Noise Transformation and Cyclotron Waves in Crossed Fields

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A transmission-line analog is developed for crossed-field space-charge flows from the simultaneous solution of Maxwell's equations, the Lorentz force equation, and the continuity equation. A thin one-dimensional injected beam is assumed. The resulting fifth-degree secular equation is solved for several values of $\omega t/\omega_p$. The five system waves consist of a pair of hybrid waves, a pair of near-cyclotron waves and a near-synchronous wave. The results indicate that noise transformers may be based on an equivalent piecewise uniform transmission line.

INTRODUCTION

An important technique developed to aid in understanding the propagation of noise phenomena in electron beams and semiconductors involves the use of a transmission-line-like analog. The analog formulation is developed rigorously from the Maxwell curl equations and the transmission-line elements are related directly to the characteristic impedance and phase velocity of the electron stream or semiconductor medium. To date, the theory of these analogs has been restricted to one-dimensional rectilinear flow systems. In this paper, the basic theory is generalized to include crossed electric and magnetic field environments directly from Maxwell's equations and the transport equations.

A comprehensive treatment of both the theoretical and experimental aspects of crossed-field interaction and the characteristics of various crossed-field electron guns are discussed in Ref. 3 although little attention is given to the reduction of the characteristic high noise in these systems.

Recent theoretical and experimental studies on the noise characteristics of crossed-field space-charge flows have indicated that current fluctuations passing the potential minimum of an electron gun are considerably reduced (space-charge smoothing) and that the high noise level characteristic of crossed-field streams is associated with large velocity fluctuations. It is shown that noise transformers based on the above transmission-line analog may be used to reduce the noise contribution due to velocity fluctuations without enhancement of the current fluctuations. Diocotron gain in the transformer region is controlled (minimized) by proper choice of operating parameters and the equilibrium stream position for a configuration as shown in Fig. 1.

ASSUMPTIONS

The following assumptions are made in this analysis: (1) Nonrelativistic mechanics, (2) the electrons drift with a constant velocity along the z axis, (3) small-amplitude conditions, (4) the beam is infinitely thin (compared to the anode-sole spacing) in the y direction and infinitely extended along the x axis, (5) the force on an electron due to the rf magnetic field is neglected, (6) the ac fluctuations vary as $\exp(j\omega t)$, (7) the discontinuity in ac fields at the top and bottom of the thin beam is not considered which amounts to neglecting space-charge effects.

MATHEMATICAL ANALYSIS

In the framework of the above assumptions and for the model outlined in Fig. 1 the component ac Lorentz equations are developed as follows after separating out and dropping the dc terms:

$$-\eta E_{1z} + \omega z \eta B = [j \omega + u_0 (\partial / \partial z)] \hat{e}_1$$

and

$$-\eta E_{1y} - \omega z \eta B = [j \omega + u_0 (\partial / \partial z)] \hat{e}_1,$$

where the suffix one indicates first-order perturbation quantities, $\omega \eta$ is the absolute value of the electron charge-to-mass ratio and there is no dc y-component velocity.

The Maxwell magnetic field curl equation yields the

![Fig. 1. Coordinate configuration for a thin laminar beam in crossed fields.](https://example.com/fig1.png)
following relationships between the current density components and the electric field components:

\[ J_{1x} + j \omega_0 E_{1x} = - \frac{1}{j \omega_0 \mu_0} \left( \frac{\partial^2 E_{1y}}{\partial y^2} - \frac{\partial^2 E_{1z}}{\partial y \partial z} \right) \]  
(3)

and

\[ J_{1y} + j \omega_0 E_{1y} = \frac{1}{j \omega_0 \mu_0} \left( \frac{\partial^2 E_{1z}}{\partial z^2} - \frac{\partial^2 E_{1x}}{\partial y \partial z} \right). \]  
(4)

After some manipulation and the introduction of convenient differential operators, the following equations for \( E_{1y} \) and \( E_{1z} \) evolve:

\[ \left( 1 - \frac{1}{k^4} P_y^{-1} P_z^{-1} \frac{\partial^2}{\partial y \partial z} \right) E_{1z} \]
\[ = - \frac{1}{j \omega_0} \frac{1}{k^3} P_y^{-1} \frac{\partial^2}{\partial y \partial z} P_y^{-1} J_{1x} - \frac{1}{j \omega_0} \frac{1}{k^3} P_y^{-1} J_{1z} \]  
(5)

and

\[ \left( 1 - \frac{1}{k^4} P_z^{-1} P_y^{-1} \frac{\partial^2}{\partial y \partial z} \right) E_{1y} \]
\[ = - \frac{1}{j \omega_0} \frac{1}{k^3} P_z^{-1} \frac{\partial^2}{\partial y \partial z} P_z^{-1} J_{1x} - \frac{1}{j \omega_0} \frac{1}{k^3} P_z^{-1} J_{1y} \]  
(6)

where

\[ k = \omega / c, \]

\[ P_y = \frac{1}{k^3} \frac{\partial^2}{\partial y^2}, \]

\[ P_z = \frac{1}{k^3} \frac{\partial^2}{\partial z^2}, \]

and

\[ P^{-1} \Delta \] the inverse differential operation.

