The Mobility Method of Computing the Vibration of Linear Mechanical and Acoustical Systems: Mechanical-Electrical Analogies*

BY F. A. FIRESTONE
Department of Physics, University of Michigan, Ann Arbor, Michigan

Introduction

THE well-established method for computing the vibrations produced in mechanical systems by impressed vibratory forces or by initial displacements or shocks, consists in writing down the differential equations applicable to the case, solving these equations, and determining the constants of integration by substituting the initial conditions. This procedure yields general equations for the motions of the different parts of the system and the slide rule can then be applied in computing the numerical values. While this method is applicable to a great variety of mechanical problems, differential equations is a fairly abstruse branch of mathematics, requiring continual practice for effective use, and the amount of time involved in solving moderately complicated problems is often prohibitive.

An alternative method of computing vibrations, which has been tried by some, is the method of electrical analogy, wherein one draws the electrical circuit which is analogous to the mechanical problem to be solved, then solves the analogous electrical problem, and finally converts the electrical answer back into mechanical terms. While there are a few meticulous souls who can arrive at correct answers by this method, most of those who have attempted it have become confused and have arrived at preposterous answers. I will show below that this blundering has not necessarily been due to lack of ability on the part of the computers, but is rather the result of imperfections in the mechanical-electrical analogy itself as taught in all the books dealing with this subject.

Furthermore, it can be shown that there is another analogy which is free of the imperfections of the old analogy, in fact, the new analogy is so close that it is no longer necessary to transfer the mechanical problem into electrical terms in order to arrive at the desired answer; the *methods* which the electrical engineer has developed for solving electrical circuit problems are taken over into mechanics and applied to the mechanical problem in purely mechanical terms. It is not necessary to make any reference to electricity.

The mobility method of computing vibration is the result of applying the electrical engineers' methods of computation to mechanical problems. The electrical engineer has developed the following distinctive tools:

1. A set of conventionalized symbols with which the essential characteristics of a circuit can be set forth in the form of a circuit diagram.

2. The concept of the potential difference *across* elements in the circuit as contrasted with the potential *of* points in the circuit relative to ground; the advantage here is that the relationship between the potential difference across an element and the current through it, depends only on the characteristics of the element itself and not on the characteristics of the rest of the system.

3. The use of complex numbers to represent simple harmonic voltages and currents, both the

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magnitude and phase of these quantities being represented by the absolute value and angle of the complex numbers.

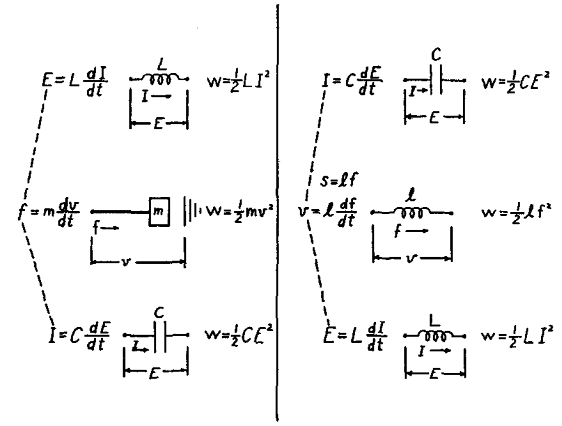

FIG. 1. The old and the new analogs of the mass and the spring.

4. The concept of the "impedance" of an element, which is defined as the complex ratio of the voltage across the element to the current through the element; rules are developed for computing the impedance of series or parallel combinations of elements and the current can then be found simply as the voltage divided by the impedance.

The mobility method provides a similar set of distinctive tools, of form suitable for application to mechanical vibration problems, as follows:

1. A set of conventionalized symbols with which the essential characteristics of a mechanical system can be set forth in the form of a schematic diagram.

2. The concept of the velocity *across* mechanical elements (velocity of one end of the element relative to the other end) as contrasted with the velocity *of* points in the system relative to ground; the advantage here is that the relationship between the velocity across an element and the force through it, depends only on the characteristics of the element itself and not on the characteristics of the rest of the system.

3. The use of complex numbers to represent simple harmonic velocities and forces, both the magnitude and phase of these quantities being represented by the absolute value and angle of the complex numbers.

4. The concept of the "mobility" of an element (ease of motion), which is defined as the complex ratio of the velocity across an element to the force through the element; simple rules are developed for computing the mobility of series or parallel combinations of elements and the force can then be found simply as the velocity divided by the mobility, or the velocity can be found as the force multiplied by the mobility.

Thus by the mobility method we can compute the velocity across each element in the system and the velocity of each point in the system, due to the action of an impressed vibratory force or velocity. Not only can the forced vibration be computed, but the natural frequencies of the free vibration of systems having but small damping, can be found. While the method is most easily applied to those systems in which the vibrations of all parts are parallel to a single line, it can be easily extended to cover many more complicated cases. Torsional vibrations may be computed as easily as linear vibrations. The presence of friction or damping adds no appreciable complication as long as it is of the type in which the force is proportional to the velocity.

Imperfections of the Old Mechanical-Electrical Analogy

Before explaining the mobility method of vibration computation, let us point out the imperfections of the old mechanical-electrical analogy, and outline the new analogy, since it is from this new analogy that the mobility method has sprung.

The left central part of Fig. 1 shows a mass whose velocity v is measured relative to the fixed point designated at the right; this may be considered either as the velocity *of* the mass or the velocity *across* the mass. If a force f of compression or tension is applied through the rod at the left of the mass, there will be a change in the velocity across the mass which can be computed from $f=mdv/dt$. The old analogy says that the inductance shown above the mass plays the same role in the electrical circuit as the mass does in the mechanical system, because an equation of similar form holds for the inductance namely, $E=LdI/dt$. Furthermore the energy stored in the mass is $\frac{1}{2}mv^2$ and the energy stored in the inductance is $\frac{1}{2}LI^2$. Likewise at the right center of Fig. 1 is shown a spring of compliance l (compliance, stretching produced by unit force, reciprocal of the stiffness) across which there is a velocity v and through which there is a force f of compression or tension. The old analogy says that the condenser shown above the spring is analogous to the spring because the velocity across the spring is $v=ldf/dt$ while an equation of similar form holds for the condenser namely, $I=CdE/dt$. Furthermore, the energy stored in the spring is $\frac{1}{2}lf^2$ while the energy stored in the condenser is $\frac{1}{2}CE^2$. A similar analogy holds between mechanical resistors and electrical resistors.

In explaining this old analogy, the books do not usually go into as much detail as we did in the above paragraph in showing figures which outline in detail the nature of the quantities which are analogous. The analogy looks very pretty as long as one merely points out that the mechanical quantities are related by equations of the same form as the equations which relate the electrical quantities. However, when one notes that velocity *across* mechanical elements is in these equations analogous to current *through* the electrical elements, while force *through* is analogous to voltage *across*, the fundamental imperfection of this analogy is obvious. This results further in mechanical elements in series being represented in the analogous circuit by electrical elements in parallel, and *vice versa*. As an example of this left-handed relationship, see the mechanical system of the central part of Fig. 2 and its old analogous electrical circuit at the top of that figure. Most of the blunders which have been made by those who tried to use the old analogy in computing vibrations, can be ascribed to this inverse relationship of series and parallel, and of through and across.

