THE
PHYSICS OF FLUIDS

VOLUME 13, NUMBER 5 May 1970

Instability of Time–Periodic Flows of Stratified Fluids

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(Received 13 January 1969; final manuscript received 18 December 1969)

The instability of two liquid layers with a free surface set in motion by an oscillatory lower boundary is analyzed for two superposed fluids with different viscosities and different densities. There are two modes of wave motion: the interfacial and the free-surface modes. When the Froude number is less than about 3, the interfacial mode governs the instability of the flow. For this case the free-surface mode is always stable. The two surfaces of discontinuity are always in the phase for the free-surface mode of disturbance, but may be in phase or 180 deg out of phase for the interfacial mode. The two modes, however, are found to compete with each other for governing the instability of the flow when the Froude number is larger than 3. The two surfaces of discontinuity can be out of phase by an angle different from 180 deg.

I. INTRODUCTION

Interest in the study of stability of unsteady flow is rather recent. Benjamin and Ursell[1] considered the stability of the free surface of an invisicid liquid contained in a vessel accelerated vertically in a simple harmonic motion. The stability chart was found to be the Mathieu function. Shen[2] and Conrad and Criminale[3,4] investigated the stability of time-dependent laminar flows. The former found that the deceleration strongly destabilizes the flow. The latter determined the lower bound for the critical Reynolds number for several cases of parallel flows and flow with curved streamlines. The stability of an interface between two inviscid fluids of different densities in parallel, time-dependent motion was studied by Kelly.[5] The equation governing the stability is also shown to be the Mathieu type. Yih[6] investigated the stability of the free surface of a horizontal layer of viscous liquid layer on an oscillating plane and found that the flow can be unstable for long waves.

In this investigation, we analyze the stability of the oscillatory motion of two superposed fluids with different viscosities and different densities and with a free surface on top. The purpose is to see how an interface, which is a second surface of discontinuity in density and viscosity, will affect the stability of a single layer studied by Yih. It is found that the oscillation of the lower boundary may stabilize or destabilize the flow, depending on the amplitude and the frequency of the oscillation as well as the viscosity and density ratios of the fluids. Detailed numerical calculations for all the cases studied show that the interfacial mode dominates the instability of the flow at least when the Froude number is less than 3. This is because the density difference between the liquid is less than the density difference between the upper liquid and air, and, therefore, the interface is easier to displace than the free surface.

II. PRIMARY FLOW

Consider two superposed fluids of depths d_1 and d_2, densities ρ_1 and ρ_2, and the viscosities μ_1 and μ_2, for the upper and the lower layer of fluid, respectively. The top surface is free while the bottom boundary is subject to a horizontal forcing oscillation described by the velocity $V_0 \cos \omega_* t$, V_0 being the amplitude, t the time, and ω_* the frequency of oscillation. The Cartesian coordinates X and Y are chosen as shown in Fig. 1, X is horizontal while Y is vertical upward measured from the mean position of the interface.

All quantities are made dimensionless with respect

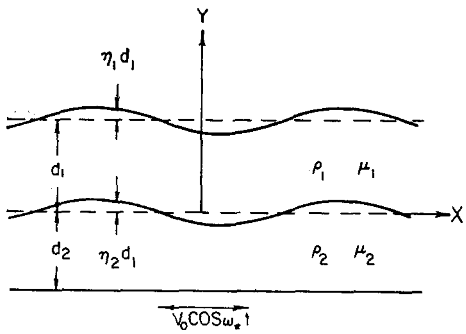

FIG. 1. Definition sketch.

to the velocity V_0, the depth d_1, the density ρ_1, the time d_1/V_0, and the pressure $\rho_1 V_0^2$. In terms of the dimensionless quantities, the equations governing the basic flow will be

$$\frac{\partial U_1}{\partial \tau} = \frac{1}{R}\frac{\partial^2 U_1}{\partial y^2} \quad \text{and} \quad \frac{\partial U_2}{\partial \tau} = \frac{m}{\gamma}\frac{1}{R}\frac{\partial^2 U_2}{\partial y^2} \tag{1}$$

for the upper and the lower fluid, respectively, in which U_1 and U_2 are the dimensionless velocities of the upper and lower layer of fluids, $y = Y/d_1(x = X/d_1$ will be used later), $\tau = tV_0/d_1$, R is the Reynolds number defined as $\rho_1 V_0\, d_1/\mu_1$, and

$$m = \frac{\mu_2}{\mu_1} \quad \text{and} \quad \gamma = \frac{\rho_2}{\rho_1}. \tag{2}$$

The U_1 and U_2 have to satisfy the boundary conditions

$$\begin{aligned} \frac{\partial}{\partial y} U_1(\tau, 1) &= 0, \\ U_1(\tau, 0) &= U_2(\tau, 0), \\ \frac{\partial}{\partial y} U_1(\tau, 0) &= m \frac{\partial}{\partial y} U_2(\tau, 0), \\ U_2(\tau, -n) &= \cos \omega_* t = \cos \omega\tau, \end{aligned} \tag{3}$$

in which $\omega = \omega_*(d_1/V_0)$, $n = d_2/d_1$.

Equations (1) and (3) can be solved for

$$\begin{aligned} U_1 = \operatorname{Re} (\{&A_0 \cosh [\beta(1 + i)y] \\ &+ B_0 \sinh [\beta(1 + i)y]\} \exp (i\omega\tau)), \\ U_2 = \operatorname{Re} \Bigg(\Bigg\{&C_0 \cosh \left[\beta\left(\frac{r}{m}\right)^{1/2}(1 + i)y\right] \\ &+ D_0 \sinh \left[\beta\left(\frac{r}{m}\right)^{1/2}(1 + i)y\right]\Bigg\} \exp (i\omega\tau)\Bigg), \end{aligned} \tag{4}$$

in which the symbol Re denotes the real part in the brackets, and

$$\beta = \left(\frac{\omega R}{2}\right)^{1/2},$$

$$\begin{aligned} A_0 &= \bar{\lambda}(m\gamma)^{1/2} \cosh [\beta(1 + i)], \\ D_0 &= -\bar{\lambda} \sinh [\beta(1 + i)], \\ C_0 &= A_0, \qquad B_0 = (m\gamma)^{1/2} D_0, \end{aligned}$$

where

$$\begin{aligned} \bar{\lambda} = \Bigg\{&(m\gamma)^{1/2} \cosh \left[\beta\left(\frac{r}{m}\right)^{1/2}(1 + i)n\right] \cosh [\beta(1 + i)] \\ &+ \sinh \left[\beta\left(\frac{r}{m}\right)^{1/2}(1 + i)n\right] \sinh [\beta(1 + i)]\Bigg\}^{-1}. \end{aligned}$$

Since the primary flow is parallel and in the X direction, the mean pressures P_1 and P_2 are hydrostatic.

III. THE DIFFERENTIAL SYSTEM GOVERNING STABILITY

The meaning of hydrodynamic stability of a general unsteady flow is ambiguous. However, for a primary flow periodic in time the usual concept of stability in the sense of Liapounov[7,8] is adequate. An infinitesimal disturbance is applied to the primary flow. We will seek the development of the disturbances in the long run, and call the flow stable if the perturbation quantities remain small for all time, unstable if they do not.

For an unsteady flow, Squire's theorem cannot be generalized in a useful way to justify the consideration of two-dimensional disturbance only. However, it is still true, as pointed out by Yih[6] that the stability of a three-dimensional disturbance can be determined from that of a two-dimensional one for a flow which differs from the original flow not only in the Reynolds number, as in the case of steady primary flows, but also in the distribution of the velocity of the primary flow. We shall treat only two-dimensional disturbances in this investigation. We shall use u_i and v_i to denote, respectively, the velocity components in the directions of increasing x and y, and p_i to denote the pressure. The subscript i is taken to be 1 for the upper fluid and 2 for the lower fluid. This convention is adopted here;

$$u_i = U_i + u_i', \qquad v_i = v_i', \qquad p_i = P_i + p_i'$$

in which the small letters with a prime refer to the perturbation quantities. The equation of continuity for the perturbation flow is

$$\frac{\partial u_i'}{\partial x} + \frac{\partial v_i'}{\partial y} = 0.$$

This permits the use of a stream function ψ_i, in terms of which

$$u_i' = (\psi_i)_y, \qquad v_i' = -(\psi_i)_x,$$

the subscripts, x and y, denoting partial differentiation. Since the coefficients of the equations of motion governing the perturbation flow depend both on y and τ, they do not permit the use of the exponential time factor to indicate the time dependence exclusively. However, as far as the variable x is concerned the equations and the boundary conditions (to be presented later) admit a solution of the form

$$\psi_i = \varphi_i(y, \tau) \exp(i\alpha x),$$
$$p_i' = f_i(y, \tau) \exp(i\alpha x),$$

in which α is the dimensionless wavenumber. The equation governing stability of the parallel flow is the well known Orr–Sommerfeld equation. Upon writing φ for $\varphi_1(y, \tau)$ and χ for $\varphi_2(y, \tau)$, the Orr–Sommerfeld equation for the unsteady primary flow can be written as

$$\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi = R\left[\left(\frac{\partial}{\partial\tau} + i\alpha U_1\right)(\varphi'' - \alpha^2\varphi) - i\alpha(U_1)_{yy}\varphi\right], \quad (5)$$

for the upper fluid, and

$$\chi'''' - 2\alpha^2\chi'' + \alpha^4\chi = \frac{\gamma}{m} R\left[\left(\frac{\partial}{\partial\tau} + i\alpha U_2\right)(\chi'' - \alpha^2\chi) - i\alpha(U_2)_{yy}\chi\right], \quad (6)$$

for the lower fluid, respectively, in which the primes on φ and on χ indicate differentiations with respect to y.

