Structure of the Combinatorial Generalization of Hypergeometric Functions for $SU(n)$ States*

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The combinatorics of the boson operator formalism in the construction of the $SU(n)$ states provides a natural scheme for the appearance of certain generalized hypergeometric functions. It is shown that, while special cases exist where the functions thus generated belong to the class of generalized hypergeometric functions defined by Gel'fand *et al.* as being the Radon transforms of products of *linear* forms, the general cases apparently do not. This is already so at the $SU(4)$ level.

1. INTRODUCTION

In the representation theory of unitary groups, much has been known on the problem of explicit construction of state vectors for the irreducible representation, in particular, with the device of the boson operator formalism.[1] One of the crucial problems lies in the structural analysis of a general state vector such that hopefully the algebraic complexities which mount extremely rapidly as the rank of the group goes up may be systematically controlled.

In recent years, some efforts have been made in, among other things, the combinatorial structure of the state vectors for the irreducible representation of $U(n)$.[2-5] In particular, it was found[2-5] that one can readily obtain the normalization constants by carrying out essentially the following steps: (i) A general state vector is obtained by applying an appropriate string of lowering operators from a so-called semimaximal state,[6] the latter being expressed as appropriate products of (antisymmetrized) determinantal factors of creation operators (each factor being raised to an appropriate power) acting on the vacuum state; (ii) the process of pushing through the lowering operators all the way to the right results in the combinatorics as a consequence of the canonical commutation relations.

One of the problems then is to study the combinatorial structure entailed in the step (ii) above. More explicitly, Baird and Biedenharn[2] have shown that for $SU(3)$ the operator-valued polynomials can, in fact, be formally expressed in terms of the well-known Gauss hypergeometric function ${}_2F_1(a, b; c; x)$; namely,

$$\begin{aligned}&|\text{general } SU(3) \text{ state}\rangle \\ &\quad = \text{const (product of antisymmetrized creation} \\ &\qquad\qquad \text{operators)}\ {}_2F_1(a, b; c; x)\,|0\rangle, \qquad (1)\end{aligned}$$

where the coefficients a, b, and c depend linearly on the Gel'fand labels[7] of the state, while the variable x is formally an operator quotient in such a way that all the denominators are cancelled eventually by appropriate multiplicative factors outside the hypergeometric function.

Attempts have been made by Ciftan and Biedenharn[4] and in particular by Ciftan[5] to generalize Eq. (1) to higher rank unitary groups.[8] Unfortunately, because of the increasing algebraic complexities, no simple expressions analogous to Eq. (1) are known. For $SU(4)$, Ciftan was able to recognize the structure of the individual block constituents, where each block of terms essentially corresponds to the action of one particular lowering operator (raised to a power), but the expression for a general $SU(4)$ state was left as a sixfold sum of operators.

The main purpose of this paper is to answer some questions raised by Ciftan's treatment of the $SU(4)$ state. The way that combinatorics of boson calculus provides a natural scheme for the appearance of certain generalized hypergeometric functions is rather intriguing. In view of considerable interest in the connection between special functions and group theory,[9-12] it is perhaps desirable to ascertain what class of functions does such generalization via combinatorics lead to.

In fact, at the $SU(n)$ level, the Gauss function ${}_2F_1$ appears as a consequence of the action of the lowering operator $(L_{n-1}^{n-2})^\alpha$. The Appell function[13,14] of the second kind, F_2, is obtained from the action of $(L_{n-1}^{n-3})^\alpha$. The Lauricella function[15] of the fourth kind, F_D, in several variables occurs as a result of the operation of $(L_{n-2}^{n-3})^\alpha$. These statements are obvious generalizations of the case $n = 4$. [See Eqs. (9)-(11) below.]

The fact that these Appell and Lauricella functions appear so rarely (if at all) elsewhere in theoretical physics perhaps warrants a systematic analysis of such functions, and generalizations thereof via the boson calculus of the $SU(n)$ state.