In the event that \( y \)-axis variations are nonexistent, Eqs. (5) and (6) considerably simplify.

\[ E_{1z} = - (1/j \omega_0) J_{1z} \]  
(7)

and

\[ E_{1y} = - (1/j \omega_0) P_z^{-1} J_{1y}. \]  
(8)

The directional velocity components and current densities are related as a result of combining Eqs. (1)-(2) and (7)-(8).

\[ u_0[\beta_s + (d/dz)] \dot{x}_1 = (-\eta/j \omega_0) J_{1z} + \omega_0 \dot{\phi}_1 \]  
(9)

and

\[ u_0[\beta_s + (d/dz)] \dot{y}_1 = (-\eta/j \omega_0) P_z^{-1} J_{1y} - \omega_0 \dot{\phi}_1, \]  
(10)

where \( \beta_s = \omega / u_0 \), the phase constant for the drifting stream. Furthermore, the component current density expressions are given by

\[ J_{1y} = \rho_0 \dot{\phi}_1, \]  
(11)

\[ J_{1z} = \rho_0 \dot{\phi}_1 + \rho_1 u_0, \]  
(12)

where \( \rho_0 \) and \( \rho_1 \) are the static and perturbation volumetric charge densities, respectively. Neglecting \( y \)-axis variations the continuity equation is then

\[ dJ_{1z} / dz = - j \omega_0 \rho_1. \]  
(13)

The theory of coupling of modes\(^7\)\(^8\) suggests a transformation facilitated by the definitions

\[ J_{1z} \Delta \frac{\partial}{\partial \theta}, \]

\[ V \Delta \frac{u \varphi}{\eta}, \]

\( = U e^{-\theta} \) (the \( z \)-component kinetic potential) \( \) \( \) (15)

and

\[ \theta = \omega \int_0^z \frac{dz}{u_0}. \]

A direct result of the transformation gives one transmission-line-like equation

\[ dQ/dz = j \beta_s (J_0/2V_0) U, \]  
(16)

where \( u_0 = 2 \eta V_0 \). An additional first-order differential equation may be developed from Maxwell's equations to complete the system. The specific form relates directly to the assumed field characteristics of the system. Two specific cases are examined below.

**Case I.** \( E_{1y} \rightarrow 0 \)

In the event that the \( y \)-directed perturbation field approaches zero then the following is obtained when Eqs. (7)-(8) and (9)-(10) are combined.

\[ (D + \beta_s^2) V = (1/j \omega) D J_{1z}, \]  
(17)

where \( D \Delta [j \beta_s + (d/dz)] \) and \( \beta_m \Delta \omega / u_0 \), the cyclotron wave phase constant. The coupled-mode amplitudes are given by \{using Eqs. (9), (10), and (14)-(16)\}

\[ U = U_0 \exp [j (\beta_m^2 + \beta_s^2) \xi] + U_- \exp [-j (\beta_m^2 + \beta_s^2) \xi], \]  
(18)

and

\[ Q = \frac{\beta_s J_0}{2V_0 (\beta_s^2 + \beta_m^2)} \] \[ \left[ U_+ \exp [j (\beta_m^2 + \beta_s^2) \xi] - U_- \exp [-j (\beta_m^2 + \beta_s^2) \xi] \right], \]  
(19)

where the kinetic potential has been decomposed to forward (+) and backward (−) traveling components and \( \beta_s \Delta \omega_0 / \eta_0 \), \( \omega_0 = \eta_0 / \eta_0 \). The analog transmission-line system is then suggested by the form of the coupled Eqs. (18) and (19). The propagation constant and line parameters are apparent in Eq. (19). \( Z_0 \) the impedance per unit length and \( V_0 \) the admittance per unit length are given by the following:

\[ Z_0 \Delta - j 2V_0 (\beta_m^2 + \beta_s^2) / \beta_s J_0 \] \[ Y_0 \Delta - j \beta_s (J_0 / 2V_0). \]  
(20)