The boundary conditions are

$$\chi(-n, \tau) = 0, \qquad \chi'(-n, \tau) = 0, \quad (7)$$

which express the conditions of adherence of the fluid to the lower rigid boundary. The prime in Eq. (7) indicates partial differentiation with respect to y. If η_1 and η_2 denote the deviations (measured in the unit of d_1) of the free surface and the interface from their mean positions, the linearized kinematic conditions at the free surface and interface are

$$\left(\frac{\partial}{\partial\tau} + U_1(1, \tau)\frac{\partial}{\partial x}\right)\eta_1 = v_1' = -(\psi_1)_x,$$

and

$$\left(\frac{\partial}{\partial\tau} + U_2(0, \tau)\frac{\partial}{\partial x}\right)\eta_2 = v_2' = -(\psi_2)_x,$$

respectively. These two equations give us the relation between η_1 and ψ_1 as well as η_2 and ψ_2. If we let

$$\eta_1 = h_1(\tau) \exp(i\alpha x) \quad \text{and} \quad \eta_2 = h_2(\tau) \exp(i\alpha x),$$

the above two equations then become

$$\left(\frac{d}{d\tau} + i\alpha U_1(1, \tau)\right)h_1 = -i\alpha\varphi(1, \tau)$$

and

$$\left(\frac{d}{d\tau} + i\alpha U_2(0, \tau)\right)h_2 = -i\alpha\chi(0, \tau). \quad (8)$$

The condition of zero shear stress at the free surface is

$$h_1 \frac{\partial^2}{\partial y^2} U_1(1, \tau) + \varphi''(1, \tau) + \alpha^2\varphi(1, \tau) = 0, \quad (9)$$

in which the first term takes care of the mean-shear increment as y varies from 1 to $1 + \eta_1$. The free surface can be considered as an interface with the upper fluid of zero density and zero viscosity. The formulations of the free normal stress at the free surface and the continuity of normal stress at the interface are similar. They demand that the difference of the quantity

$$\rho g\, d_1\eta - p'\rho_1 V_0^2 + \frac{2\mu V_0}{d_1}\frac{\partial v'}{\partial y} \quad (10)$$

evaluated for the upper fluid from its value for the lower fluid be equal to

$$-\frac{T}{d_1}\frac{\partial^2\eta}{\partial x^2},$$

in which T is the surface tension. T_1 and T_2 will be used to denote the surface tension at the free surface and at the interface, respectively. In Eq. (10), the first term takes care of the contribution of hydrostatic pressure to the actual pressure at the interface. The last two terms in (10) are evaluated at the mean positions of the interface and the free surface. The p' can be evaluated from the x component of the Navier–Stokes' equation governing the perturbation flow. The condition that the normal stress should vanish at the free surface then yields

$$-i\alpha(F_1^{-2} + S_1\alpha^2)h_1 + \frac{1}{R}(\varphi''' - 3\alpha^2\varphi') - \frac{\partial\varphi'}{\partial\tau} - i\alpha U_1\varphi' = 0, \quad (11)$$

to be evaluated at $y = 1$, in which $F_1 = V_0/(g\, d_1)^{1/2}$ is the Froude number, and $S_1 = T_1/\rho\, d_1 V_0^2$. The continuity of v' at the interface demands

$$\varphi(0, \tau) = \chi(0, \tau). \quad (12)$$

The continuity of the x-component velocity at the interface demands that

$$\varphi'(0, \tau) + h_2(\tau)\frac{\partial}{\partial y} U_1(0, \tau) = \chi'(0, \tau) + h_2(\tau)\frac{\partial}{\partial y} U_2(0, \tau). \quad (13)$$

The continuity of shear stress at the interface is expressed by

$$\varphi''(0,\tau)+\alpha^2\varphi(0,\tau)+h_2(\tau)\frac{\partial^2}{\partial y^2}U_1(0,\tau) = m\left(\chi''(0,\tau)+\alpha^2\chi(0,\tau)+h_2(\tau)\frac{\partial^2}{\partial y^2}U_2(0,\tau)\right). \quad (14)$$

Finally, the continuity of normal stress at the interface demands

$$-i\alpha R\left(-U_1\varphi'+\frac{i}{\alpha}\frac{\partial}{\partial\tau}\varphi'+\varphi\frac{\partial}{\partial y}U_1\right)-(\varphi'''-\alpha^2\varphi')+i\gamma\alpha R\left(-U_2\chi'+\frac{i}{\alpha}\frac{\partial}{\partial\tau}\chi'+\chi\frac{\partial}{\partial y}U_2\right)+m(\chi'''-\alpha^2\chi')+2\alpha^2\varphi'-2\alpha^2 m\chi' = i\alpha R(F_2^{-2}+\alpha^2 S_2)h_2, \quad (15)$$

in which all variables are evaluated at $y = 0$, and

$$F_2^{-2}=\frac{\rho_2-\rho_1}{\rho_1}\frac{g\,d_1}{V_0^2}=(\gamma-1)F_1^{-2},$$

$$S_2=\frac{T_2}{\rho_1\,d_1V_0^2}.$$

Equations (5) and (6) together with the boundary conditions govern the stability of the flow.

IV. SOLUTION OF THE STABILITY PROBLEM

An extended Floquet theorem is adopted to solve the Orr–Sommerfeld equation with time-periodic coefficients. We shall assume the roots of that equation to be simple. Hence, the solutions φ and χ in Eqs. (5) and (6) have the form

$$\varphi(y,\tau)=\exp(\mu\tau)\Phi(y,\tau),\qquad \chi(y,\tau)=\exp(\mu\tau)\mathrm{X}(y,\tau). \quad (16)$$

Correspondingly,

$$h_1(\tau)=\exp(\mu\tau)H_1(\tau),\qquad h_2(\tau)=\exp(\mu\tau)H_2(\tau), \quad (17)$$

in which $\Phi(y,\tau)$, $\mathrm{X}(y,\tau)$, $H_1(\tau)$, and $H_2(\tau)$ are periodic in τ.

The method of regular perturbation of Yih for long wave, wavenumber $\alpha \ll 1$, will be adopted to solve the eigenvalue problem. The functions $\Phi(y,\tau)$, $\mathrm{X}(y,\tau)$, $H_1(\tau)$, $H_2(\tau)$, and μ can be expanded in power series of α:

$$\Phi(y,\tau)=\sum_{i=0}^{\infty}\alpha^i\varphi_i(y,\tau),\quad \mathrm{X}(y,\tau)=\sum_{i=0}^{\infty}\alpha^i\chi_i(y,\tau),\quad H_1(\tau)=\sum_{i=0}^{\infty}\alpha^i h_{1i}(\tau),\quad H_2(\tau)=\sum_{i=0}^{\infty}\alpha^i h_{2i}(\tau),\quad \mu=\sum_{i=0}^{\infty}\alpha^i\theta_i \quad (18)$$

in which φ's, χ's, and h's are all periodic functions in τ. The superscript i on α denotes the power. Upon substituting these expansions into (5), (6), and (8), as well as all boundary conditions, and collecting the powers of α and letting the coefficients be zero, we obtain a series of equations for each order of approximation. First of all, we have from (8), corresponding to the order of α^0, the kinematic conditions

$$\frac{dh_{10}}{d\tau}+\theta_0h_{10}=0 \quad (19)$$

and

$$\frac{dh_{20}}{d\tau}+\theta_0h_{20}=0, \quad (20)$$

for the free surface and for the interface, respectively. Since h_{10} and h_{20} are periodic time functions, the only way to satisfy (19) and (20) is $\theta_0 = 0$. Consequently, h_{10} and h_{20} are two constants. Without loss of generality, take

$$h_{10}=1 \quad\text{and}\quad h_{20}=\sigma_1, \quad (21)$$

where σ_1 is a constant of integration to be determined, and represents the ratio between h_{20} and h_{10} or the amplitude ratio between the interface wave and the free surface wave.