It is obvious that the combined action of products of lowering operators will lead to a multiple sum of products of folded blocks of terms. For example, at the $SU(4)$ level, one already has to deal with a folded product of the Gauss, Appell, and Lauricella functions.

In analyzing the structure of these new functions beyond the known repertory, one criterion used is to check whether their integral representations admit a definite pattern of generalization. One class of generalized hypergeometric functions has been defined by Gel'fand *et al.*[16] as the Radon transforms[16] of products of *linear* forms. It so happens that all the known hypergeometric and generalized hypergeometric functions such as the Gauss, Appell, and Lauricella functions have this property.[17] The question arises whether this feature holds for all functions generated for the general $SU(n)$ states. This question was hitherto unsettled even for $n = 4$. We show that, in general, these combinatorially generated functions for the $SU(n)$ states are not confined to the class of generalized hypergeometric functions which are Radon transforms of products of linear forms. Already at the $SU(4)$ level, while special cases may possess such property, the general $SU(4)$ states do not. This in essence answers the question[5] posed by Ciftan in a negative way.

2. COMBINATORIAL STRUCTURE FOR GENERAL $SU(4)$ STATES

There are obvious advantages in reaching a general $SU(n)$ state by approximate application of lowering operators from the semimaximal state. Besides the apparent ease of getting the normalization constants,[4] a subject which will not concern us here, the structure of the semimaximal state is sufficiently simple so that the combinatorics ensued in pushing through the set of lowering operators can be systematically controlled.

We shall assume that the reader is reasonably familiar with the boson operator formalism[2] for the representations for the $U(n)$ groups. Thus we shall merely sketch the necessary expressions for the sake of getting the notations straight.

In terms of the creation and annihilation operators $a_i^{(\lambda)}$ and $\bar{a}_j^{(\lambda)}$, the elements of the Lie algebra read

$$E_{ij} = \sum a_i^{(\lambda)}\bar{a}_j^{(\lambda)}, \quad i, j = 1, \cdots, n. \tag{2}$$

The commutation relations

$$[E_{ij}, E_{rs}] = \delta_{jr}E_{is} - \delta_{is}E_{rj} \tag{3}$$

follow from the canonical commutation relations for the a's,

$$[\bar{a}_i^{(\lambda)}, a_j^{(\lambda')}] = \delta_{ij}\delta^{\lambda\lambda'}. \tag{4}$$

In terms of the (antisymmetrized) determinantal factors

$$a_{i_1 i_2 \cdots i_s} = \sum \epsilon(i_1 i_2 \cdots i_s) a_{i_1} a_{i_2} \cdots a_{i_s}, \tag{5}$$

where $\epsilon(i_1 i_2 \cdots i_s)$ is $+1$ if the set of indices is an even permutation of $(1, 2, \cdots, s)$ and is -1 otherwise, we have thus

$$\begin{aligned}
&|\text{general } SU(4) \text{ state}\rangle \\
&\equiv \left|\begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & 0 \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{pmatrix}\right\rangle \\
&= \text{const}\,(L_2^1)^{n_{12}}(L_3^2)^{n_{23}}(L_3^1)^{n_{13}} \\
&\times \left|\begin{pmatrix} m_{14} & & m_{24} & & m_{34} & & 0 \\ & m_{13} & & m_{23} & & m_{33} & \\ & & m_{13} & & m_{23} & & \\ & & & m_{13} & & & \end{pmatrix}\right\rangle \\
&= \text{const}\,(L_2^1)^{n_{12}}(L_3^2)^{n_{23}}(L_3^1)^{n_{13}}(a_{123})^{\nu_{33}}(a_{124})^{n_{34}} \\
&\times (a_{12})^{\nu_{23}}(a_{14})^{n_{24}}(a_1)^{\nu_{13}}(a_4)^{n_{14}}\,|0\rangle,
\end{aligned} \tag{6}$$