\(^8\) M. C. Pease, J. Appl. Phys. 31, 2028 (1960).
This degenerate case \( (E_{1y}=0) \) indicates that the presence of an orthogonal static magnetic field serves to modify the electron natural oscillation frequency giving an effective plasma frequency of
\[
\omega_{pe} = (\omega_{pe}^2 + \omega_c^2)^{1/2}.
\]  
\( (21) \)

**Case II.** \( E_{1y} \neq 0 \)

Under more general conditions the \( \gamma \)-directed perturbation field does not vanish and we are led to a generalization of Eq. (17) from the Maxwell equations,
\[
(AD + \beta_m P_e) V = (1/j \omega \epsilon_0) AL_{1z},
\]  
\( (22) \)
where the operator
\[
A = P_e D + (\omega_p/j \omega) \beta_p
\]

Following a format similar to that in Case I and assuming the harmonic nature of \( Q \) and \( U \),
\[
\begin{bmatrix}
Q \\
U
\end{bmatrix} = \begin{bmatrix}
Q_0 \\
U_0
\end{bmatrix} e^{\gamma t}
\]
yields a fifth-degree secular equation
\[
\gamma^2 \frac{2}{\omega_p'} \left[ 1 + \left( \frac{\omega_p'}{\omega_p} \right)^2 - \frac{1 + \omega_p'}{\omega_p} \right] \gamma^2
\]
\[+ \frac{2}{\omega_p} \left[ 1 + \left( \frac{\omega_c'}{\omega_p} \right)^2 + 2 \omega_p' \right] \gamma^2
\]
\[- \frac{1}{\omega_p} \left[ 1 + \left( \frac{\omega_c'}{\omega_p} \right)^2 \right] \gamma^2
\]
\[= 0,
\]  
\( (23) \)
where \( \omega_p' = \omega_p/\omega_c, \omega_c' = \omega_c/\omega_p \). The \( k^2/\beta_p^2 \) terms vanish in the nonrelativistic limit \( [(u_0/v) \rightarrow 0] \) yielding a reduction by one degree. In the very low velocity limit it is preferable to redefine the secular equation directly from Maxwell's equations and the equations of motion. The result is a fourth degree equation. The secular equation eigenvalues are examined in some detail below for the often studied open-circuit diode. In the zero-field limit of \( \beta_m \rightarrow 0 \) the secular equation is of second degree and the Bloom and Peter results prevail.

It is noted that the result of neglecting space-charge effects leads to a degeneration of the two cyclotron waves to one synchronous unchanging wave. A consideration of space-charge effects as treated by Gould\(^*\) would yield a six-wave system. The neglect of space-charge effects is justified since their consideration adds only a pair of growing and declining waves in place of the synchronous wave on a drifting beam and the cyclotron-wave propagation constants are not appreciably changed.


**Case III. Open-Circuit Diode**

The open-circuit diode often examined in perturbation noise theory is characterized by zero total alternating current and thus
\[
J_{1x} + j \omega \epsilon_0 E_{1z} = 0
\]
\( (24) \)

Equation (24) is a special case of Eq. (8) occurring when \( P_e = 1 \) and is valid at high frequencies where displacement currents are important. Again following the earlier method a third-degree secular equation evolves.
\[
\gamma^2 - \omega_p' \gamma - \left[ 1 + (\omega_p'/\omega_c')^2 \right] \gamma + \omega_p' = 0.
\]  
\( (25) \)
The above corresponds to large \( k \) for \( \omega \) large and may be obtained directly from Eq. (23) by neglecting appropriate terms. If \( \omega_p'/\omega_c \rightarrow 0 \) then Eq. (25) reduces to Case I supporting the assumption that \( E_{1y} = 0 \).

The eigenvalues of Eq. (24) then facilitate the calculation of the perturbation charge density and velocity through
\[
d \rho_1(0)/dz = -(\rho_0 \eta / u_0 \epsilon_0) [(dU/\epsilon_0 U)] - j \beta_p U
\]  
\( (26) \)
and
\[
y_1 = -(u_0 + u_0 (dU/\epsilon_0 U) - (\eta_0 / j \omega \epsilon_0) \beta_p^{-1}) y\epsilon_0 \delta y,
\]  
\( (27) \)
where
\[
\rho_1 = \rho_0 e^{-i \delta y}.
\]

**GENERAL RESULTS FOR CASE II**

The normalized propagation constant written as
\[
\beta_p = 1 - \gamma (\omega_p/\omega)
\]
is calculated from the eigenvalues of Eq. (23). These normalized values are shown in Figs. 2 through 5 for space-charge flows in orthogonal static electric and magnetic fields under the low-velocity assumption \( (u_0/c) = 0.1 \) and varying \( \omega_c/\omega_p \). If \( \omega_c = \omega_p \), then we have

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**Fig. 2. Case II propagation constants vs \( \omega_c/\omega_p \). (\( \omega_c/\omega_p = 0.5 \)).**
laminar Brillouin flow. The degeneracy to a third-degree system for large $k^2$ is evident and to the first order $y_2$ and $y_3$ correspond to the cyclotron mode propagation constants.