The differential system of the zeroth-order approximation is

$$\varphi_0''''=R\frac{\partial}{\partial\tau}\varphi_0'',\qquad \chi_0''''=R\frac{\gamma}{m}\frac{\partial}{\partial\tau}\chi_0'', \quad (22)$$

$$\chi_0(-n,\tau)=0,\qquad \chi_0'(-n,\tau)=0,$$

$$\frac{\partial^2}{\partial y^2}U_1(1,\tau)+\varphi_0''(1,\tau)=0,$$

$$\varphi_0'''(1,\tau)-R\frac{\partial}{\partial\tau}\varphi_0'(1,\tau)=0,$$

$$\varphi_0(0, \tau) = \chi_0(0, \tau),$$

$$\varphi_0'(0, \tau) + \sigma_1 \frac{\partial}{\partial y} U_1(0, \tau) = \chi_0'(0, \tau) + \sigma_1 \frac{\partial}{\partial y} U_2(0, \tau), \quad (23)$$

$$\varphi_0''(0, \tau) + \sigma_1 \frac{\partial^2}{\partial y^2} U_1(0, \tau) = m\left(\chi_0''(0, \tau) + \sigma_1 \frac{\partial^2}{\partial y^2} U_2(0, \tau)\right),$$

$$R \frac{\partial}{\partial \tau} \varphi_0'(0, \tau) - \varphi_0'''(0, \tau) - \gamma R \frac{\partial}{\partial \tau} \chi_0'(0, \tau) + m\chi_0'''(0, \tau) = 0.$$

Equations (22) and (23) can be solved for

$$\varphi_0 = \mathrm{Re}\,(\{B_1 + C_1 \cosh[\beta(1+i)y] + D_1 \sinh[\beta(1+i)y]\} \exp(i\omega\tau)), \quad (24)$$

$$\chi_0 = \mathrm{Re}\left(\left\{B + C \cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right] + D \sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right]\right\} \exp(i\omega\tau)\right) \quad (25)$$

in which B_1, B_2, C_1, C_2, D_1, and D_2 are given in the Appendix.

The next approximation, (8) gives

$$\frac{d}{d\tau} h_{11}(\tau) + \theta_1 = -i[\varphi_0(1, \tau) + U_1(1, \tau)] \quad (26)$$
$$= -\frac{i}{2}[B_1 \exp(i\omega\tau) + B_1^* \exp(-i\omega\tau)],$$

and

$$\frac{d}{d\tau} h_{21}(\tau) + \sigma_1\theta_1 = -\frac{i}{2}[(B_1 + C_1 + \sigma_1 C_0) \exp(i\omega\tau) + (B_1^* + C_1^* + \sigma_1 C_0^*) \exp(-i\omega\tau)], \quad (27)$$

respectively, in which the asterisk in the superscript indicates the complex conjugate. This convention will be adopted from here on. The right-hand sides of (26) and (27) consist of the periodic time functions only; furthermore, h_{11} and h_{21} are also periodic, so that

$$\theta_1 = 0. \quad (28)$$

Then, h_{11} and h_{21} are found to be

$$h_{11} = -\frac{1}{2\omega}[B_1 \exp(i\omega\tau) - B_1^* \exp(-i\omega\tau)], \quad (29)$$

$$h_{21} = -\frac{1}{2\omega}[(B_1 + C_1 + \sigma_1 C_0) \exp(i\omega\tau) - (B_1^* + C_1^* + \sigma_1 C_0^*) \exp(-i\omega\tau)] + \sigma_2, \quad (30)$$

in which the constant of integration for h_{11} is taken to be zero, since we want to keep the constant part of $h_1 = 1$. The constant of integration for h_{21} is σ_2, which can be determined from the third-order approximation to the kinematic boundary conditions at the surfaces of discontinuity. This point will be illustrated later.

The differential system of the first order approximation is

$$\varphi_1'''' - R \frac{\partial}{\partial \tau} \varphi_1'' = iR[U_1\varphi_0'' - (U_1)_{yy}\varphi_0], \quad (31)$$

$$\chi_1'''' - \frac{\gamma}{m} R \frac{\partial}{\partial \tau} \chi_1'' = iR \frac{\gamma}{m}[U_2\chi_0'' - (U_2)_{yy}\chi_0], \quad (32)$$

with

$$\chi_1(-n, \tau) = 0, \qquad \chi_1'(-n, \tau) = 0,$$

$$U_1''(1, \tau)h_{11}(\tau) + \varphi_1''(1, \tau) = 0,$$

$$\frac{1}{R}\varphi_1''' - \frac{\partial}{\partial \tau}\varphi_1' - iU_1\varphi_0' - iF_1^{-2} = 0 \text{ at } y = 1,$$

$$\varphi_1(0, \tau) = \chi_1(0, \tau),$$

$$\varphi_1'(0, \tau) + h_{21}(\tau) \frac{\partial}{\partial y} U_1(0, \tau) = \chi_1'(0, \tau) + h_{21}(\tau) \frac{\partial}{\partial y} U_2(0, \tau) \quad (33)$$

$$\varphi_1''(0, \tau) + h_{21}(\tau) \frac{\partial^2}{\partial y^2} U_1(0, \tau) = m\left(\chi_1''(0, \tau) + h_{21}(\tau) \frac{\partial^2}{\partial y^2} U_2(0, \tau)\right),$$

$$\varphi_1''' - R \frac{\partial}{\partial \tau}\varphi_1' - m\chi_1''' + \gamma R \frac{\partial}{\partial \tau}\chi_1' + iR\{-U_1\varphi_0' + (U_1)_y\varphi_0 + \gamma[U_2\chi_0' - (U_2)_y\chi_0] + \sigma_1 F_2^{-2}\} = 0 \text{ at } y = 0.$$

The particular solutions of (31) and (32) are found to be

$$\varphi_{1P} = iR\left[\frac{1}{2}\iint\left(\varphi_0' \int U_1\,dy - U_1' \cdot \int \{\varphi_0 - \tfrac{1}{2}[B_1 \exp(i\omega\tau) + B_1^* \exp(-i\omega\tau)]\}\,dy\right)dy\,dy - K_1\right] \quad (34)$$

and

$$\chi_{1P} = i\frac{\gamma}{m}R\left[\frac{1}{2}\iint\left(\chi_0'\int U_2\,dy - U_2'\right.\right.$$
$$\cdot\int\{\chi_0 - \tfrac{1}{2}[B_2\exp(i\omega\tau)$$
$$\left.\left.+ B_2^*\exp(-i\omega\tau)]\}\,dy\right)dy\,dy - K_2\right], \quad (35)$$

respectively, in which K_1 and K_2 have to satisfy, respectively, the equations

$$L_1(K_1'') = \tfrac{1}{2}U_1''[B_1\exp(i\omega\tau) + B_1^*\exp(-i\omega\tau)]$$
and
$$L_2(K_2'') = \tfrac{1}{2}U_2''[B_2\exp(i\omega\tau) + B_2^*\exp(-i\omega\tau)], \quad (36)$$

where the operators L_1 and L_2 are defined as

$$L_1 = \frac{\partial^2}{\partial y^2} - R\frac{\partial}{\partial\tau}, \qquad L_2 = \frac{\partial^2}{\partial y^2} - \frac{\gamma}{m}R\frac{\partial}{\partial\tau}.$$

Solving K_1 and K_2 from the above two equations, and evaluating φ_{1p} and χ_{1p} from (34) and (35), respectively, we obtain

$$\varphi_{1P} = iR(I_1 + I_2 + I_3 + I_1^* + I_2^* + I_3^*),$$

and

$$\chi_{1P} = iR(Q_1 + Q_2 + Q_3 + Q_1^* + Q_2^* + Q_3^*),$$

in which I_1, I_2, I_3, Q_1, Q_2, and Q_3 are given in the Appendix. The complementary solutions of (31) and (32) are in the forms

$$\varphi_{1c} = E_1 + F_1y + G_1y^2 + J_1y^3 + M_1V_a + N_1W_a$$
$$+ M_1^*V_a^* + N_1^*W_a^* + \Lambda_1(\tau) + \Xi_1(\tau)y, \quad (37)$$
$$\chi_{1c} = E_2 + F_2y + G_2y^2 + J_2y^3 + M_2V_b + N_2W_b$$
$$+ M_2^*V_b^* + N_2^*W_b^* + \Lambda_2(\tau) + \Xi_2(\tau)y, \quad (38)$$

in which the 12 coefficients are constants of integration, and the expressions for V_a, W_a, $\Lambda_1(\tau)$, $\Xi_1(\tau)$, V_b, W_b, $\Lambda_2(\tau)$, and $\Xi_2(\tau)$ are given in the Appendix. Combining the particular and complementary solutions, one obtains the general solutions of φ_1 and χ_1 as

$$\varphi_1 = \varphi_{1P} + \varphi_{1c} \quad\text{and}\quad \chi_1 = \chi_{1P} + \chi_{1c}, \quad (39, 40)$$

respectively.

The second-order approximation of kinematic boundary conditions is

$$\frac{dh_{12}}{d\tau} = -\theta_2 - i[U_1(1, \tau)h_{11}(\tau) + \varphi_1(1, \tau)] \quad (41)$$

and

$$\frac{dh_{22}}{d\tau} = -\sigma_1\theta_2 - i[U_2(0, \tau)h_{21}(\tau) + \chi_1(0, \tau)], \quad (42)$$

for the free surface and the interface, respectively. Since h_{12} and h_{22} are purely periodic time functions, the time-independent part in the right-hand side of both Eqs. (41) and (42) should vanish. This reveals that as far as the search for the eigenvalue up to the second-order approximation is concerned, only the time-independent forms in

$$\varphi_1,\ \chi_1,\ U_1(1, \tau)h_{11}(\tau), \quad\text{and}\quad U_2(0, \tau)h_{21}(\tau)$$

need to be considered in predicting the stability for long waves. If we want to go a step farther, to determine the sign and magnitude of θ_3, we should consider both the time-dependent and time-independent terms in φ_1, χ_1, $\cdots$, etc., in order to determine the 12 constants of integration in (37) and (38). This will not be carried out in this investigation. If $\Phi(y)$ and $X(y)$ denote, respectively, the time-independent parts of $\varphi_1(y, \tau)$ and $\chi_1(y, \tau)$, we have

$$\Phi(y) = iR(I_1 + I_2 + I_1^* + I_2^*)$$
$$+ E_1 + F_1y + G_1y^2 + J_1y^3,$$
$$X(y) = iR(Q_1 + Q_2 + Q_1^* + Q_2^*)$$
$$+ E_2 + F_2y + G_2y^2 + J_2y^3, \quad (43)$$

in which E_1, F_1, G_1, J_1, E_2, F_2, G_2, and J_2 are given in the Appendix.