Fortunately, there is another analogy, detailed in the lower part of Fig. 1, which is free from these imperfections. Here the condenser is put analogous to the mass, and the inductance analogous to the spring. Not only is the formal similarity of the equations complete as before, but velocity *across* is analogous to voltage *across*, and force *through* is analogous to current *through*. This results in mechanical elements in series being represented by electrical elements in series; similarly for elements in parallel. The analogous electrical circuit is therefore of a form very similar to the mechanical system itself, as illustrated by the new analogous electrical circuit at the bottom of Fig. 2 which represents the mechanical system in the center of the figure. We could now work out the analogous electrical problem and transfer our answer into mechanical terms, but with the analogy as close as is shown, we ask ourselves what advantage is to be gained

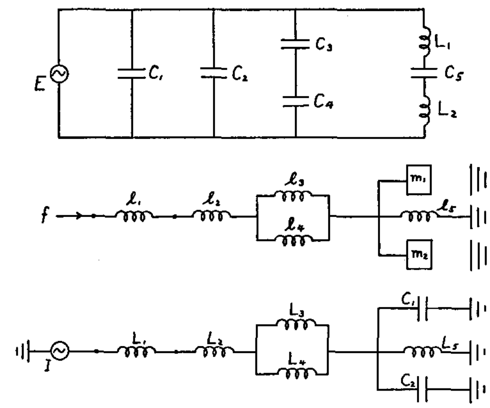

FIG. 2. A mechanical system (center) with its old analogous electrical circuit above and its new analogous electrical circuit below.

through working with the capital letters in the electrical circuit when we could as well work with the small letters in the mechanical system itself, while carrying out the computation in the same manner. The mobility method of computation is the final result of this investigation and consists in applying the new analogy merely in the development of the method of computation; in making computations by the mobility method, no reference is made to the analogy or to electrical systems.

In order that the equations may have an instructive and familiar appearance to those who know their electrical circuit theory we will change the letter symbols for force, velocity, and mass, from the usual f, v, and m to i, e, and c as shown below.

	Usual Symbol	Symbol Used Here
force	f	i
velocity	v	e
mass	m	c

Obviously the meaning of equations is not changed by a change in the letters which represent the quantities appearing in the equations, so anyone may feel free to use the usual letter symbols if the equations appeal to him more in

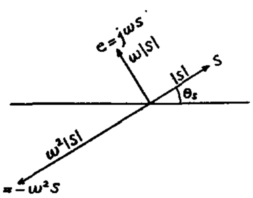

FIG. 2a. The relationship between the vectors representing displacement, velocity, and acceleration, in simple harmonic motion.

that form. (i and e above do not represent current and voltage; they represent force and velocity.)*

The Mobility Method of Vibration Computation **

For our purposes we may consider mechanical systems as being constructed of only three kinds of elements: springs, mechanical resistors, and masses. These correspond to the three fundamental mechanical properties of matter in bulk: elasticity, viscosity, and mass. A spring has a *displacement* across it proportional to the force through it (Hooke's law); a mechanical resistor has a *velocity* across it proportional to the force through it; and a mass has an *acceleration* across it proportional to the force through it (definition of force). The displacement produced across a spring by unit force through it is a constant of its structure called the *compliance* l of the spring and is measured in cm/dyne or inches/pound; the velocity produced across a mechanical resistor by unit force through it is called the *responsiveness* r of the resistor and is measured in kines/dyne (1 kine equals 1 cm/sec.) or inches/sec./pound; the acceleration produced across a mass by unit force acting on it is called the *mass* of the mass and is measured in grams or pounds/g ($g=386$ inches/sec.²). The mechanical resistor may be visualized as a pair of massless concentric tubes having a viscous oil in the space between the tubes so that the velocity across the combination (velocity of one tube relative to the other) is proportional to the force through the combination; substances like oil, rubber, felt, and deadening materials generally, have to a considerable extent the properties of a mechanical resistance, in addition to their elastic or mass properties. While any actual spring has some mass, and every mass has some elasticity, and every actual mechanical resistor has some mass, for purposes of analysis of mechanical systems we consider our mechanical elements as being "pure" and we take into account any important additional properties of any structure by adding elements to the schematic diagram. Regardless of the actual structure of our elements, we will represent them in our schematic diagrams by the symbols shown in the figures which follow; these same symbols will be used whether the vibration is linear or torsional.

If a simple harmonic force is sent through our mechanical elements, it will produce a simple harmonic displacement across the spring, a simple harmonic velocity across the resistor, and a simple harmonic acceleration across the the mass, in fact, if we confine our attention to the oscillating components of the motion we may say that each of these elements will have a simple harmonic displacement, velocity, and acceleration across it. Representing the vibratory motion as the real part of a vector rotating in the complex plane (as is common in electrical circuit theory)† we have the following relationships between the displacement, velocity, and acceleration, across any element:

	Instantaneous Value	Amplitude	Units c.g.s.	Units English
Displacement	$s_i = s\epsilon^{j\omega t}$	s	cm	inches
Velocity	$e_i = ds_i/dt = e\epsilon^{j\omega t}$	$e = j\omega s$	kines	inches/sec.
Acceleration	$a_i = d^2s_i/dt^2 = a\epsilon^{j\omega t}$	$a = j\omega e = -\omega^2 s$	kines/sec.	inches/sec.²
Force	$i_i = i\epsilon^{j\omega t}$	i	dynes	pounds

* For a further discussion of the relative advantages of the old and new analogies, see F. A. Firestone, "A New Analogy Between Mechanical and Electrical Systems," J. Acous. Soc. Am. **4**, 249–267 (1933). This paper was anticipated by a paper appearing in Germany in 1932 pointing out the new analogy, and which was called to my attention by its author in 1933; Walter Hahnle, "Die Darstellung Elektromechanischer Gebilde durch rein elektrische Schaltbilder," Wiss. Veröff. a. d. Siemens-Konzern, Vol. XI, No. 1 (Julius Springer, Berlin, 1932). A recent paper dealing exclusively with the old analogy is Myron Pawley, "The Design of a Mechanical Analogy for the General Linear Electrical Network with Lumped Parameters," J. Frank. Inst., **223**, 179–198 (1937).

** Anyone particularly interested in this subject will find a more detailed discussion in a mimeographed booklet of about 150 pages which the author has prepared for his students. This can be obtained from the author after July 1938 at a cost not to exceed $1.00.