where the set of lowering operators are related to the E_{ij} as follows[3]:

$$L_k^{k-1} \equiv E_{k,k-1}, \quad k = 2, \cdots, n-1, \tag{7a}$$

$$L_k^{k-2} \equiv \mathcal{E}_{k-2,k-1}E_{k,k-2} + E_{k,k-1}E_{k-1,k-2}, \text{ etc.}, \tag{7b}$$

$$\mathcal{E}_{ij} \equiv (E_{ii} - E_{jj}) + (j - i)1. \tag{7c}$$

In the exponent, the following short-hand notation is used:

$$n_{ij} \equiv m_{ij} - m_{ij-1}, \quad \nu_{ij} \equiv m_{ij} - m_{i+1,j+1}. \tag{7d}$$

We note first the action of each operator separately. On the $SU(4)$ level, $(L_3^1)^{n_{13}}$ leads to the Appell function of the second kind F_2; $(L_3^2)^{n_{23}}$ gives the Gauss function ${}_2F_1$, and finally $(L_2^1)^{n_{12}}$ yields the Lauricella function of the fourth kind $F_D^{(3)}$ in 3-variables.[2,5] Since these are crucial ingredients, they have been rederived and recorded here below for the sake of readability. The defining power series expansions and their integral representations are given in the Appendix. We have

$$\begin{aligned}
&(L_3^1)^{n_{13}}(a_{124})^{n_{34}}(a_{12})^{\nu_{23}}(a_{14})^{n_{24}}(a_1)^{\nu_{13}}\,|0\rangle \\
&= \frac{\Gamma(\nu_{13}+1)\Gamma(s+2)}{\Gamma(\nu_{13}-n_{13}+1)\Gamma(s+2-n_{13})} \\
&\quad \times (a_{124})^{n_{34}}(a_{12})^{\nu_{23}}(a_{14})^{n_{24}}(a_1)^{\nu_{13}-n_{13}}(a_3)^{n_{13}} \\
&\quad \times F_2(-n_{13}; -n_{34}, -n_{24}; \\
&\qquad -s-1, \nu_{13}-n_{13}+1; w_1, w_2)\,|0\rangle,
\end{aligned} \tag{8}$$

where $s = \nu_{13} + \nu_{23} + n_{24} + n_{34}$,

$$(L_3^2)^{n_{23}}(a_{124})^{n_{34}-k_1}(a_{12})^{\nu_{23}}|0\rangle = \frac{\Gamma(\nu_{23}+1)}{\Gamma(\nu_{23}-n_{23}+1)}(a_{124})^{n_{34}-k_1}(a_{12})^{\nu_{23}-n_{23}}(a_{13})^{n_{23}} \times {}_2F_1(-n_{23}, -n_{34}+k_1; \nu_{23}-n_{23}+1; w_3)\,|0\rangle, \tag{9}$$

$$\begin{aligned}(L_2^1)^{n_{12}}(a_{134})^{k_3}(a_{13})^{n_{23}-k_3}(a_{14})^{n_{24}-k_2}(a_1)^{\nu_{13}-n_{13}+k_2}|0\rangle &= \frac{\Gamma(n_{23}+1)}{\Gamma(n_{23}-n_{12}+1)}\frac{(n_{12}-n_{23})_{k_3}}{(-n_{23})_{k_3}}(a_{134})^{k_3}(a_{13})^{n_{23}-n_{12}-k_3}\\ &\times (a_{14})^{n_{24}-k_2}(a_1)^{\nu_{13}-n_{13}+k_2}(a_{23})^{n_{12}}\\ &\times F_D^{(3)}(-n_{12}; -n_{24}+k_2, -\nu_{13}+n_{13}-k_2, -k_3;\\ &\qquad 1+n_{23}-n_{12}-k_3; w_4, w_5, w_6)\,|0\rangle,\end{aligned} \tag{10}$$