It is interesting to note that the propagation constants $\beta_4$ and $\beta_5$ have imaginary parts, plotted as $x$, for small values of $\omega/\omega_p$ only. A comparison of the three cases indicates that there is a critical value of $\omega/\omega_p$, given by $(\omega/\omega_p)_{cr}$, such that if $\omega/\omega_p < (\omega/\omega_p)_{cr}$, $x_4 = -x_5: 0$ and for $\omega/\omega_p > (\omega/\omega_p)_{cr}$, $x_4 = -x_5 = 0$. The value of $(\omega/\omega_p)_{cr}$ is inversely proportional to $\omega_c/\omega_p$ to the first order. The corresponding values of $y_4$ and $y_5$ when $x_4 = -x_5: 0$ are negative, indicating backward waves in the stream. The positive and negative values of $x$ indicate the presence of a growing backward wave and a declining backward wave, respectively. The exact physical significance of these waves is not clear. The fact that the value of $x$ decreases as $\omega/\omega_p$ is increased is in agreement with Gould's results, which indicate that the space-charge effects in crossed fields tend to increase the impressed signal growth rate. It may also be noticed that the value of $x$ is decreased as $\omega_c$ is increased.

The waves denoted by $y_2$ and $y_3$ are referred to as cyclotron modes and are discussed further in the next section. The waves denoted by $y_4$ and $y_5$ may be called hybrid waves, since they represent a combination of space-charge wave, fast wave, and cyclotron wave propagation constants. These hybrid waves are not excited for large values of $\omega/\omega_p$.

It can be seen in comparing Figs. 3 and 5 that there is no appreciable change in the values of the propagation constants when the electron velocity is changed. However, the value of $x$ for small values of $\omega/\omega_p$ seems to decrease as the velocity is decreased.

**CYCLOTRON MODE EXCITATION**

When a stream of the type illustrated in Fig. 1 interacts with a slow electromagnetic wave the excitation of cyclotron waves will be slight provided that the initial velocity, position, and space-charge density perturbations are extremely small. This condition, however, is not generally the case and thus cyclotron wave excitation becomes important. Significant initial velocity fluctuations (noise) can lead to an appreciable excitation of cyclotron waves.\textsuperscript{10}

Under the restriction of $k > 1$ or presuming that the modes designated by $y_4$ and $y_5$ are weakly excited the solution of Eq. (23) is written as

$$Q = \sum_{i=1}^{3} Q_0 e^{i(\theta-e-y_i)} t,$$

(28)

which reduces for large values of $\omega/\omega_p$ (which implies

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the neglect of hybrid waves) to
\[ Q = Q_1 + Q_2 e^{-jk_1 z} + Q_3 e^{jk_1 z}, \]  
(29)

where
\[ y_1 = \beta, \]
\[ y_2 = y_1 - k_1, \]
\[ y_3 = y_1 + k_1, \]
and
\[ k_1 \approx (\omega_0/\omega) \beta. \]

Equations (16) and (29) lead to
\[ dQ/dz = -jk_1 (Q_2 e^{-jk_1 z} - Q_3 e^{jk_1 z}), \]  
(30)
\[ dU/dz = (j2V_0/\beta J_0) k_1^2 (Q - Q_1), \]  
(31)

where \((Q - Q_1)\) denotes the contribution from the first-order cyclotron modes of Eq. (29).

The following definitions of transformation variables
\[ W \approx (2V_0/J_0) k_1/\beta, \]
and
\[ \varphi \approx \int_0^z k_1 dz, \]
lead to the desired differential equations
\[ (d^2 U/d\varphi^2) + U = 0, \]  
(32)
\[ (\partial^2 Q/d\varphi^2) + \dot{Q} = 0, \]  
(33)

where \(\dot{Q} \approx (Q - Q_1)\). The solutions of Eqs. (32) and (33) are
\[ U(\varphi) = U_i \cos \varphi + \dot{Q}_i \sin \varphi, \]  
(34)
\[ \dot{Q}(\varphi) = j(U_i/W) \sin \varphi + \dot{Q}_i \sin \varphi. \]  
(35)

This result has the same form as Bloom and Peters except that the plasma phase constant is here replaced by the cyclotron-wave phase constant to the first order. It is thus apparent that perturbation signals may be amplified or attenuated by periodic changes in \(W\).