† If the reader is not familiar with the use of complex numbers in representing vibrations, he can easily pick up the necessary knowledge by reading the first four pages of Appendix B of K. S. Johnson's book, *Transmission Circuits*

Here the angular frequency $\omega = 2\pi f$ where f is the frequency of the vibration in cycles per second. $j = (-1)^{\frac{1}{2}}$. The displacement amplitude s is a complex constant whose norm or absolute value $|s|$ represents the actual displacement amplitude across the element (maximum displacement in the vibration) and whose angle θ_s represents the epoch angle of the displacement (portion of the cycle of motion in which the displacement started when $t=0$). The velocity amplitude e and the acceleration amplitude a are complex constants whose absolute values and angles are the actual amplitudes and epoch angles of these quantities. The relationships expressed in the above amplitude equations are set forth in Fig. 2a. The acceleration leads the velocity by 90° and the velocity leads the displacement by 90°. The multiplication of each of these vectors by $\epsilon^{j\omega t}$ causes the set of vectors to rotate with angular speed ω radians/sec. and the real part of each vector, or its projection on the horizontal real axis, is the instantaneous acceleration, velocity, and displacement, respectively. If any one of the three quantities s, e, or a is known, the other two can be found immediately from the equations at the right above. We will find it most convenient to compute e first, even though we may be primarily interested in finding s or a.

The mobility z of an element (or group of elements) is by definition the ratio of the velocity amplitude across the element to the force amplitude through the element;

$$z \equiv e/i \text{ kines/dyne or inches/sec./pound.}$$

The mobility is a complex number, its absolute value $|z|$ being the ease of motion, the amount of velocity produced by unit force, and its angle θ_z being the angle by which the velocity leads

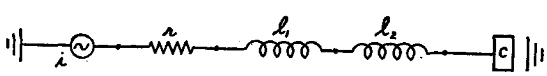

FIG. 3. A series mechanical system. The mobility across a number of elements in series is the sum of their separate mobilities; $z = z_1 + z_2 + z_3 +$.

the force. The mobility depends only on the structure of the element and on the frequency of the impressed force or velocity, and is independent of the amount of force or velocity which is impressed since these are proportional to each other. For convenience, we may give names to the real and imaginary parts of the mobility, calling them the *responsiveness* r and *excitability* x, respectively. Thus

$$\text{mobility} = \text{responsiveness} + j \text{ excitability};$$
$$z = r + jx.$$

If a system has some responsiveness in its mobility it absorbs mechanical energy and turns it into heat or waves or some other form of

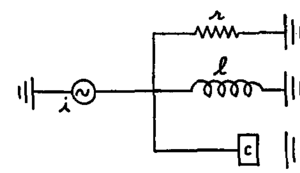

FIG. 4. A parallel mechanical system; $z = 1/(1/z_1 + 1/z_2 + 1/z_3 +)$.

energy, that is, it really *responds* to the vibration; while if the mobility of a system is a pure excitability, energy oscillates in and out of the system but no energy is permanently extracted from the source, that is, the system merely gets *excited*.

The mobility of each kind of element can be easily computed from a knowledge of the fundamental constant of the element and the frequency of the impressed vibration. If an oscillating force of amplitude i is sent through a spring of compliance l, the displacement amplitude across the spring will be $s = li$ (from the definition of compliance). The velocity amplitude across the spring will therefore be $e = j\omega s = j\omega li$. The mobility of a spring is therefore

$$z = e/i = j\omega li/i = j\omega l$$

and is a pure excitability proportional to the frequency ω. On the other hand, if an oscillating force of amplitude i is sent through a mechanical resistor of responsiveness r, it will produce a velocity across the resistor $e = ri$ (from the definition of the responsiveness of a resistor). The mobility of a resistor is therefore.

$$z = e/i = ri/i = r$$

and is a pure responsiveness independent of the frequency. If an oscillating force of amplitude i is impressed on a mass c, it will produce an

for Telephonic Communication (Van Nostrand); or consulting the instruction book accompanying the Keuffel and Esser log log vector slide rule.

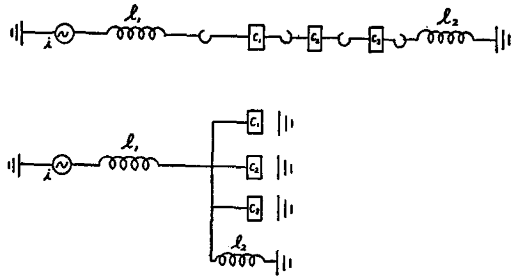

FIG. 5. How to make a blunder in analyzing a mechanical system: the three masses shown in the upper diagram are in parallel since one terminal of each is the ground and their movable terminals are connected together; the two hooks on opposite sides of each mass are not two terminals, they are the same terminal. The schematic diagram of the structure at the top is shown in the lower part of the figure.

acceleration across the mass of amplitude $a=i/c$ (from the definition of force, otherwise known as Newton's laws of motion). The velocity amplitude across the mass will therefore be $e=a/j\omega=i/j\omega c$ and the mobility will be

$$z=e/i=1/j\omega c=-j/\omega c,$$

a pure negative excitability inversely proportional to the frequency. Summarizing, the mobility of each of our three mechanical elements is simply:

ELEMENT	MOBILITY	C.G.S.	ENGLISH
Spring	$z=j\omega l$	kines/dyne	inches/sec./pound
Resistor	$z=r$	"	"
Mass	$z=-j/\omega c$	"	"

In the above equations the units are as follows:

		C.G.S.	ENGLISH
Spring	has compliance	of l cm/dyne	or inches/pound
Resistor	has responsiveness	of r kines/dyne	or inches/sec./pound
Mass	has mass	of c grams	or pound sec.^{2}/inch (or w/g)

Elements are said to be connected in series when their terminals are connected end to end (with not more than two terminals to any junction point) as shown in Fig. 3. Elements are connected in parallel when their terminals are connected to two common junction points as shown in Fig. 4. If there is any question as to whether in any specific example the elements are connected in series or in parallel, one should note that functionally a series connection results in the same force acting through all the elements, while the velocity across the combination is the sum of the velocities across the individual elements; a parallel connection results in the same velocity across all the elements while the force through the combination is the sum of the forces through the individual elements. The mobility of a series combination of elements is therefore

$$z=e/i=(e_1+e_2+e_3+)/i=z_1+z_2+z_3+,$$

and is simply the sum of the mobilities of the individual elements. The mobility of a parallel combination of elements is

$$z=\frac{e}{i}=\frac{e}{i_1+i_2+i_3+}=\frac{1}{1/z_1+1/z_2+1/z_3+}$$

and is the reciprocal of the sum of the reciprocals of the mobilities of the individual elements. It should be remembered that one terminal of every mass is the fixed point relative to which the velocity of the mass is measured, otherwise one may look at Fig. 5 and conclude erroneously that the masses shown there are in series because they are hooked end to end; however, the two hooks on each mass are not the two terminals of the mass, they are the same terminal, there is no relative velocity between them, they move together, and the other terminal of each mass is the floor of the laboratory. The schematic diagram of these masses is shown in the lower part of Fig. 5, indicating that the masses are in parallel and that the force on the parallel combination of elements is the sum of the forces required by the individual elements. Since all masses have one terminal in common (the frame of reference), it is not possible to connect a number of them in series in any simple manner.