where $(c)_r \equiv \Gamma(c+r)/\Gamma(c)$. The w's are defined in Eq. (13) below. Expressing the ${}_2F_1$, F_2, and $F_D^{(3)}$ in the standard single, double, and triple power-series expansions, respectively (see Appendix), we get from Eq. (6)

$$\begin{aligned}|\text{general } SU(4)\text{ state}\rangle &= \text{const} \times (a_{123})^{\nu_{33}}(a_{124})^{n_{34}}(a_{12})^{\nu_{23}-n_{23}}(a_{13})^{n_{23}-n_{12}}\\ &\times (a_{14})^{n_{24}}(a_{23})^{n_{12}}(a_1)^{\nu_{13}-n_{13}}(a_3)^{n_{13}}(a_4)^{n_{14}}S^{(4)}\,|0\rangle,\end{aligned} \tag{11}$$

$$\begin{aligned}S^{(4)} &= \sum_{k_1,\cdots,k_6}\frac{(-n_{13})_{k_1+k_2}(-n_{34})_{k_1}(-n_{24})_{k_2}}{(-s-1)_{k_1}(1+\nu_{13}-n_{13})_{k_2}}\frac{(n_{12}-n_{23})_{k_3}(-n_{34}+k_1)_{k_3}}{(1+\nu_{23}-n_{23})_{k_3}}\\ &\times \frac{(-n_{12})_{k_4+k_5+k_6}(-n_{24}+k_2)_{k_4}(-\nu_{13}+n_{13}-k_2)_{k_5}(-k_3)_{k_6}}{(1+n_{23}-n_{12}-k_3)_{k_4+k_5+k_6}}\prod_{j=1}^{6}\frac{(w_j)^{k_j}}{k_j!},\end{aligned} \tag{12}$$

where

$$\begin{aligned} w_1 &= \frac{a_{123}a_4}{a_{124}a_3}, & w_2 &= \frac{a_1 a_{34}}{a_3 a_{14}},\\ w_3 &= \frac{a_{12}a_{134}}{a_{13}a_{124}}, & w_4 &= \frac{a_{13}a_{24}}{a_{14}a_{23}},\\ w_5 &= \frac{a_2 a_{13}}{a_1 a_{23}}, & w_6 &= \frac{a_{13}a_{234}}{a_{23}a_{134}}.\end{aligned} \tag{13}$$

Equation (11) is essentially equivalent to Eq. (4.6c) of Ref. 5, apart from some obvious misprints there.

Using the standard integral representations[13,14] for those block constituents in (12), one can easily convert the sixfold summation into a fourfold integral representation[18]:

$$\begin{aligned}S^{(4)} &= \text{const} \times \iiiint_0^1 dt_1\,dt_2\,dt_3\,dt_4\\ &\times t_1^{-n_{34}-1}t_2^{-n_{24}-1}t_3^{-n_{23}-1}t_4^{-n_{12}-1}\\ &\times (1-t_1)^{n_{34}-s-2}(1-t_2)^{n_{24}+\nu_{13}-n_{13}}(1-t_3)^{\nu_{23}}\\ &\times (1-t_4)^{n_{23}}(1-w_4t_4)^{n_{24}}(1-w_5t_4)^{\nu_{13}-n_{13}}\\ &\times [1-w_3t_3(1-w_6t_4)/(1-t_4)]^{n_{34}}\\ &\times \{1-w_1t_1[1-w_3t_3(1-w_6t_4)/(1-t_4)]^{-1}\\ &\quad - w_2t_2(1-w_5t_4)/(1-w_4t_4)\}^{n_{13}}.\end{aligned} \tag{14}$$

By inspection, one easily convinces oneself that, because of the delicate folding of variables involved, it does not seem likely that Eq. (14) in general can be brought to be the Radon transform of products of *linear* forms. In the next section, we shall see that only special cases of Eq. (14) will have such a simple property.

