0417	1.0182	0.9956	0.9738	0.9527	0.9324	0.9128	0.8937	0.8754	0.8576
70	0.8404	0.8237	0.8075	0.7918	0.7767	0.7619	0.7476	0.7337	0.7203	0.7072
80	0.6944	0.6821	0.6701	0.6584	0.6470	0.6359	0.6252	0.6147	0.6045	0.5945
90	0.5848	0.5753	0.5661	0.5572	0.5484	0.5398	0.5315	0.5233	0.5154	0.5076
					k = 3					
00	—	99.0000	49.0004	32.3342	24.0015	19.0024	15.6701	13.2903	11.5059	10.1185
10	9.0090	8.1049	7.3460	6.7070	6.1598	5.6859	5.2716	4.9065	4.5823	4.2926
20	4.0323	3.7971	3.5836	3.3890	3.2111	3.0476	2.8971	2.7580	2.6291	2.5095
30	2.3981	2.2941	2.1970	2.1060	2.0206	1.9403	1.8648	1.7935	1.7263	1.6627
40	1.6026	1.5455	1.4914	1.4401	1.3912	1.3447	1.3005	1.2583	1.2180	1.1796
50	1.1429	1.1077	1.0741	1.0419	1.0111	0.9815	0.9531	0.9259	0.8997	0.8745
60	0.8504	0.8271	0.8047	0.7831	0.7623	0.7423	0.7230	0.7044	0.6864	0.6691
70	0.6523	0.6361	0.6205	0.6054	0.5907	0.5766	0.5629	0.5496	0.5368	0.5243
80	0.5123	0.5006	0.4893	0.4783	0.4622	0.4573	0.4473	0.4375	0.4281	0.4189
90	0.4160	0.4013	0.3929	0.3847	0.3768	0.3690	0.3615	0.3542	0.3470	0.3401

TABLE VII

μ OF LOGARITHMIC SERIES DISTRIBUTION

θ = .01(.01).99

θ	00	01	02	03	04	05	06	07	08	09
00	—	1.005043	1.010172	1.015383	1.020690	1.026091	1.031587	1.037177	1.042873	1.048672
10	1.054580	1.060598	1.066730	1.072980	1.079351	1.085846	1.092472	1.099231	1.106129	1.113168
20	1.120355	1.127694	1.135191	1.142852	1.150681	1.158686	1.166872	1.175248	1.183817	1.192590
30	1.201574	1.210777	1.220207	1.229874	1.239789	1.249960	1.260400	1.271117	1.282128	1.293442
40	1.305076	1.317043	1.329358	1.342039	1.355104	1.368569	1.382458	1.396791	1.411590	1.426882
50	1.442695	1.459053	1.475991	1.493543	1.511744	1.530632	1.550251	1.570649	1.591874	1.613982
60	1.637035	1.661095	1.686240	1.712564	1.740101	1.769007	1.799368	1.831307	1.864961	1.900476
70	1.938027	1.977805	2.020029	2.064946	2.112839	2.164042	2.218924	2.277935	2.341581	2.379962
80	2.485340	2.567035	2.656609	2.755341	2.864811	2.986979	3.124362	3.280188	3.458686	3.665562
90	3.908651	4.199053	4.671580	4.996013	5.568560	6.342356	7.456020	9.220787	12.525489	21.497578