The aforementioned argument that the time-independent parts on the right-hand side of Eqs. (41) and (42) should vanish individually leads to

$$\theta_2 + i\left(\frac{(m\gamma)^{1/2}}{4\omega}(B_1^*\bar{\lambda} - B_1\bar{\lambda}^*)\right.$$
$$+ iR[I_1(1) + I_2(1) + I_1^*(1) + I_2^*(1)]$$
$$\left.+ E_1 + F_1 + G_1 + J_1\right) = 0 \quad (44)$$

and

$$\sigma_1\theta_2 + i\left(\frac{1}{4\omega}[C_0(B_1^* + C_1^*) - C_0^*(B_1 + C_1)]\right.$$
$$\left.+ iR[\theta_1(0) + Q_2(0) + Q_1^*(0) + Q_2^*(0)] + E_2\right) = 0. \quad (45)$$

The last two equations can be solved simultaneously for σ_1 and θ_2. Since (45) is a quadratic equation, two roots, corresponding to two modes of disturbance can be found for both σ_1 and θ_2. The flow is stable if $\theta < 0$, unstable if $\theta > 0$. We know that each term in the bracket of Eqs. (44) and (45) contains two parts: one with a factor σ_1 and the other without. We shall not write them out explicitly, since the calculation, while straightforward, is lengthy. In order to make the computer programming possible while numerical calculations are done, Eqs. (44) and (45) will be rewritten as follows:

$$\theta_2 = R\Big[\Big(-\frac{1}{3} - \frac{n}{3m}(n^2 + 3n + 3) - \frac{n^2}{6m}(3 + 2n)(\gamma - 1)\sigma_1\Big)F_1^{-2} + I_1(1) + I_2(1) + I_1^*(1) + I_2^*(1) - \tilde{A} - (1 + n)\tilde{B} - \tilde{C} - 2\tilde{D}_1 - \tilde{E} - \tilde{F} + \frac{1}{m}[n(2 + n)\tilde{G} - 2n(n^2 + 3n + 3)\tilde{D}_1 - n(n + 2)\tilde{C}] + \frac{i(m\gamma)^{1/2}}{8\beta^2}(B_1\tilde{\lambda}^* - B_1^*\tilde{\lambda})\Big] = R(\xi_a + \sigma_1\xi_b), \tag{46}$$

$$\sigma_1\theta_2 = R\Big(\frac{n^2}{6m}[-3 - 2n + 2n(1 - \gamma)\sigma_1]F_1^{-2} + \frac{1}{m}[n^2\tilde{G} - n^2(3 + 2n)\tilde{D}_1 - n^2\tilde{C} - m(\tilde{A} + n\tilde{B})] + \frac{1}{16\beta^2}[i(C_2C_0^* - C_2^*C_0) + 2iC_0^*(B_1 + C_1 - B_2) - 2iC_0(B_1^* + C_1^* - B_2^*)]\Big) = R(\eta_a + \sigma_1\eta_b), \tag{47}$$

in which ξ_a and η_a denote the terms in brackets on the right-hand side of (46) and (47), respectively, which do not contain a factor σ_1 while ξ_b and η_b denote the terms which contain a factor σ_1. We shall not write them out explicitly. As soon as ξ_a, ξ_b, η_a, and η_b are evaluated, we can quickly solve for σ_1

$$\sigma_1 = \frac{\eta_b - \xi_a \pm [(\eta_b - \xi_a)^2 + 4\xi_b\eta_a]^{1/2}}{2\xi_b}. \tag{48}$$

This gives us two roots for σ_1. As soon as we have σ_1, we obtain from (46)

$$\theta_2 = R\Theta_2(\gamma, n, m, R, F_1), \tag{49}$$

in which

$$\Theta_2 = \xi_a + \sigma_1\xi_b. \tag{50}$$

Obviously, corresponding to two roots of σ_1, we also have two roots for Θ_2 from (50).

To ascertain the possibility of going to a higher approximation in order to determine σ_2 in (30) and θ_3 for a larger α, consider the third-order approximation in the kinematic conditions (8), namely,

$$\frac{dh_{13}}{d\tau} + \theta_2 h_{11} + \theta_3 + iU_1(1, \tau)h_{12}(\tau) + i\varphi_2(1, \tau) = 0 \tag{51}$$

and

$$\frac{dh_{23}}{d\tau} + \theta_2 h_{21} + \theta_3\sigma_1 + iU_2(0, \tau)h_{22}(\tau) + i\chi_2(0, \tau) = 0 \tag{52}$$

σ_2 appears only in the term $\theta_2 h_{21}$. When we integrate Eqs. (41) and (42), we take zero as a constant of integration for h_{12} and a constant, say σ_3, for h_{22}. But we know σ_3 makes no contribution to the time-independent part in (52). Again, the time-independent parts in (51) and in (52) should vanish individually. These lead to two simultaneous equations in which σ_2 and θ_3 are the only two unknowns which can be found. Since σ_2 and θ_3 do not appear in products in the simultaneous equations just mentioned, we only have one root for both σ_2 and θ_3 corresponding to each set of values of θ_2 and σ_1. This not only indicates that we can determine σ_2 and θ_3 to a higher order approximation, but also confirms that there are only two modes in disturbances that can be found.

V. GENERAL CONSIDERATION OF THE RESULTS

Equations (48)–(50) give us the first nonvanishing value of the θ's which will be used to determine the stability or the instability of the flow. Although these three equations appear short, in reality they are very involved. We are unable to see the feature of the stability of the flow merely through the algebraic results. The numerical calculations have been carried out by means of an IBM 360 computer. The results are illustrated graphically and discussed in this section and in the section immediately following.

The plus and minus signs in front of the radical in (48) indicate that there are two possible modes of disturbance. We call it the "first" mode of disturbance if the "plus" sign is taken in the calculation and the "second" mode if the "minus" sign is taken. The stability of the flow is dominated by the interfacial waves if the absolute value of σ_1 is larger than 1 and by the free surface waves if σ_1 is less than 1.

When the Froude number F_1 is less than about 3, the first mode is always the interfacial mode while the second mode is the free-surface mode. The latter is stable. However, the former can be stable or unstable. Therefore, we conclude that it is the interfacial mode which governs the instability of the flow when $F_1 < 3$. This is to be expected since the energy required to distort the interface is less than that required to distort the free surface. The interfacial mode reduces to a neutral mode when γ and m become unity. But the long-wave mode considered here is different from that usually considered in hydrodynamic stability problems cited in Lin's[9] book. In the latter, whether in the stability of parallel flow or nearly parallel flow, the long-wave modes are damped, at least for small Reynolds numbers. Following Yih's[10] arguments, we will classify the waves we are now considering as the "soft" waves to distinguish them from the Tollmien–Schlichting waves which were considered to be "hard." Therefore, the unstable modes discussed in this investigation are not near the damped modes considered in the usual theory, but near the soft modes ignored by the usual theory. When $m = 1 = \gamma$, the interfacial mode is physically insignificant since the flow reduces to the single fluid case. The stability or instability of the flow is governed by the free-surface mode alone.

The first and second modes correspond to the free-surface and interfacial modes, respectively, or vice versa, when the Froude number is larger than 3. Either the interfacial or the free-surface mode can be unstable depending on the values of the parameters m, n, β, and F_1. This would then indicate that for $F_1 > 3$, there will be a competition between the interfacial and free-surface modes in governing the instability of the flow. This is physically plausible because we know from Yih's[6] results that the free-surface mode for the case of a single fluid becomes unstable when the Froude number is large.

The results also show that the two surfaces of discontinuity oscillate in phase with each other for the free-surface mode, but may be in phase or 180 deg out of phase for the interfacial mode. Furthermore, both Θ_2 and σ_1 can be complex for some values of γ, m, n, and β when $F_1 > 3$. For instance, when $\gamma = 1$, $n = 1$, $\beta = 0.2$, and $F_1^{-2} = 0.1$. For such a case, we know from Eq. (18), that the two surface oscillations are out of phase by an angle different from 180 deg.

We are interested in investigating the instability due to the oscillatory motion of the lower boundary. The calculations on the variations of growth rate of disturbance with β for various values of γ have been carried out for the special value of m where the flow with homogeneous density approximately has a maximum growth rate of disturbance. The results also show that the free-surface mode is stable while the interface mode is unstable. The growth rate of disturbance increases with increasing values of γ when β is less than 2.5. The flow with the upper fluid heavier than the lower one, which is hydrostatically unstable, can become stable when β is small. The results just mentioned agree with Kao[11,12] in his study of the two-layer density stratified flow down an inclined plane. In Kao's problem, however, the source of energy for the unstable disturbance is the longitudinal component of gravity while in the present problem the energy is supplied by the oscillatory motion of the lower boundary.