3. RADON STRUCTURE OF THE SEMISEMIMAXIMAL $SU(4)$ STATE

This is another case which was started by Ciftan but was left unsettled. For the semisemimaximal state, i.e., the state with $n_{12} = 0$, or $m_{12} = m_{11}$, the net effect is that the threefold summation involving the Lauricella function is now absent. The analog of Eq. (14) here is thus considerably simpler:

$$\begin{aligned}&S^{(4)}(n_{12}=0, w_4=w_5=w_6=1)\\ &\quad = \text{const} \times \iiint_0^1 dt_1\,dt_2\,dt_3 t_1^{-n_{34}-1}t_2^{-n_{24}-1}t_3^{-n_{23}-1}\\ &\quad\times (1-t_1)^{n_{34}-s-2}(1-t_2)^{n_{24}+\nu_{13}-n_{13}}\\ &\quad\times (1-t_3)^{\nu_{23}}(1-w_3t_3)^{n_{34}}\\ &\quad\times [1-w_1t_1/(1-w_3t_3)-w_2t_2]^{n_{13}}.\end{aligned} \tag{15}$$

Equation (15) can be easily cast into the Radon form with the aid of the following change of variables:

$$\begin{aligned} x_1 &= w_1t_1, \quad x_2 = 1 - w_2t_2,\\ x_3 &= (1-w_2t_2)(1-w_3t_3).\end{aligned} \tag{16}$$

We have

$$S^{(4)}(n_{12}=0, w_4=w_5=w_6=1) = \text{const} \times w_1^{s+2} w_2^{n_{13}-\nu_{13}} w_3^{n_{23}-\nu_{23}} \times \iiiint dx_1\, dx_2\, dx_3\, dx_4 \delta\Big(1-\sum_{i=1}^{4} x_i\Big) \times \prod_{k=1}^{9} (\xi^{(k)}, x)^{b_k}, \tag{17}$$

where

$$(\xi, x) \equiv \sum_{i=1}^{4} \xi_i x_i$$

denotes a linear form and the coefficients are

$$\begin{aligned} &\xi^{(1)} = (1,0,0,0), && \xi^{(2)} = (0,1,0,0), \\ &\xi^{(3)} = (0,0,1,0), && \xi^{(4)} = (-1,0,1,0), \\ &\xi^{(5)} = (0,1,-1,0), && \xi^{(6)} = (0, w_3-1, 1, 0), \\ &\xi^{(7)} = (1,0,1,1), && \xi^{(8)} = (w_1-1, w_1, w_1, w_1), \\ &\xi^{(9)} = (w_2-1, w_2, w_2-1, w_2-1), \end{aligned} \tag{18a}$$

$$\begin{aligned} &b_1 = -n_{34}-1, && b_2 = n_{13}+n_{23}-n_{34}-\nu_{23}, \\ &b_3 = n_{34}-n_{13}, && b_4 = n_{13}, \\ &b_5 = -n_{23}-1, && b_6 = \nu_{23}, \\ &b_7 = -n_{24}-1, && b_8 = n_{34}-s-2, \\ &b_9 = n_{24}+\nu_{13}-n_{13}. \end{aligned} \tag{18b}$$

The right-hand side of Eq. (17) is readily recognized as the Radon transform of products of linear forms in a 4-dimension space. However, as mentioned earlier, unfortunately the general case Eq. (14) does not share this property.

We thus conclude that the class of functions generated by the boson combinatorics, in general, is not confined to the class of generalized hypergeometric functions defined as Radon transforms of products of linear forms.