CHARTS 1 - 4

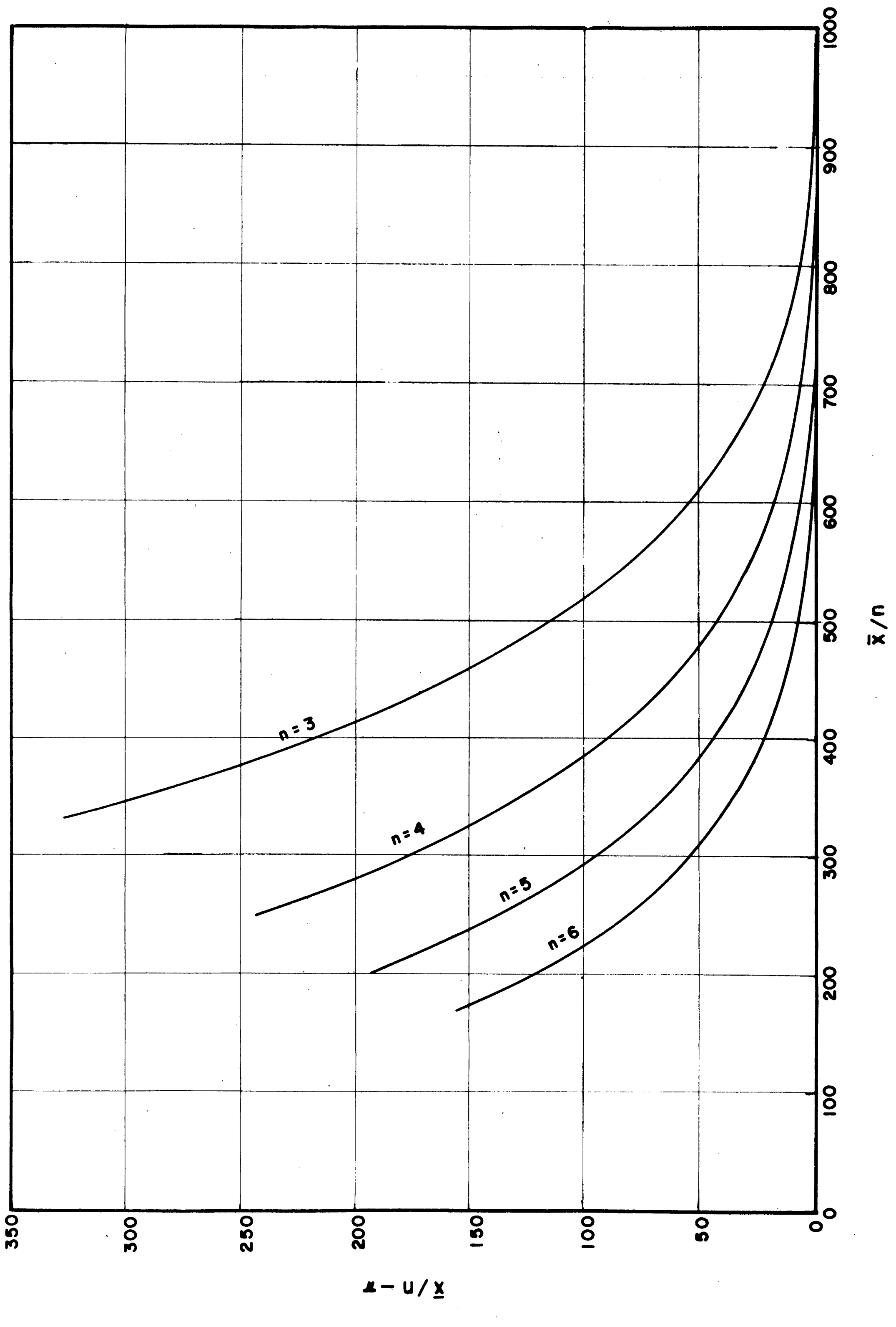

Chart 1. Estimation of π of Singly Truncated Binomial Distribution at c = 1 for n = 3(1)6