VI. DISCUSSION OF GRAPHS

The general considerations discussed above are borne out by detailed calculations of the stability problem. Since there are too many parameters included in this investigation, it is very inconvenient to present all the numerical results in graphs. Only those of the typical cases are selected and described in this section.

First of all, we will describe the results of the Θ_2 for the flows with γ and n fixed to be unity. Results for the cases of $F_1^{-2} = 10$ and 1 are chosen to demonstrate the feature of the stability or instability for the flow with $F_1 < 3$. The variation of Θ_2 as a function of m for the first and second modes for these two values of F_1^{-2} are plotted in Figs. 2(a) and (b) and 3(a) and (b). The curves in Figs. 2(a) and 3(a) show the values of Θ_2 for the first mode, which is known to be the interfacial mode. Owing to the similarity in these two sets of curves and other results obtained which for brevity are not presented here, we conclude that as long as the Froude number is less than 3, its variation has a negligible influence on the damping or growth rate of the interfacial mode of disturbance. Figures 2(a) and 3(a) also show that the flow is predominantly stable for $m < 1$. If it is unstable, the growth rate of disturbance is shown to be small. But for $m > 1$, the flow is unstable and the growth rate of disturbance increases with increasing β. The maximum growth rate occurs when β is equal to about 0.8. Curves in Figs. 2(b) and 3(b) give the values of Θ_2 for the second mode which is known to be the free-surface mode. These two sets of curves and other results obtained which are not presented here all show that the free-surface is stable for any value of m and β as long as $F_1 < 3$. The damping rate of disturbance is decreased by increasing the value of m. However, it varies only slightly with β and is almost constant with respect to β when F_1 is of the order of $(10)^{-1/2}$. After carefully studying the dependence of Θ_2 on F_1 from the results shown in Figs. 2(b) and 3(b) as well as those which are not plotted, we find that for the free-surface mode of disturbance the damping rate of disturbance increases almost linearly with increasing values of F_1^{-2} as long as $F_1 < 3$. This is not only true for the case of $n = 1$, but also for n other than 1, as we will show later. For $m = 1$, the flow reduces to a single fluid system and Yih's[6] result is reproduced.

The result for the case of $F_1 = 10$ is chosen to demonstrate the nature of the stability or instability for the flow with $F_1 > 3$. The variations of Θ_2 as a function of m and β for the first and second modes for this particular value of F_1^{-2} are given in Figs. 4(a) and 4(b), respectively. Figure 4(a) shows that when the Froude number becomes as large as 10, the first mode of disturbance is unstable no matter what the values of β and m are. The dashed lines in Fig. 4(a) indicate the maximum growth rate of disturbance for various values of β when m varies from 0.1 to 10. The value of maximum growth rate is decreased by increasing β. The larger the β the greater the m at which the growth rate has a maximum. Figure 4(b) shows that for the second mode of disturbance the flow can be either stable or unstable, depending on the values of m and β. When β increases gradually, the flow changes from stable to unstable if m is in the range of 1–5 and from unstable if $m < 1$. A flow with a high Froude number, such as 10, has the maximum growth rate for the disturbance when the viscosity of the lower fluid is around twice that of the upper fluid. As we have mentioned, the first and the second modes correspond to the interfacial and the free-surface modes, respectively, or vice versa, when $F_1 > 3$. Since the first mode is always unstable, we see from this part alone that when $F_1 > 3$ the interfacial and the free-surface modes compete in governing the instability of the flow.

So far we have only discussed the results for the cases of n and γ being unity. In order to understand the role played by n in the instability consideration of the flow, results with n as a variable parameter (γ remaining unity) will be shown. Since we stress the instability of the flow, β is fixed at 0.8 because for $F_1 < 3$ the flow with $n = 1$ has a maximum amplification rate at approximately that point. The results for the cases of $F_1^{-2} = 10$ and 1 will be illus-

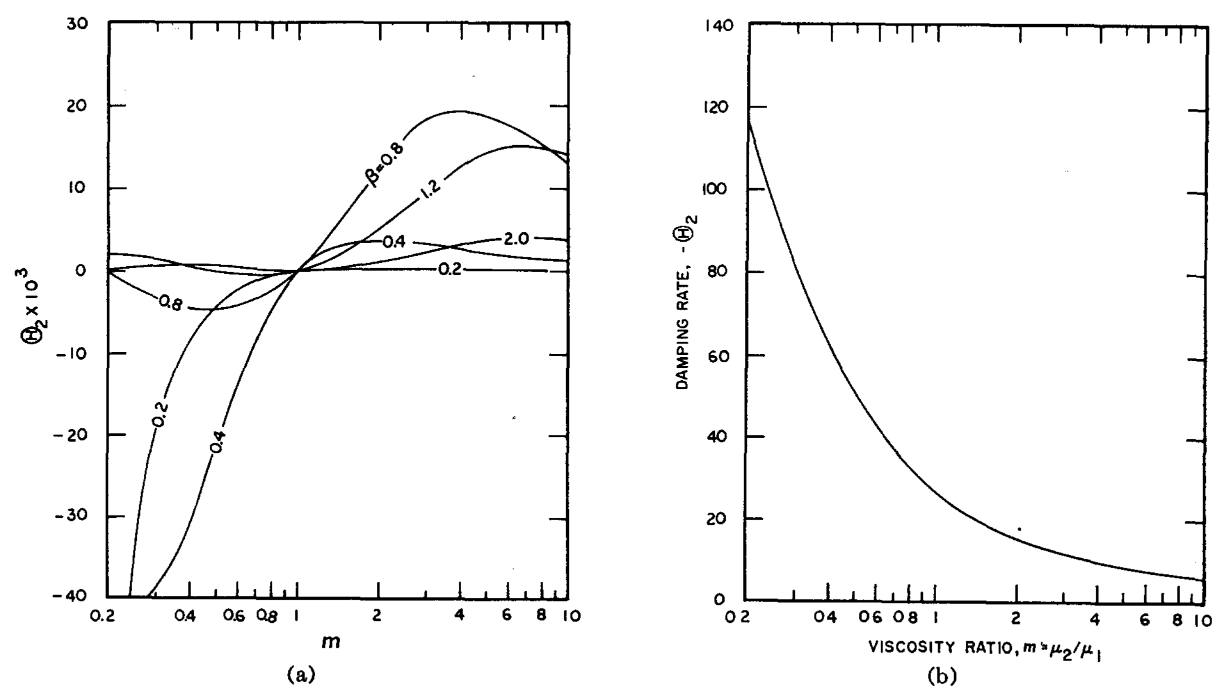

FIG. 2. (a) Variation of the growth rate of disturbance Θ_2 as a function of the viscosity ratio m for various values of β [$= (\omega R/2)^{1/2}$] for the first mode with Froude number $F_1^{-2} = 10$, depth ratio $n = 1$, and density ratio $\gamma = 1$. (b) Variation of the damping rate of disturbance $-\Theta_2$ as a function of the viscosity ratio m for various values of β for the second mode. $F_1^{-2} = 10$, $n = \gamma = 1$. (Curves for all β coincide.)

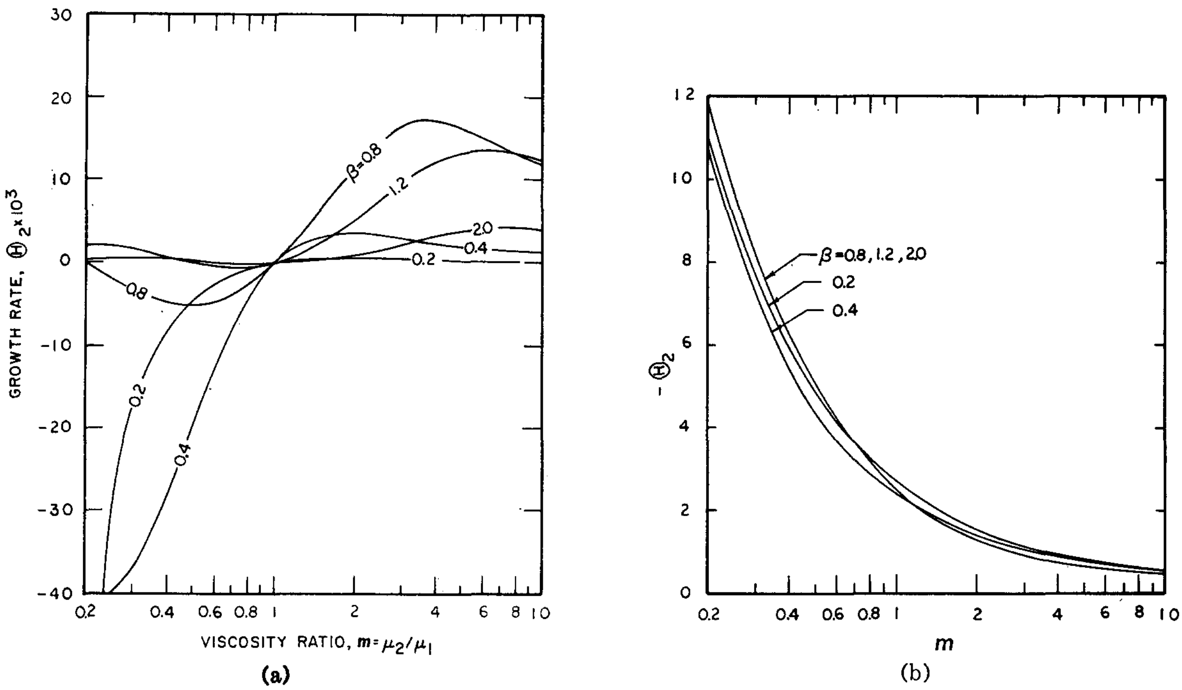

FIG. 3. (a) Variation of the growth rate Θ_2 with the viscosity ratio m for various values of β for the interfacial mode. $F_1 = n = \gamma = 1$. (b) Variation of the damping rate $-\Theta_2$ with the viscosity ratio m for various values of β for the free-surface mode. $F_1 = n = \gamma = 1$.