The mobility method of vibration computation consists in drawing the schematic diagram of the mechanical system and applying the simple formulas of the last two paragraphs. The system will usually be merely a series-parallel arrangement of elements, and with the aid of the above formulas we can easily compute the mobility of the combination through which the given oscillating force is applied. The velocity amplitude across this combination can then be found merely as $e=iz$. The details of such computations will be most quickly grasped by reference to the fairly complicated example shown in Fig. 6. Here a one-pound mass is supported by a vertical leaf spring of compliance 0.054 inches/pound. A 2-pound mass sits upon the 1-pound mass and is separated from it by a film of very viscous oil of responsiveness 0.1 inches/sec./pound so that the motion of the lower mass transmits a force to the upper mass due to the viscosity. The 2-pound mass is connected to the frame of the machine through a spring of compliance 0.012 inches/pound and to a 3-pound mass through a spring of compliance 0.005 inches/pound. The 3-pound mass is supported by the frame of the machine through a film of very viscous oil of responsiveness 0.4 inches/sec./pound. A sinusoidal force of amplitude 10 pounds and frequency 15.91 cycles per second acts upon the 1 pound mass and reacts against the frame of the machine. We wish to find the velocity amplitude and displacement amplitude of each mass.

In drawing the schematic diagram of the system we identify the two terminals of each element of the system and connect together in the diagram those terminals which move together in the system. Those terminals which are stationary are connected to the ground symbol —||ı. Thus the force $i=10$ is shown at the left, acting between the ground and the movable terminal of the mass 1; the leaf spring is connected from this terminal to ground. The resistor representing the oil film is connected between the movable terminals of masses 1 and 2. We proceed similarly throughout the remainder of the system, remembering that one terminal of every mass is the ground, relative to which we measure the velocity of the mass. It is to be understood that regardless of the directions in which the vibrations might take place in any

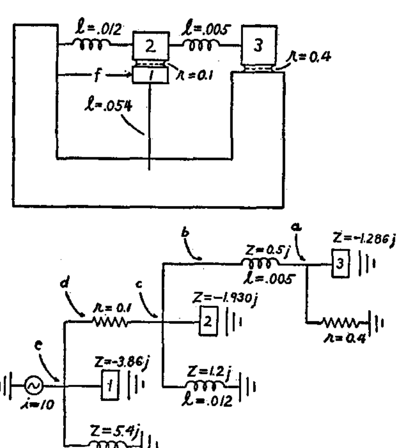

FIG. 6. A mechanical structure and its schematic diagram, a numerical problem solved by the mobility method in the text.

actual system, the vibrations in the schematic diagram are along a horizontal line, and the vertical lines in the diagram remain vertical throughout the vibration cycle. When the schematic diagram has been drawn the problem is already analyzed and we are as far along as we would have been if we had written down the differential equations of the system and had found their solutions, for we are now ready to start numerical computation.

Using the formulas given above we first compute the mobility of each element at the impressed angular frequency $\omega=2\pi 15.91=100$ radians/sec. This has been done and the results are tabulated above the elements in the figure (remember that $c=w/g$), except for the resistors, whose mobilities equal their responsivenesses. The mobility of the point a looking toward the right, is the mobility of the parallel combination of mass 3 and the resistor;

$$z_a=\frac{1}{1/0.4+1/-1.286j}=\frac{1}{2.5+0.778j}$$
$$=\frac{2.5-0.778j}{2.5^2+0.778^2}=0.3640-0.1133j.$$

The mobility of the point b looking toward the right, is the mobility of the spring, in series with the mobility z_a;

$$z_b=0.5j+(0.3640-0.1133j)=0.3640+0.3867j.$$

The mobility of the point c looking toward the right, is the mobility of the parallel combination of z_b, the mass 2, and the spring;

$$z_c=\frac{1}{1/1.2j+1/-1.930j+1/(0.3640+0.3867j)}$$
$$=0.2864+0.3743j.$$

Similarly

$$z_d=0.1+(0.2864+0.3743j)=0.3864+0.3743j$$

and $$z_e=1\Big/\left(\frac{1}{5.4j}+\frac{1}{-3.86j}+\frac{1}{0.3864+0.3743j}\right)$$
$$=0.409+0.3737j.$$

Since the mobility of point e is by definition the velocity which unit force would produce at the point e, we can immediately find the velocity of the point e due to the impressed force $i=10$ simply as

$$e_e = iz_e = 10(0.409+0.3737j) = 4.09+3.737j$$
$$= 5.54 \angle 42^\circ.4 \text{ inches/sec.}$$

This means that the maximum velocity of mass 1 is 5.54 inches/sec. while its motion leads the impressed force by a phase angle of 42°.4. $(5.54 = (4.09^2+3.737^2)^{\frac{1}{2}}.)$ The velocity of the point d is the same as the velocity of e, $e_d = e_e$, so the force through the point d is*

$$i_d = \frac{e_d}{z_d} = \frac{5.54 \angle 42^\circ.40}{0.3864+0.3743j} = \frac{5.54 \angle 42.40}{0.538 \angle 44.08}$$
$$= 10.28 \angle -1^\circ.68 \text{ pounds.}$$

One should not be disturbed to find that the force through d is greater than the impressed force of 10 pounds, since it is the *vector* sum of the forces through the spring, the mass, and d, which equals the impressed force. The force through the point c equals the force through the point d, $i_c = i_d$, so the velocity of c

$$e_c = i_c z_c = 10.28 \angle -1.68(0.28648+0.3743j)$$
$$= 4.845 \angle 51.00 \text{ inches/sec.}$$

Thus the velocity amplitude of the mass 2 is 4.845 inches/sec. and leads the impressed force $i=10$ by 51°. Similarly, since $e_b = e_c$

$$i_b = \frac{e_b}{z_b} = \frac{4.845 \angle 51.00}{0.3640+0.3867j} = 9.10 \angle 4^\circ.23 \text{ pounds,}$$

and since $i_a = i_b$

$$e_a = i_a z_a = 9.1 \angle 04.23(0.3640-0.1133j)$$
$$= 3.465 \angle -13.08 \text{ inches/sec.}$$

Thus the velocity amplitude of mass 3 is 3.465 inches/sec. and lags the impressed force $i=10$ by 13°.08. The displacement amplitudes of the masses can be found by dividing their velocity amplitudes by $100j$ since $s = e/j\omega$. We summarize our answers as follows:

Velocity Amplitudes	Displacement Amplitudes
$e_1 = e_e = 5.54 \angle 42.40$ inches/sec.	$s_1 = 0.0554 \angle -47.60$ inches
$e_2 = e_c = 4.845 \angle 51.00$ inches/sec.	$s_2 = 0.04845 \angle -39.00$ inches
$e_3 = e_a = 3.465 \angle -13.08$ inches/sec.	$s_3 = 0.03465 \angle -103.08$ inches.