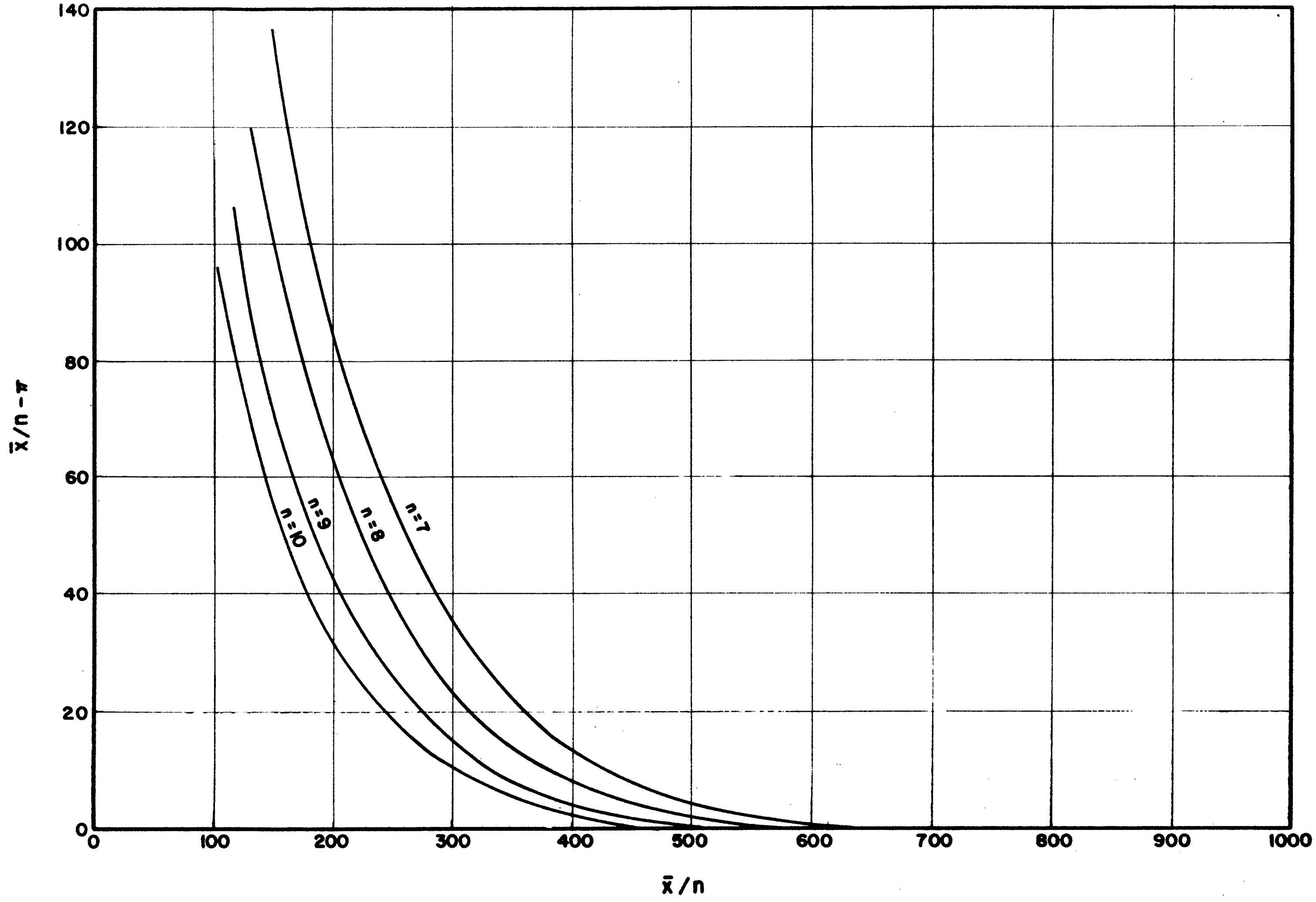

Chart 2. Estimation of π of Singly Truncated Binomial Distribution at c = 1 for n = 7(1)10