trated. In Figs. 5(a) and 6(a) the values of Θ_2 for the interfacial mode are plotted against m with n as a parameter for the cases $F_1^{-2} = 10$ and 1. Again the curves in these two figures are similar; this confirms the aforesaid statement that the Froude number makes a negligible influence on the stability or instability of the interfacial mode (for $F_1 < 3$) for any values of m, β, and n. These two figures also show that for any value of n the flow is unstable for $m > 1$ up to at least $m = 10$, and the growth rate

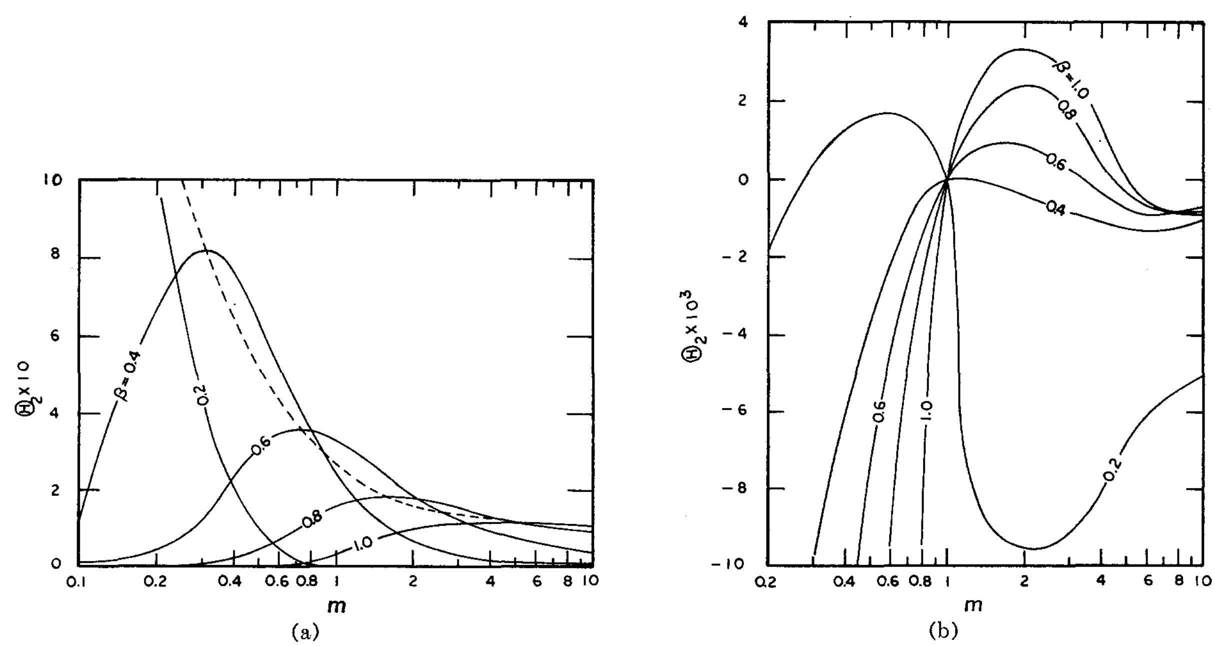

FIG. 4. (a) Variation of the growth rate Θ_2 with the viscosity ratio m for various values of β for the first mode for the case of Froude number $F_1 = 10$, depth ratio $n = 1$, and density ratio $\gamma = 1$. (b) Variation of the growth rate Θ_2 with the viscosity ratio m for various values of β for the second mode. $F_1 = 10$, $n = \gamma = 1$.

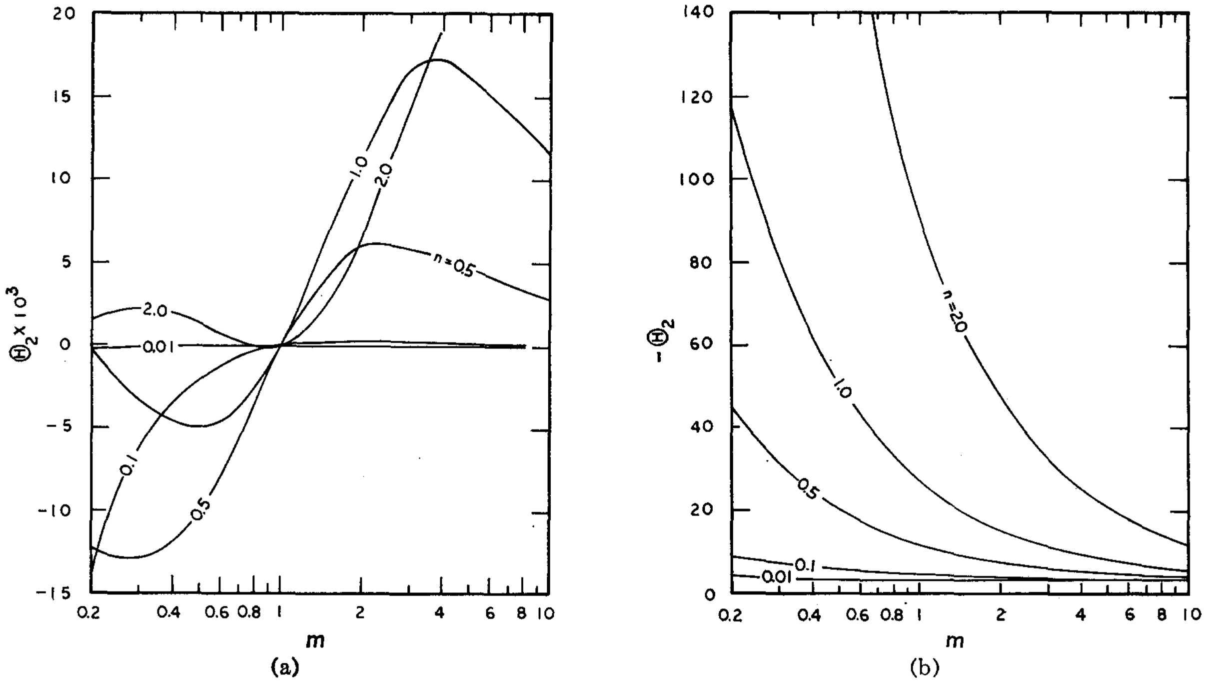

FIG. 5. (a) Variation of the growth rate Θ_2 as a function of the viscosity ratio m for various values of the depth ratio n for the interfacial mode. $F_1^{-2} = 10$, $\beta = 0.8$, $\gamma = 1$. (b) Variation of the damping rate $-\Theta_2$ as a function of the viscosity ratio m for various values of the depth ratio n for the free-surface mode. $F_1^{-2} = 10$, $\beta = 0.8$, $\gamma = 1$.

increases with increasing n. For $m < 1$ the flow is predominantly stable. It is, however, unstable when $n > 2$. These results indicate that increasing the depth ratio might cause the interfacial mode to become more unstable. Figures 5(b) ($F_1^{-2} = 10$) and 6(b) ($F_1 = 1$) show variations of Θ_2 for the free-surface mode. The free-surface mode is always stable for any value of n and m as long as $F_1 < 3$. This result, together with the result mentioned in the discussion of the case of $\gamma = n = 1$, shows that the