If the frequency of the impressed force were to be changed, the mobilities of the individual elements would be changed and the computation would have to be repeated from the first, in order to find the new velocities and displacements of the masses.

Reviewing the method applied above, the mobility at the driving point has been computed by the application of the series and parallel formulas, and the velocity of the driving point is found from $e = iz$. The force applied to the next junction point is found from $i = e/z$, and the velocity of that point from $e = iz$. By continuing this process, all force and velocity amplitudes can be found in any system which consists of a series-parallel connection of elements.

It can be shown quite simply that the average power dissipated in a mobility $z = |z| \angle \theta_z = r + jx$ when a simple harmonic force i is sent through it or a simple harmonic velocity e is impressed across it, can be computed from any of the following expressions:

$$p_{\text{Av}} = \frac{|e||i|}{2} \cos\theta_z = \frac{|e|^2}{2|z|} \cos\theta_z = \frac{|e|^2}{2r} \cos^2\theta_z,$$
$$p_{\text{Av}} = \frac{|i|^2}{2} r = \bar{\imath}^2 r \text{ ergs/sec. or inch pounds/sec.}$$

The factor of 2 in the denominator of these equations becomes unity if we use the effective value of e and i, namely $\bar{e}$ and $\bar{\imath}$, which for simple harmonic motion are equal to $\bar{e} = 0.707|e|$ and $\bar{\imath} = 0.707|i|$. In the above example, the average power delivered by the force $i=10$ to the point e on which it acts, is

$$p_{\text{Av}} = \frac{|i|^2}{2} r_e = \frac{10^2}{2} 0.409 = 20.45 \text{ inch pounds/sec.}$$

The average power dissipated by the resistor under mass 3 is

$$p_{\text{Av}} = \frac{|e_a|^2}{2r} = \frac{3.465^2}{2 \times 0.4} = 15.00 \text{ inch pounds/sec.}$$

The remainder of the power supplied by the source, namely 5.45 inch pounds/sec., must be dissipated by the other resistor since the springs and masses do not dissipate power; this value checks approximately with that which we obtain by direct computation;

$$\frac{|i|^2}{2} r = \frac{10.28^2}{2} 0.1 = 5.28 \text{ inch pounds/sec.}$$

* In carrying out computations of this sort in which the complex numbers are changed from rectangular form to polar form, or *vice versa*, the log log vector slide rule is very convenient as it permits this operation to be performed with a single setting of the rule.

This is one method of checking the accuracy of the entire calculation.

Force and Velocity Laws

Some mechanical systems do not consist simply of a series-parallel arrangement of elements, for instance, if we add to the system discussed above, a spring connecting mass 1 directly to mass 3, the schematic diagram will look as shown in Fig. 7 and the mobility at the driving point cannot be computed merely from the series and parallel formulas. In order to find the forces and velocities in such a system we must use two general rules which we may call the

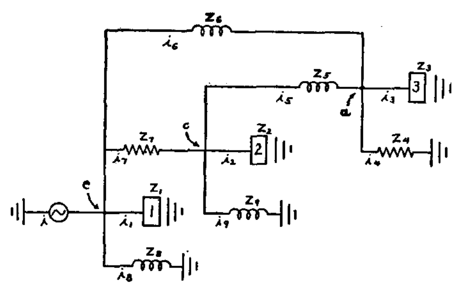

FIG. 7. A modification of the mechanical system shown in Fig. 6, which requires the application of the force and velocity laws for its solution.

force and velocity laws (Kirchhoff's laws in electricity):

Force law. The sum of all the forces acting on any junction point is zero.

Velocity law. The sum of all the velocities across the structures included in any closed mechanical circuit is zero.

Since the junction point to which a number of terminals of elements may be connected is considered as being massless, no force is required to move it, so the vector sum of all the force amplitudes acting on the junction must be zero. Forces through elements lying on the left of the junction may be taken with the signs given, and those on the right with their signs changed. (The force through an element is considered positive when it is a compressional force.) Since in going around a closed mechanical circuit we return to the starting point, or go from a grounded point to another grounded point, the sum of all the velocity differences across the structures included in the circuit must be zero. When writing down such a velocity equation, we progress around the circuit in a clockwise direction and write the velocity across each element passed through in the positive direction (to the right) with the sign given, and write each passed through in the negative direction, with its sign changed. (Velocity across an element is considered positive when the element is growing shorter; displacement is positive when the element is shorter than its normal length.) In applying the force and velocity laws we designate each of the unknown forces through series of elements by a letter, and then write down the equations indicated by the laws. This will yield a sufficient number of equations for determining the unknown forces. When the forces are known, velocities can be found at once from $e=iz$.

As an example, let us write down the force and velocity equations for the system of Fig. 7. At the junctions a, c, and e, respectively, we find the following complex equations:

$$i_5+i_6-i_3-i_4=0,$$
$$i_7-i_5-i_2-i_9=0,$$
$$i\ -i_6-i_7-i_1-i_8=0.$$

We now write down velocity laws around various closed circuits as follows:

$$i_3z_3-i_4z_4=0,$$
$$i_2z_2-i_9z_9=0,$$
$$i_1z_1-i_8z_8=0,$$
$$i_6z_6-i_5z_5-i_7z_7=0,$$
$$-i_1z_1+i_7z_7+i_2z_2=0,$$
$$-i_2z_2+i_5z_5+i_3z_3=0.$$

Since at any frequency the z's in the above equations are known from the constants of the elements, we have 9 simultaneous equations which can be solved (perhaps by determinants) for the 9 unknown forces, or as many of them as we wish to know. The velocity of mass 1 is then $e_1=i_1z_1$; similarly for the other masses.

While this method will work for any system whose schematic diagram can be drawn, the series-parallel method is simpler when it is applicable.

Other General Laws

There are a number of other dynamical laws which are well known in electrical circuit theory but which can be advantageously applied to the mechanical systems as represented in our schematic diagrams. These laws will merely be stated here in mechanical terms without proof.