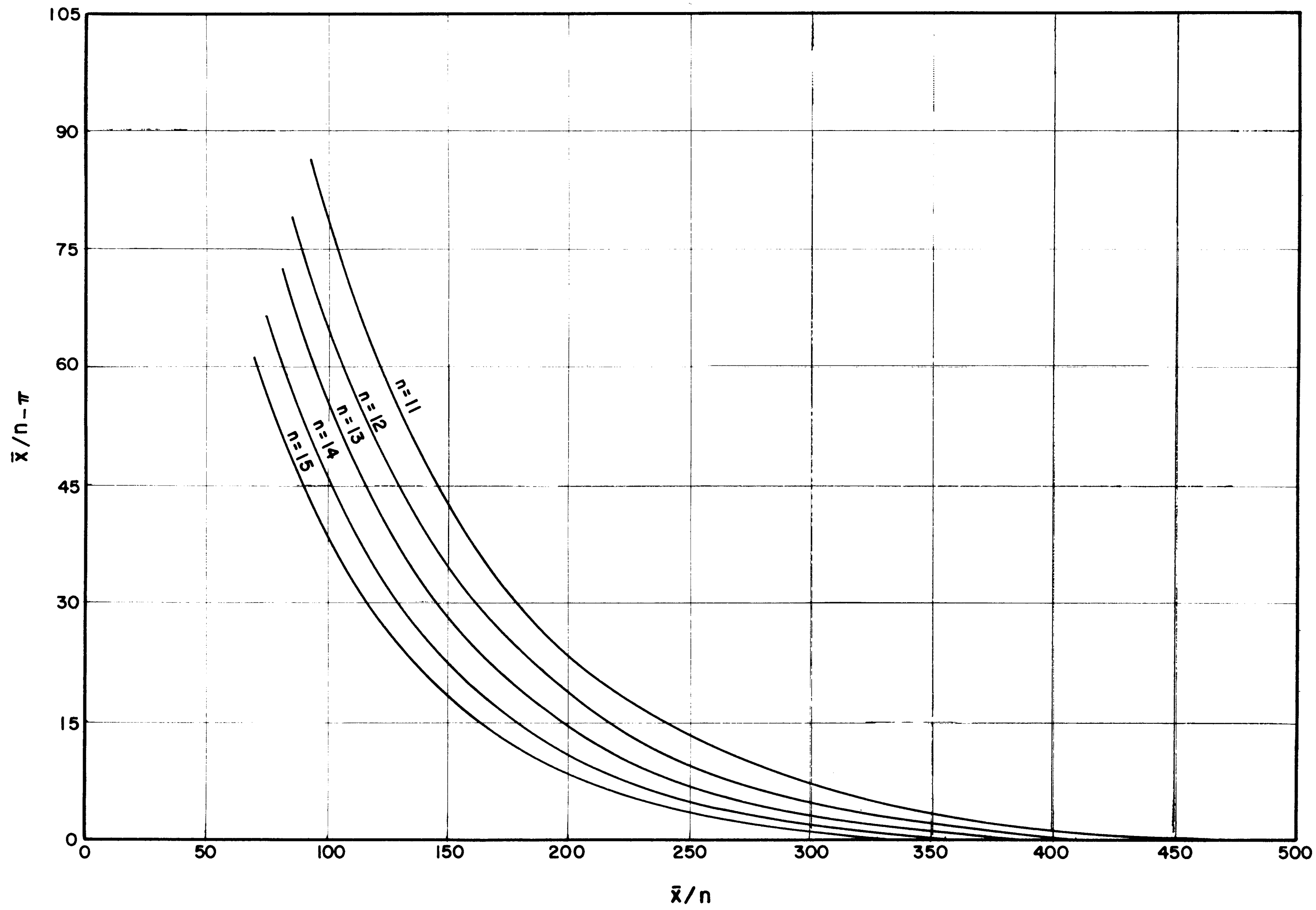

Chart 3. Estimation of π of Singly Truncated Binomial Distribution at $c = 1$ for $n = 11(1)15$