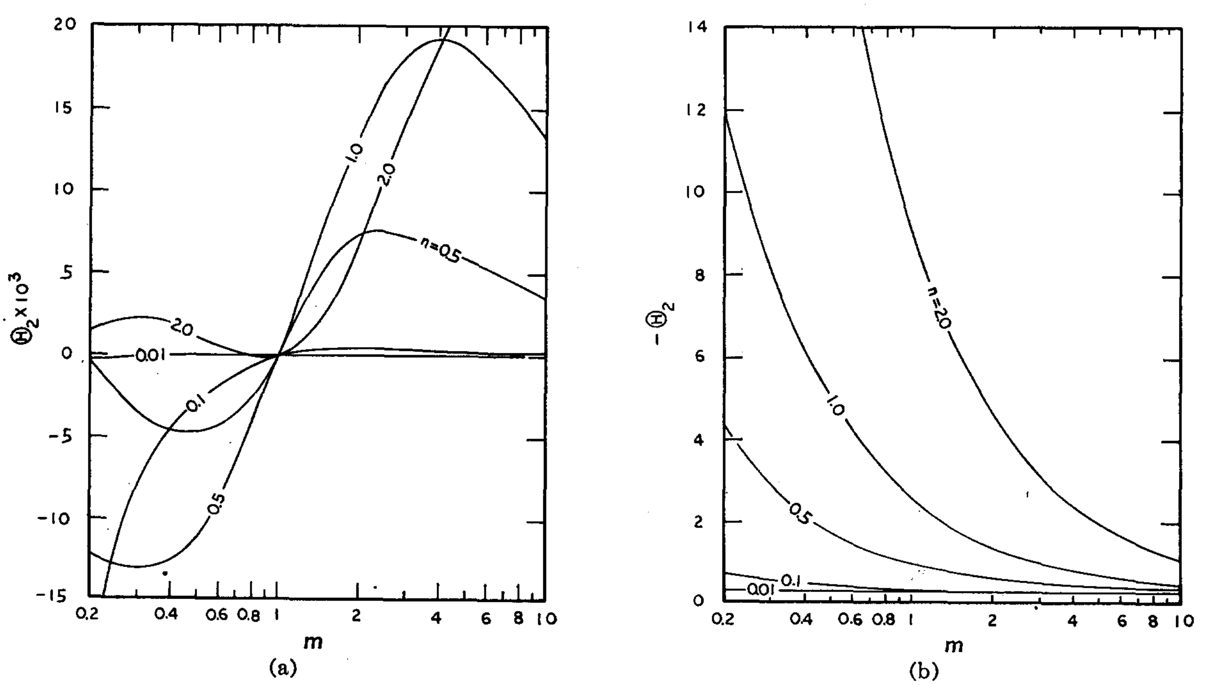

FIG. 6. (a) Variation of the growth rate Θ_2 as a function of the viscosity ratio m for various values of the depth ratio n for the interfacial mode. $F_1 = 1$, $\beta = 0.8$, $\gamma = 1$. (b) Variation of the damping rate $-\Theta_2$ as a function of the viscosity ratio m for various values of the depth ratio n for the free-surface mode. $F_1 = 1$, $\beta = 0.8$, $\gamma = 1$.

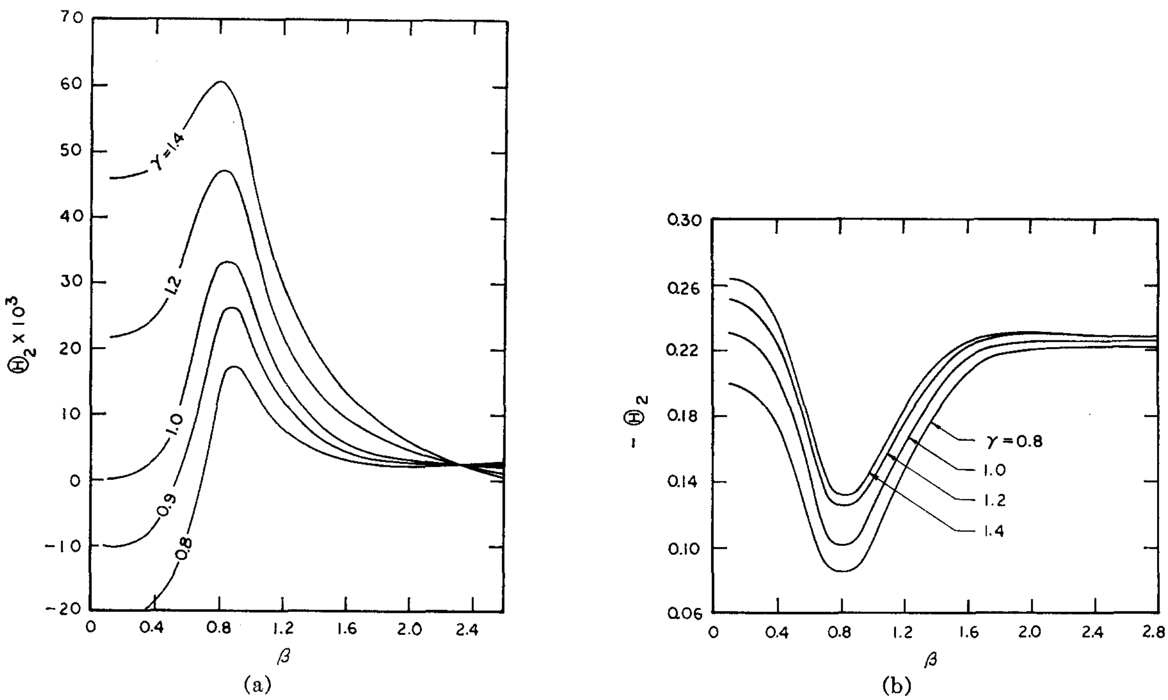

FIG. 7. (a) Dependence of the growth rate of disturbance Θ_2 upon β [$= (\omega R/2)^{1/2}$] for various values of the density ratio γ for the interfacial mode. $F_1 = 2$, $m = 4$, $n = 1$. (b) Dependence of the damping rate of disturbance $-\Theta_2$ upon β [$= (\omega R/2)^{1/2}$] for various values of the density ratio γ for the free-surface mode. $F_1 = 2$, $m = 4$, $n = 1$.

free-surface mode for flow with $F_1 < 3$ is always stable no matter what the values of m, β, and n. Figures 5(b) and 6(b) also show that the larger the values of n and the smaller the values of m, the larger the damping rate. The similarity of the curves in these two figures also confirms the previous statement that the damping rate of disturbance, for any value of m, β, and n, is increased almost linearly by increasing the value of F_1^{-1}, as long as $F_1 < 3$, for the free-surface mode.

So far we have considered only the results for equal densities. In order to see the role played by γ in the instability consideration of the flow, the dependence of Θ_2 upon β and various values of γ for the interfacial and the free-surface modes, for the case $F_1 = 2$, $m = 4$, and $n = 1$, is shown in Figs. 7(a) and (b). We have chosen $m = 4$ because the flow with uniform density has a high growth rate of disturbance in the interfacial mode. Figure 7(a) shows that if the interfacial mode is an unstable mode for $\gamma = 1$, it is also an unstable mode for γ other than 1. The larger the value of γ, the greater the growth rate of disturbance when β is equal to about 0.8. The feature that the flow with the heavier fluid on top can be stable can be seen from Fig. 7(a). On the other hand, Fig. 7(b) shows that if the free-surface mode is stable for the case of $\gamma = 1$, it will be stable for γ other than 1. At first sight it is surprising that the flow with the heavier fluid on top is shown to be stable for any value of β for the free-surface mode. However, it is reasonable, since it is only the interfacial mode which may be unstable. The damping rate of disturbance is apparently increased by increasing γ and the flow has a maximum damping rate when β is almost 0.8. The feature just mentioned is not only true for $F_1 = 2$, but also for F_1 other than 2 (as long as $F_1 < 3$).

VII. CONCLUSIONS

As a result of this investigation, the following conclusions can be drawn for the unsteady flow of two liquid layers due to an oscillating plate:

(1) The interfacial mode and the free-surface mode are found to govern the stability or instability of the flow.

(2) When the Froude number F_1 is less than about 3, the interfacial mode governs the instability of the flow. The free-surface mode is always stable in this case. The two modes are, however, found to compete with each other in governing the instability of the flow when the Froude number is larger than about 3.

(3) For the Froude number less than about 3, the interfacial mode is unstable if the upper fluid is more viscous than the lower one and is either stable or unstable if the upper fluid is less viscous than the

lower one. The growth or damping rate is almost constant with respect to the Froude number. The free-surface mode is always stable as mentioned before. The damping rate increases with increasing the depth ratio n or the density ratio γ or with decreasing the viscosity ratio m. The damping rate increases almost linearly with inverse square of the Froude number, F_1^{-2}, but the effect of $\beta[=(\omega R/2)^{1/2}]$ on it is negligible.

(4) For the Froude number F_1 less than about 3, the two surfaces of discontinuity are always in phase for the free-surface mode of disturbance, but may be in phase or 180 deg out of phase for the interfacial mode. For $F_1 > 3$ (about), the two surfaces can be out of phase by an angle different from 180 deg.

(5) Without considering the diffusion between the two fluids, the flow with the heavier fluid on top can become stable when β is small.

ACKNOWLEDGMENTS

This work was included in the second part of the author's doctoral thesis submitted to the Department of Engineering Mechanics at the University of Michigan. Professor Chia-Shun Yih suggested the topic and provided guidance during the period of this research, and his help is gratefully acknowledged.

The sponsorship of the Army Research Office (Durham) and the National Science Foundation is much appreciated.