Thevenin's theorem. While we often think of a force vibrator as being capable of impressing a certain constant amplitude of oscillating force regardless of the mobility of the mechanical system on which it acts, there is a limit to the velocity with which any practical force producing device can move its terminals even when it is not connected to anything, so it cannot send as much force through a system of high mobility as through one of low mobility. Likewise, a practical velocity producing vibrator is limited as regards the force amplitude which it can send through systems of even very low mobility, so it cannot produce as much velocity across a system of low mobility as across one of high mobility. The *internal mobility* of a vibrator is defined as the velocity amplitude which it can produce between its own terminals when not connected to anything, divided by the force amplitude which it can produce through an immovable object;

$$z_s = \frac{e \text{ free}}{i \text{ blocked}}.$$

The velocity form of Thevenin's theorem states that any vibrator is equivalent to an ideal velocity vibrator of velocity amplitude equal to *e free*, in series with its internal mobility z_s, and the force sent through any system of mobility z to which this vibrator might be connected is (see Fig. 8)

$$i = \frac{e \text{ free}}{z_s + z}.$$

The velocity produced across the system can be found as $e = iz$. The force form of Thevenin's theorem states that any vibrator is equivalent to an ideal force vibrator of force amplitude

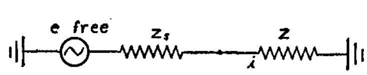

FIG. 8. According to the velocity form of Thevenin's theorem, any complicated vibrator is equivalent to an ideal velocity vibrator of velocity amplitude equal to e free, in series with the internal mobility of the vibrator $z_s = e$ free/i blocked. The force i sent through a load of mobility z connected to the actual vibrator is $i = e$ free/$(z_s + z)$.

equal to *i blocked*, in parallel with its internal mobility z_s, and the velocity produced across any system of mobility z to which this vibrator might be connected is (see Fig. 9)

$$e = \frac{i \text{ blocked}}{1/z_s + 1/z}.$$

Even a complicated portion of a system may be considered as the vibrator and its internal mobility determined by observing or computing its e free and i blocked; the effect of this portion of the system on the balance of the system can be computed from either of the formulas given above. Such a procedure would be advantageous when one wished to compute the vibration which a given portion of the system would produce in a number of different driven systems; the entire system would not have to be recomputed for each trial, only the changed driven systems.

The reciprocity theorem. If a force of high internal mobility is connected between terminals AB of a mechanical system and produces a certain velocity between terminals CD, it will produce that same velocity, both in magnitude and phase, between terminals AB if connected between terminals CD. Or, if a velocity of low internal mobility is impressed in series with an arm X in a mechanical system and produces a certain force through an arm Y, it will produce that same force through X if connected in series with Y. These are the two forms of the reciprocity theorem and permit one to immediately compute the vibration which will be transmitted in one direction through a mechanical system

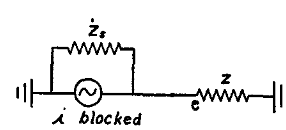

FIG. 9. According to the force form of Thevenin's theorem, any complicated vibrator is equivalent to an ideal force vibrator of force amplitude equal to i blocked, in parallel with the internal mobility z_s. The velocity e across a load of mobility z connected to the actual vibrator is $e = i$ blocked$/(1/z_s + 1/z)$.

by a given force or velocity when he knows how much is transmitted in the other direction. This theorem is true for vibrations of any wave form, and for both the transients and the forced vibration.

The principle of superposition. If a number of vibratory forces and/or velocities act simultaneously in various parts of a mechanical system, the instantaneous forces and velocities produced in the different parts of the system are simply the sum of the instantaneous forces and velocities which the separate vibrators would produce when acting alone. If the frequencies of all the vibrators are the same, the resultant force or velocity amplitude at any point is simply the vector sum of the force and velocity amplitudes produced by the vibrators when acting alone. In computing the vibration produced by one of the vibrators acting alone, the internal mobilities of the other vibrators must remain connected in the system even though these vibrators are not introducing any energy into the system.

Compensation theorem. If a system is modified by making a change Δz in the mobility of one of its branches, the force increment thereby pro-

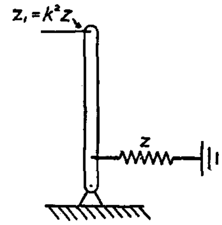

FIG. 10. A lever of ratio k, changes the mobility by a factor k^2.

duced through any arm in the system is equal to the force that would be produced through that arm by a compensating velocity acting in series with the *modified* branch, whose value is $-i\Delta z$, where i is the original force through the modified branch.

Mobility matching with levers or gears. If we have given a vibrator of internal mobility $z_s = r_s + jx_s$, we can extract maximum power from this vibrator by connecting it to a load whose mobility is the conjugate complex of z_s, namely to $z = r_s - jx_s$. If the angle of the mobility of the load is fixed while only the absolute value of the mobility of the load is subject to choice, maximum power will be delivered by the vibrator to the load when the absolute value of the load mobility equals the absolute value of the internal mobility of the vibrator; when $|z| = |z_s|$. As shown in Fig. 10 a massless lever whose arms have lengths of ratio k : 1, increases the velocity by a factor k and decreases the force by a factor k, so the mobility looking toward the lever is k^2 times the mobility of the load to which the lever is connected; $z_1 = k^2 z$. Thus a massless lever or gear can be used to connect the vibrator to the load, thereby changing the absolute value of the load mobility without changing its angle and bringing about a mobility match which will cause maximum power to be delivered by the vibrator to the load; further increase of power

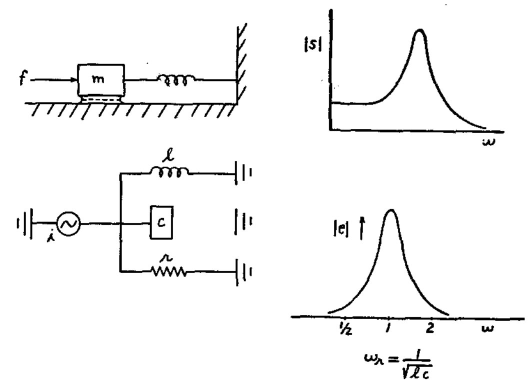

FIG. 11. The elementary problem of the forced vibration of the parallel mechanical system of one degree of freedom, the first problem discussed by most books on vibration (and unfortunately always considered as analogous to a *series* electrical circuit).

can be brought about only by resonating the system by the addition of a suitable excitability.

Examples

The first problem dealt with by most books is the simple system shown in Fig. 11 wherein a mass connected to a fixed point through a spring and supported on a fixed surface by an oil film, is acted on by a vibrating force. The schematic diagram shows the elements in parallel since one terminal of each is the fixed point, and their other terminals are fastened together. Since this is a parallel system we can at once write down the velocity across the system, or the velocity of the mass, as

$$e = iz = \frac{i}{1/r + 1/(-j/\omega c) + 1/j\omega l} = \frac{i}{1/r + j(\omega c - 1/\omega l)}$$

and the displacement amplitude

$$s = e/j\omega.$$

If $|e|$ and $|s|$ are plotted against the frequency of the impressed force they yield the familiar resonance curves of the form shown in the figure; the velocity resonance curve is symmetrical when plotted against a logarithmic frequency scale. The actual velocity amplitude is maximum when $\omega c = 1/\omega l$ giving $\omega_r = 1/(lc)^{\frac{1}{2}}$. An example of this kind of a system is not easy to find on an automobile.