APPENDIX

For the sake of readability, we give here the relevant definitions and integral representations for the Gauss, Appell and Lauricella functions involved[13,14]:

(a) The Gauss function ${}_2F_1$:

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{A1}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}; \tag{A2}$$

(b) The Appell function of the second kind, F_2:

$$F_2(a; b_1, b_2; c_1, c_2; z_1, z_2) = \sum_{k_1,k_2} \frac{(a)_{k_1+k_2}(b_1)_{k_1}(b_2)_{k_2}}{(c_1)_{k_1}(c_2)_{k_2}} \frac{z_1^{k_1}}{k_1!} \frac{z_2^{k_2}}{k_2!} \tag{A3}$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(c_1-b_1)\Gamma(b_2)\Gamma(c_2-b_2)} \times \iint_0^1 dt_1\, dt_2 t_1^{b_1-1} t_2^{b_2-1}(1-t_1)^{c_1-b_1-1} \times (1-t_2)^{c_2-b_2-1}(1-z_1t_1-z_2t_2)^{-a}; \tag{A4}$$

(c) The Lauricella function of the fourth kind, $F_D^{(n)}$, in n-variables:

$$F_D(a; b_1, \cdots, b_n; c; z_1, \cdots, z_n) = \sum_{k_i} \frac{(a)_{\Sigma k_i} \prod_{j=1}^{n} (b_j)_{k_j}}{(c)_{\Sigma k_i}} \prod_{j=1}^{n} \frac{z_j^{k_j}}{k_j!} \tag{A5}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt\, t^{a-1}(1-t)^{c-a-1} \prod_{i=1}^{n} (1-z_it)^{-b_i}. \tag{A6}$$

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[1] Among earlier references on the boson operator formalism, we mention the following: J. Schwinger, "On Angular Momentum," U.S. Atomic Energy Commission Report NYO-3071, 1952, reprinted in *Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. Van Dam (Academic, New York, 1965), p. 229; V. Bargmann and M. Moshinsky, Nucl. Phys. **18**, 697 (1960); **23**, 177 (1961); M. Moshinsky, *ibid.* **31**, 384 (1962); Rev. Mod. Phys. **34**, 813 (1962); J. Math. Phys. **4**, 1128 (1963); also Refs. 2–4 below.

[2] G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

[3] J. G. Nagel and M. Moshinsky, J. Math. Phys. **6**, 682 (1965).

[4] M. Ciftan and L. C. Biedenharn, J. Math. Phys. **10**, 221 (1969).

[5] M. Ciftan, J. Math. Phys. **10**, 1635 (1969).

[6] I.e., a $U(n)$ state such that it is a maximal state on the $U(n-1)$ level; a maximal state is one where the entries in the Gel'fand pattern take on maximal values $m_{i\,j-1} = m_{ij}$ for $j = 2, \cdots, n$ and $i = 1, \cdots, n-1$.

[7] I. M. Gel'fand and M. L. Zeltin, Doklady Akad. Nauk SSSR **71**, 825 (1950).

[8] For similar considerations in the extension to the groups $Sp(4)$ and $O(5)$, see, e.g., W. J. Holman III, J. Math. Phys. **10**, 1710 (1969).

[9] E. P. Wigner, "Applications of Group Theory to the Special Functions of Mathematical Physics," unpublished lecture notes, Princeton University, 1955.

[10] N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (Transl. Math. Mono. Vol. 22) (American Mathematical Society, Providence, R.I., 1968).

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[13] P. Appell and J. Kampé de Fériet, *Fonctions hypergeometriques et hyperspheriques* (Gauthier-Villars, Paris, 1926), esp. Chap. 7; G. Lauricella, Rend. Circ. Mat. Palermo **7**, 111 (1893).

[14] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge U.P., Cambridge, 1966); esp. Chap. 8.

[15] The apparent twist in nomenclature may be unfortunate. The Lauricella function of the fourth kind, F_D, actually corresponds to the generalization of the Appell function of the first kind, F_1, to several variables.

[16] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. V.

[17] Except, of course, the Appell function of the fourth kind, F_4, for which no analogous integral representation is known.

[18] Here, the t_1, t_2 integrals come from the F_2 or the k_1, k_2 sum; the t_3 integral from the ${}_2F_1$ or the k_3 sum; finally the t_4 integral from $F_D^{(3)}$ or the k_4, k_5, k_6 sum.