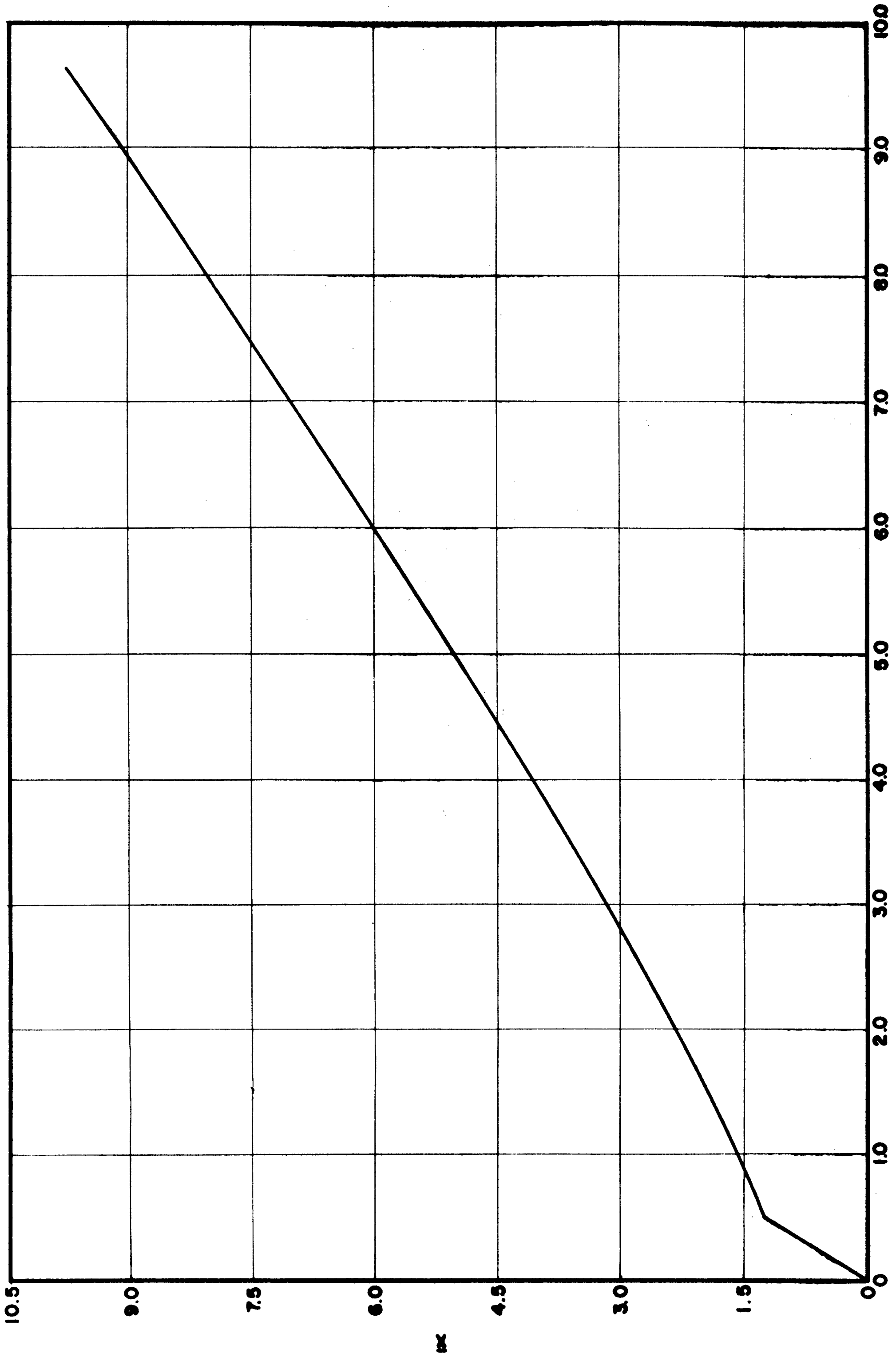

Chart 4. Estimation of μ of Singly Truncated Poisson Distribution at c = 1

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