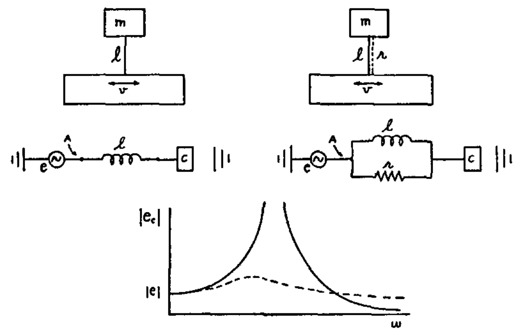

FIG. 12. A simple mechanical system, both without and with resistance, subjected to an impressed vibration.

A more common type of system is shown at the left of Fig. 12. A massive base which is vibrating with a given velocity, supports a leaf spring which carries a mass on its outer end. This might represent some accessory which is fastened to the motor by an elastic bracket, or the knob on the end of the shift lever, or a radio amplifier which we wish to protect from vibration by an elastic suspension, or the vibrating element of a tuned vibration microphone such as is screwed to a gear test stand in order to indicate the amount of gear note, or very crudely the center of a door panel which receives vibration from the door frame. Since the velocities of the base and the mass are measured relative to ground, one terminal of the velocity vibrator and of the mass is shown as the ground in the schematic diagram, while the spring has its terminals connected to the movable terminals of these elements. The mobility at the point A is the mobility of the spring and mass in series, and the force through the combination is

$$i = \frac{e}{z_A} = \frac{e}{j\omega l - j/\omega c}$$

and the velocity of the mass is

$$e_c = iz_c = i\left(\frac{-j}{\omega c}\right) = \frac{e}{1-\omega^2/\omega_r^2} \quad \text{where } \omega_r = 1/(lc)^{\frac{1}{2}}.$$

If the actual velocity amplitude across the mass is plotted against frequency of the impressed velocity, the solid curve shown in the figure is obtained; this indicates that below the resonant frequency the sprung mass vibrates with a greater velocity than the source of vibration, while considerably above resonance the sprung mass has a very low velocity. At resonance the sprung mass has a very high velocity and the system reacts on the vibrator with a large force. If now we add a resistor in parallel with the spring by making the spring out of rubber or adding a damping material to the door panel, we get the system shown at the right of Fig. 12. Here again

$$i = \frac{e}{z_a} = e\bigg/\left(\frac{1}{1/r + 1/j\omega l} - \frac{j}{\omega c}\right)$$

and

$$e_c = i(-j/\omega c).$$

This yields the dotted curve shown in the figure which indicates that the addition of the damping has materially reduced the velocity of the sprung mass in the neighborhood of the resonance frequency, but at higher frequencies the damping has increased the velocity above what it was with a purely elastic spring. (At the higher frequencies, however, a door panel would vibrate in more complicated modes so this analysis would no longer apply to it.)

The mobility method of computation is not limited to linear vibrations but is equally applicable to torsional vibrations if the units are properly chosen. The same symbols can be used in the schematic diagram, but with the understanding that the vibrations in the elements are torsional. e is the angular velocity across an element in radians/sec. i is the torque through an element in dyne cm or pound inches. z is the torsional mobility in rad./sec./dyne cm or rad./sec./pound inch. l is the torsional compliance in rad./dyne cm or rad./pound inch. r is the torsional responsiveness in the same units as mobility. c is the moment of inertia in gram cm^2 or pound inches^2/g (or pound inch sec.^{2}). An example is shown in Fig. 13 which represents a six cylinder crank shaft with flywheel. Each crank and connecting rod has been replaced by its equivalent disk as explained in the books on such matters and we wish to compute the oscillation of each crank and the flywheel due to a certain sinusoidal component of the torque applied to the first crank. The schematic diagram is as shown in the figure. As in our first example discussed on p. 379 we start with the mobility of c_7 and compute the mobilities at successive junction points until we obtain the mobility at the driving point at the first crank. We then compute the forces through the springs and the velocities of the other cranks and flywheel. In this example, the computation will be simpler than in our first example because all of the mobilities are pure imaginary instead of complex, there being no resistors in the system. The vibration of the flywheel due to the vibratory torques supplied at all the cranks can be found by applying the principle of superposition mentioned above.

Suppose further that we wished to find the natural frequencies of this crank shaft. At any natural frequency, any mass can be maintained in vibration with a finite velocity without maintaining any vibrating force on the mass. At a natural frequency, the mobility of any vibrating mass and the system to which it is connected, is infinite. Therefore, we can locate the natural frequencies by computing the mobility of any point in the system at a number of different frequencies and noting which frequency makes the mobility infinite. This is not as difficult as might at first sight appear since the mobility changes sign on passing through infinity, so at least one natural frequency lies between any two frequencies of computation if the mobilities at these frequencies are of different sign. There will be as many natural frequencies as there are

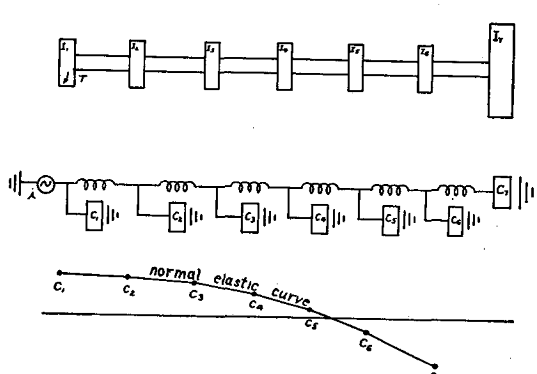

FIG. 13. The torsional vibrations of a crank shaft may be analyzed by the mobility method; not only can the forced vibration be found but the natural frequencies and normal modes of vibration can be computed.

active degrees of freedom of the system, six in this example. When a natural frequency has been found, the normal mode of vibration can be computed merely by applying a vibratory velocity of this frequency to one point in the system and computing the velocities of the other points. In this example the lowest natural frequency results in a normal elastic curve somewhat as shown in the lower part of the figure, the first five cranks swinging against the last crank and the flywheel.

Acoustical Mobility

The mobility method is also applicable to the computation of the performance of certain types of enclosed acoustical systems, such as intake silencers. Just as the vibration of mechanical systems is expressed in terms of the force through elements and the velocity across them, a simple harmonic sound traveling along the inside of a tube is described at any cross section of the tube by giving its *sound pressure* and its *volume velocity*. The sound pressure is the variation of the pressure at the cross section of the tube above and below the normal atmospheric pressure, and the maximum value of this variation is called the *sound pressure amplitude* I measured in dynes/cm^2. The volume velocity is the rate at which air is flowing through the cross section due to the sound and the maximum value of this oscillating quantity is called the *volume velocity amplitude* E measured in cm^3/sec.

The acoustical mobility Z of an area lying in the wave front, is defined as the volume velocity amplitude through the area divided by the sound pressure amplitude at the area;

$$Z \equiv E/I, \text{ cm}^5/\text{dyne sec.}$$

The acoustical mobility is the ease of motion of the air lying in the area, the amount of oscillatory volume velocity produced by unit sound pressure amplitude.