**METHOD FOR VARYING W**

**A. Variation of Voltage**

Reference to Fig. 1 indicates that for laminar trajectories the \(z\)-axis electron velocity is governed by
\[ u_0 = |E|/B = V_0/Bd, \]
assuming a thin stream and no space-charge depression.

![Fig. 6. Crossed-field voltage jump configuration.](image)

Thus
\[ V_0 = \eta(Bd)^2 \]  
(36)

so that \(V_0\) and \(d\) may not be varied independently, while maintaining laminarity. In the case of a Brillouin stream and assuming constant stream thickness, \(t \ll d\), the following dependence evolves
\[ V_0/\omega = (d_0/d_1)^3 \]  
(37)

for the geometry of Fig. 6.

**B. Variation of \(\omega_p\)**

The normalized characteristic impedance \(W\) also depends on \(\omega_p\) through \(J_0\). Thus an alternative method for varying \(W\) is to change the stream thickness, say by the factor \(T\). Then \(\omega_p\) changes by \(1/\sqrt{T}\) and \(J_0\) by \(1/T\). Such a change in the plasma frequency can lead to space amplification or deamplification effects.

A plot of the normalized value of \(k_1(J_0/J_0)\) vs \(T\) is shown in Fig. 7 where \(J_0\) corresponds to the \(T = 1\) condition. These results assume \(\omega_c = 2\omega_p\) and are strictly valid only for \(\omega > \omega_p\).

**CONCLUSIONS**

The propagation constants of a crossed-field space-charge flow are evaluated for different values of \(\omega_c/\omega_p\) and beam velocity. For small values of \(\omega/\omega_p\), the growth rate in a drift region is evaluated and the variation with \(\omega_p\) is in qualitative agreement with earlier results.

A comparison is made between the modes denoted by \(y_2\) and \(y_3\) and the cyclotron modes. The effect of a finite value of \(E_{1y}\) is to reduce the absolute values of the cyclotron mode differential propagation constants. It is noticed that the cyclotron modes may be excited strongly by velocity fluctuations. In the transmission-line analogy, the cyclotron phase constant (as a first-order approximation) plays a role similar to that of the
Effect of Cesium Vapor on the Emission Characteristics of Uranium Carbide at Elevated Pressures

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The saturated emission currents from uranium carbide in cesium vapor were measured for four different bath temperatures $7\,^\circ$ 167, 197, 227, and 250$^\circ$C (the cesium arrival rates $\mu_n$ in $10^{12}$ atoms/cm$^2$ were 0.3, 1, 3, and 6, respectively). The measurements were made with a “plasma anode” tube at constant temperature. The plot of $\log \mu_n$ vs $1000/T$ was used to extrapolate the electron emission “S” curves to arrival rates that differed by a factor of ten. Since calculations based on the assumption of a homogeneous surface yields unexpectedly high hindering energies and surface coverages, it seems likely that some surface patchiness exists.

INTRODUCTION

THE work function of uranium carbide in a vacuum has been measured by several investigators. Haas and Thomas, in a recent paper, report that uranium carbide has a surface consisting of patches that range in work function from 3.25 to 4.5 eV, which upon thermal activation becomes covered with patches in the narrower range of 3.0 to 3.25 eV.

Because little information has been published on the emission characteristics of uranium carbide in cesium vapor, and because several investigators have found that cesium vapor can affect the emission from a refractory metal, it was decided to determine the effect of cesium vapor on the emission characteristics of uranium carbide.

EXPERIMENTAL METHOD

A “plasma anode” tube similar to that used by Marchuk and Houston was used for the experimental work (see Fig. 1). This tube used a plasma, produced by the discharge between a hot cathode and a cold anode, to neutralize space charge and act as a collector for any emitter immersed in it. There were three emitters in the tube; two were made of a 0.25 mm (10 mil) tungsten wire electrophoretically coated with uranium carbide 0.13 mm (5 mils) thick, and the other was a bare 0.25-mm tungsten wire; each emitter was shielded by an alumina insulator so that only a small, uniform temperature area of the loop was exposed to the plasma. The bare tungsten emitter was used to check the method of measurement. When data were being taken, the tube and cesium reservoir were immersed in a constant temperature oil bath (see Fig. 2 for the relation be-

Fig. 1. Design of the experimental tube.

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3. F. Cranston and J. W. Barger, Los Alamos Scientific Laboratory, New Mexico, LADC 5128 (1960).


9. For cesium on W, Nb, and Rh.

10. The emission curves for cesium on W were compared with those already published by J. B. Taylor and I. Langmuir (see Ref. 5 above).