APPENDIX. EXPRESSION OF THE DETERMINED FUNCTIONS

$$B_1 = B_{1a} + \sigma_1 B_{1b}, \qquad B_2 = B_{2a} + \sigma_1 B_{2b},$$

$$C_1 = C_{1a} + \sigma_1 C_{1b}, \qquad C_2 = C_{2a} + \sigma_1 C_{2b},$$

$$D_1 = D_{1a} + \sigma_1 D_{1b}, \qquad D_2 = D_{2a} + \sigma_1 D_{2b},$$

where

$$C_{2a} = -m\bar{\lambda}^2 \cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)n\right],$$

$$C_{2b} = \bar{\lambda} V_1 \cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)n\right],$$

$$D_{2a} = -m\bar{\lambda}^2 \sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)n\right],$$

$$D_{2b} = \bar{\lambda} V_1 \sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)n\right],$$

$$D_{1a} = \left(\frac{\gamma}{m}\right)^{1/2} D_{2a},$$

$$D_{1b} = \left(\frac{\gamma}{m}\right)^{1/2}[(1-m)D_0 + D_{2b}],$$

$$B_{2a} = m\bar{\lambda}^2, \qquad B_{2b} = -\bar{\lambda} V_1,$$

$$C_{1a} = \gamma C_{2a}, \qquad C_{1b} = \gamma C_{2b} + A_0(\gamma - 1),$$

$$B_{1a} = B_{2a} + (1-\gamma)C_{2a},$$

$$B_{1b} = B_{2b} + (1-\gamma)C_{2b} + A_0(1-\gamma),$$

where

$$V_1 = D_0(m-1)\sinh[\beta(1+i)] + A_0(1-\gamma)\cosh[\beta(1+i)].$$

Note that the constant factors B_1, B_2, C_1, C_2, D_1, and D_2 are divided into two parts, one contains a factor σ_1 and the other does not. This is done for convenience in performing the numerical calculations.

$$I_1 = \frac{i}{32\beta^2}[(C_1A_0^* + D_1B_0^*)\cosh 2\beta y + (C_1A_0^* - D_1B_0^*)\cos 2\beta y + (D_1A_0^* + C_1B_0^*)\sinh 2\beta y + i(D_1A_0^* - C_1B_0^*)\sin 2\beta y],$$

$$I_2 = \frac{i}{8\beta^2}B_1^*\{A_0\cosh[\beta(1+i)y] + B_0\sinh[\beta(1+i)y]\},$$

$$I_3 = -\frac{i}{8\beta^2}B_1\{A_0\cosh[\beta(1+i)y] + B_0\sinh[\beta(1+i)y]\}\exp(2i\omega\tau),$$

$$Q_1 = \frac{i}{32\beta^2}\left[(C_2C_0^* + D_2D_0^*)\cosh 2\beta\left(\frac{\gamma}{m}\right)^{1/2}y - (D_2D_0^* - C_2C_0^*)\cos 2\beta\left(\frac{\gamma}{m}\right)^{1/2}y + (D_2C_0^* + C_2D_0^*)\sinh 2\beta\left(\frac{\gamma}{m}\right)^{1/2}y - i(C_2D_0^* - D_2C_0^*)\sin 2\beta\left(\frac{\gamma}{m}\right)^{1/2}y\right],$$

$$Q_2 = \frac{i}{8\beta^2}B_2^*\left\{C_0\cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right] + D_0\sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right]\right\},$$

$$Q_3 = -\frac{i}{8\beta^2}B_2\left\{C_0\cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right] + D_0\sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right]\right\}\exp(2i\omega\tau),$$

$$V_a = \sinh[\beta(1+i)y]\exp(2i\omega\tau),$$

$$W_a = \cosh[\beta(1+i)y]\exp(2i\omega\tau),$$
$$\Lambda_1(\tau) = \alpha_1\exp(2i\omega\tau) + \alpha_1^*\exp(-2i\omega\tau),$$
$$\Xi_1(\tau) = \beta_1\exp(2i\omega\tau) + \beta_1^*\exp(-2i\omega\tau),$$
$$V_b = \sinh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right]\exp(2i\omega\tau),$$
$$W_b = \cosh\left[\beta\left(\frac{\gamma}{m}\right)^{1/2}(1+i)y\right]\exp(2i\omega\tau),$$
$$\Lambda_2(\tau) = \alpha_2\exp(2i\omega\tau) + \alpha_2^*\exp(-2i\omega\tau),$$
$$\Xi_2(\tau) = \beta_2\exp(2i\omega\tau) + \beta_2^*\exp(-2i\omega\tau),$$
$$E_2 = \frac{iR}{m}[n^2\tilde{G} - n^2(3+2n)\tilde{D} - n^2\tilde{C} + 2n^3\tilde{H} - m(\tilde{A} + n\tilde{B})],$$
$$F_2 = \frac{iR}{m}[2n\tilde{G} - 3n(2+n)\tilde{D} - 2n\tilde{C} + 3n^2\tilde{H} - m\tilde{B}],$$
$$G_2 = \frac{iR}{m}(\tilde{G} - 3\tilde{D} - \tilde{C}),$$
$$J_2 = \frac{iR}{m}(\tilde{D} - \tilde{H}),$$
$$E_1 = E_2 - iR\tilde{E}, \qquad F_1 = F_2 - iR\tilde{F},$$
$$G_1 = -iR(3\tilde{D} + \tilde{C}), \qquad J_1 = iR\tilde{D},$$

in which

$$\tilde{A} = Q_1(-n) + Q_2(-n) + Q_1^*(-n) + Q_2^*(-n),$$
$$\tilde{B} = Q_1'(-n) + Q_2'(-n) + (Q_1^*)'(-n) + (Q_2^*)'(-n),$$
$$\tilde{C} = \tfrac{1}{2}[I_1''(1) + I_2''(1) + (I_1^*)''(1) + (I_2^*)''(1)] + \tfrac{1}{8}(m\gamma)^{1/2}(B_1^*\bar{\lambda} + B_1\bar{\lambda}^*),$$
$$\tilde{D} = \tilde{D}_1 + \tfrac{1}{6}F_1^{-2},$$
$$\tilde{E} = \mathrm{Re}\left(\frac{i}{8\beta^2}(C_1A_0^* + 2B_1^*A_0 - C_2C_0^* - 2B_2^*C_0)\right),$$
$$\tilde{F} = \mathrm{Re}\left(\frac{i}{8\beta^2}\left\{(1+i)\left[D_1A_0^* + 2B_1^*B_0 - D_2C_0^*\left(\frac{\gamma}{m}\right)^{1/2} - 2B_2^*D_0\left(\frac{\gamma}{m}\right)^{1/2}\right] + (1-i)\left[C_1B_0^* - C_2D_0^*\left(\frac{\gamma}{m}\right)^{1/2}\right] + 2(1-i)(B_1 + C_1 + \sigma_1C_0)\left[B_0^* - \left(\frac{\gamma}{m}\right)^{1/2}D_0^*\right]\right\}\right),$$
$$\tilde{G} = \mathrm{Re}\left(\frac{1}{4}(iD_1B_0^* - B_1^*A_0) - \frac{\gamma}{4}(iD_2D_0^* - B_2^*C_0) - \frac{1}{4}(\gamma-1)A_0(B_1^* + C_1^* + \sigma_1C_0^*)\right),$$
$$\tilde{H} = -\tfrac{1}{6}(\gamma-1)F_1^{-2}\sigma_1,$$

where

$$\tilde{D}_1 = \tfrac{1}{48}\beta(1-i)[A_0C_1^*(\sinh 2\beta - i\sin 2\beta) + A_0D_1^*(\cosh 2\beta + \cos 2\beta) + B_0C_1^*(\cosh 2\beta - \cos 2\beta) + B_0D_1^*(\sinh 2\beta + i\sin 2\beta)] + \tfrac{1}{48}\beta(1+i)[A_0^*C_1(\sinh 2\beta + i\sin 2\beta) + A_0^*D_1(\cosh 2\beta + \cos 2\beta) + B_0^*C_1(\cosh 2\beta - \cos 2\beta) + B_0^*D_1(\sinh 2\beta - i\sin 2\beta)] - \tfrac{1}{6}[I_1'''(1) + I_2'''(1) + (I_1^*)'''(1) + (I_2^*)'''(1)],$$

in which $\bar{\lambda}^*$ is the complex conjugate of $\bar{\lambda}$. The primes on the I's and Q's and their complex conjugates denote the differentiation with respect to y. The argument in the parentheses following I's, Q's, (I^*)'s, and (Q^*)'s and their derivatives indicates the position where they are evaluated.

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[1] T. B. Benjamin and F. Ursell, Proc. Roy. Soc. (London) **A225**, 505 (1954).
[2] S. F. Shen, J. Aerospace Sci. **28**, 397 (1961).
[3] P. W. Conrad and W. O. Criminale, Jr., Z. Angew. Math. Phys. **16**, 233 (1965).
[4] P. W. Conrad and W. O. Criminale, Jr., Z. Angew. Math. Phys. **16**, 569 (1965).
[5] R. E. Kelly, J. Fluid Mech. **22**, 547 (1965).
[6] C.-S. Yih, J. Fluid Mech. **31**, 737 (1968).
[7] N. N. Krasovskii, *Stability of Motion* (Stanford University Press, Stanford, California, 1963), p. 2.
[8] N. G. Chetayer, *The Stability of Motion*, A. W. Babister and J. Burlak, Translation Eds. (Pergamon Press, Inc., New York, 1961), p. 4.
[9] C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, London, 1966), 2nd ed., Chap. 1.
[10] C.-S. Yih, J. Fluid Mech. **27**, 337 (1967).
[11] T. W. Kao, Phys. Fluids **8**, 812 (1965).
[12] T. W. Kao, Phys. Fluids **8**, 2190 (1965).