The acoustical mobility depends on the structure of the acoustical system lying beyond the area and on the frequency of the impressed sound pressure; it can be computed from simple formulas for a number of structures and combinations of structures. The formulas are simplified in form if the frequency enters them in the form of a wave number k (the number of waves in 2π cm) which is defined as $k \equiv \omega/c$ where c is the speed of propagation of sound in the medium in cm/sec. Another constant of the medium which enters the equations is called the specific acoustic responsiveness of the medium and is defined as $R_s \equiv 1/\rho_0 c$ where ρ_0 is the normal density of the medium in grams/cm^3; for air $R_s \equiv 0.025$ cm^3/dyne sec. under average conditions. The acoustical elements are necks, vol-

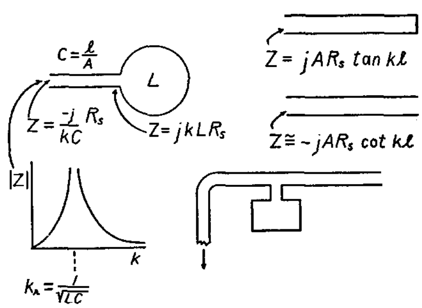

FIG. 14. The concept of acoustical mobility can be applied to the computation of the effectiveness of an intake silencer.

umes, and acoustical resistors. In Fig. 14 a neck and a volume, whose linear dimensions must be small compared to the wave-length, are combined into a bottle shaped structure called a Helmholtz resonator. The acoustical mobility of the area which opens into the wall of the volume of L cm^3 can be shown to be

$$Z_L = jkL\ R_s.$$

The mobility of the area at one end of the neck, assuming that the other end opens into free space and calling the *slenderness* of the neck $C = l/A$ = length/area, in cm^{-1}, can be shown to be

$$Z_C = (-j/kC)R_s.$$

The effective length of a neck is greater than its measured length by an end correction of approximately $\pi r/4$ at each end, where r is the radius of the neck in cm. If now the neck is connected to the volume to make the Helmholtz resonator, the air in the neck will move as a whole due to the sound pressure which might be applied at its outer end and the volume velocity E will be the same into the volume as into the neck, while the applied pressure will have to overcome the pressure built up in the volume and the pressure required to overcome the inertia of the air in the neck. The mobility of the combination, looking into the area at the outer end of the neck, is therefore

$$Z = \frac{1}{1/Z_L + 1/Z_C} = 1\Big/\left(\frac{1}{jkLR_s} + \frac{1}{-jR_s/kC}\right) = \frac{jkLR_s}{1-k^2LC}.$$

This indicates a very high mobility or resonance when $k = 1/(LC)^{\frac{1}{2}}$ as shown in the graph of Fig. 14, at which frequency the air in the neck will move with maximum velocity and the sound pressure in the volume will be a considerable factor times the sound pressure impressed at the outer end of the neck.

Simple formulas have also been derived for the mobility looking into the open end of a tube of constant cross-sectional area A cm^2 and length l cm (no restriction on l as regards its ratio to the wave-length). If the tube is closed at the distant end so as to give perfect reflection

$$Z = jAR_s \tan kl.$$

If the tube is open at the distant end but the diameter of the tube is small compared to the wave-length, there will be approximately perfect reflection at the open end though with change of phase in the sound pressure, and the mobility at the sending end will be approximately

$$Z \cong -jAR_s \cot kl.$$

With this much theory we can understand the principle of the intake silencer which is shown in its simplest form in Fig. 14. The air comes through the horizontal tube and flows in the direction of the arrow into the carburetor, the pulsations in this flow constituting the sound which we wish to keep from radiating from the open end of the tube. A Helmholtz resonator is arranged as a side branch and so proportioned that its resonance frequency lies in the midst of the frequencies to be attenuated. The mobility of the resonator is therefore much higher than the mobility of the tube at the right, and the pulsations flow in and out of the resonator rather than out of the tube. The resonator is effective over a comparatively wide range of frequencies since its mobility is considerably less than the mobility of the tube over a fairly wide range. If the wave-length becomes twice the length of the tube at the right, the mobility of the tube becomes high and there is no silencing action. The principle of the silencer can also be described by saying that the presence of the high mobility resonator cuts down the sound pressure at that point in the system, so that a smaller sound pressure actuates the tube at the right. The behavior of this acoustical system cannot be computed in detail unless the internal mobility of the sending end of the system to the left of the resonator, is known either by computation or measurement. Double resonators are often used on silencers, utilizing a second Helmholtz working out of the volume of the first resonator. The acoustical mobility of this double resonator can be computed from these same principles if the dimensions of the resonators are small compared to the wave-length.

Methods of experimentally measuring acoustical mobility have been developed and can be used to supplement or replace computation in design problems. While much more can be written on this subject, the above outline should serve to indicate the usefulness of the concept of acoustical mobility, the acoustical counterpart of the mechanical mobility. The mechanical mobility z of an area A cm^2 whose acoustical mobility is Z, is

$$z = Z/A^2 \text{ kines/dyne.}$$

Acoustical mobility has the useful properties that it is unchanged at a change of cross section of a tube, and the acoustical mobility of the combination of a number of systems which enter a common junction point small compared to the wave-length, is the sum of their separate acoustical mobilities;

$$Z = Z_1 + Z_2 + Z_3 + .$$

In Conclusion

The mobility method has its limitations. It is not applicable to the detailed computation of the transient vibrations excited in systems by arbitrary impulses; it is limited to the computation of forced vibrations or free vibration in normal modes. It is not so easily applied to systems like diaphragms, door panels, or stretched strings, in which the mass and elasticity are distributed rather uniformly throughout the system; it is most convenient when the masses, springs, and resistors are concentrated. It is not strictly applicable to systems containing springs whose force is not proportional to the displacement, or resistors whose force is not proportional to the velocity; were it not for the shock abosrber being a resistor with force not proportional to the velocity, the mobility method would be very convenient for computing the vibrations of the car body when driving over a road of sinusoidal contour. Such an analysis of riding qualities would require one schematic diagram referring to the vertical motion of the center of mass of the body, and another diagram referring to the pitching motion of the body. One must be careful when applying the mobility method to systems in which the vibrations are not all in the same line.

However, the mobility method is particularly advantageous in computing the forced vibration of linear mechanical systems, a kind of problem very important in practice. Those who know their electrical circuit theory will have noticed that due to our choice of symbols, the equations and methods have become of the same form as those used in electricity, although no reference is made to electricity in the solving of the problems. This similarity is helpful in remembering the equations and methods, although a lack of knowledge of electricity need not be considered a handicap in applying the mobility method. Noting the analogy which exists between mechanics as set forth in the mobility method, and electrical circuit theory as set forth in a book like Shea's *Transmission Networks and Wave Filters*, we may translate the conclusions of that book into mechanical terms with the assurance that they apply to every mechanical system whose schematic diagram is of the same form as the circuit diagram given in the book.