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Computations of three-dimensional Rayleigh–Taylor instability

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The fully three-dimensional deformation of an interface between two fluids resulting from a Rayleigh–Taylor instability is studied numerically in the limit of weak stratification. The Navier–Stokes equations are solved by a finite difference method, and the interface is kept sharp by front tracking. The difference between the large-amplitude stages of flows initiated by two- and three-dimensional perturbations is discussed.

The Rayleigh–Taylor instability, where a heavy fluid falls into a lighter, underlying fluid, is one of the classical instabilities of fluid mechanics. It is the prototype problem for fluid mixing induced by unstable stratification, and as such, is of similar importance as the Kelvin–Helmholtz instability is to fluid mixing induced by a shear flow. Because of its fundamental importance, it has been the focus of many studies in the past, starting with Rayleigh late in the last century.[1] In addition to a number of analytical studies, most confined to early times and small amplitudes, several excellent experiments have been performed to investigate various aspects of the problem. We only mention the work of Read,[2] who focused on the statistical properties of mixing as a result of the growth of many initial waves, and Talbot and Jackson,[3] who considered viscous fluids as a model for the formations of salt domes.

The Rayleigh–Taylor instability has also been a favorite topic of computational scientists, and a large number of simulations have been reported. Most calculations have assumed inviscid flow (or at least very small viscosity), and the evolution of a single initial wave to a large amplitude is now fairly well understood for two-dimensional or axisymmetric flows. The early calculations by Harlow and coworkers at Los Alamos, e.g., Daly,[4] laid the foundations for such an understanding, and later work has confirmed and extended their findings. For the so-called single fluid case, where the lighter fluid is of negligible density, the heavy fluid falls down in thin pointed spikes, whereas the light fluid penetrates upward in big rounded bubbles. For a finite density difference, vortices form on the side of the spike. These vortices are stronger and have less downward motion as the density difference is reduced; in the limit of a vanishing density difference, they remain stationary at the original interface. More recent numerical studies have focused on the interaction of the initial "structures" (bubbles and vortices) and how the number and size of such structures change as the flow evolves. Youngs,[5] using a method related to the marker and cell (MAC) method of Harlow and Welch[6] and Tryggvason,[7] using a vortex method, show simulations of moderate stratification where the flow is more vortex dominated; Glimm *et al.*,[8] using a front-tracking method and Zufiria,[9] using a modified form of the vortex algorithm described by Tryggvason,[7] study bubble competition for strongly stratified flows. In addition to numerical studies of the large-amplitude evolution, a number of models have been proposed to explain the general trend. Such models include Gardner *et al.*[10] and Zufiria[11] for the single fluid case, and Aref and Tryggvason[12] for weakly stratified flows. For a review of the Rayleigh–Taylor problem, we refer the reader to Sharp.[13]

Experiments on the Rayleigh–Taylor instability almost always involve a fully three-dimensional problem, and unlike, say, the Kelvin–Helmholtz instability, there is usually no phase observed to be predominantly two dimensional. Although the importance of three-dimensionality is widely recognized, see, for example, the discussion by Sharp,[13] numerical investigations of the large-amplitude stage have been limited to two-dimensional (or axisymmetric) studies. Here, we present fully three-dimensional simulations for weakly stratified, viscous fluids and compare the evolution resulting from a simple three-dimensional disturbance to the evolution initiated by a strictly two-dimensional disturbance.

We solve the Navier–Stokes equation in vorticity form by a finite difference method, second order in both time and space. The density stratification is assumed to be small so that the vorticity (vector) streamfunction equations can be solved by a fast Poisson solver. The computational domain is a rectangular box, periodic in the horizontal directions, and with rigid, stress-free, top and bottom. The density interface is assumed to remain sharp throughout the calculations. To

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keep the interface sharp, and prevent numerical diffusion or spurious density oscillations, we explicitly track the interface. The interface is divided into triangular elements whose corner points are advected with the flow. At each time step, the density field is constructed from the new position of the interface and used to calculate the baroclinic generation of vorticity. The front-tracking technique will be discussed in detail elsewhere.[14] In the simulations presented here, the distribution of elements on the interface, i.e., the front grid, is not modified during the simulations. Such regridding is essential for long time simulations, since stretched portions of the interface are depleted of computational points and (although usually less troublesome) elements accumulate on compressed portions. The grid is kept fairly coarse, i.e., $17\times17\times33$ meshes. Two-dimensional simulations suggest that this adequately captures the characteristics of the evolution, and although the actual numbers do change slightly under grid refinement, the qualitative structure of the solution remains the same.

Here we limit ourselves to simple initial conditions. In two dimensions, the addition of two waves of the same length corresponds simply to a phase shift, so the interaction must be between waves of different wavelength. In three dimensions, the waves can have a different orientation, so the initial conditions can contain waves of the same wavelength but different direction. To address the effect of such additional modes, we compare the large-amplitude evolution of an interface perturbed by a single harmonic wave, $\cos(2\pi x/L)$, to the evolution of an interface perturbed by $0.5[\cos(2\pi x/L_x)+\cos(2\pi y/L_y)]$. In order to keep the initial growth rate the same, the total wavenumber must be the same, so we select $L=1$ and $L_x=L_y=\sqrt{2}$. For a wave of this dimension, corresponding to the most unstable wavelength, the nondimensional viscosity is $\nu/\sqrt{AgL^3}=0.024$ (see Chandrasekhar[14]).

In Fig. 1, we show the evolution of the interface for the three-dimensional perturbations at nondimensional times $(t\sqrt{Ag/L})$ 2.0, 2.5, and 3.0 for a viscosity corresponding to the most unstable wave. The initial disturbance of the interface is such that a "blob" of heavy fluid falls down in the center, and the light fluid rises in the corners. As the top fluid penetrates the light one, baroclinically generated vorticity forms a closed vortex ring around this blob and starts to roll up the interface. Since the evolution is entirely symmetric with respect to the heavy and the light fluid in the weakly stratified limit, the same process takes place for the rising light fluid.

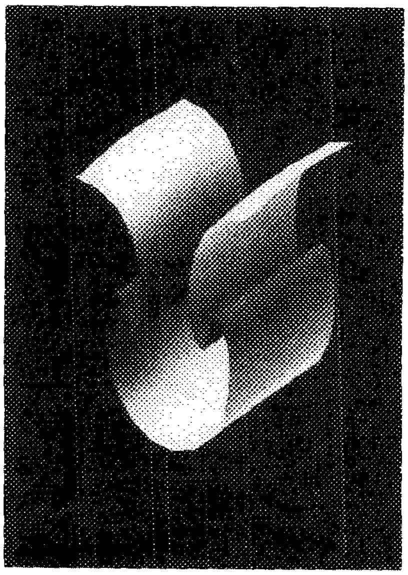

FIG. 2. The large-amplitude stage for a single initial mode. Here $t=2.75$.

The large-amplitude stage for an interface perturbed by a two-dimensional disturbance, but with the same linear growth rate, is shown in Fig. 2, at time 2.75. The development of the initial wave follows the pattern familiar from two-dimensional studies; two counter-rotating vortices form at the original interface and remain stationary as the heavy and the light fluid penetrate each other. The amplitude versus time for both cases is plotted in Fig. 3. Initially, the growth is the same, since the waves in both cases are still almost linear. As the interface enters the nonlinear stage, the

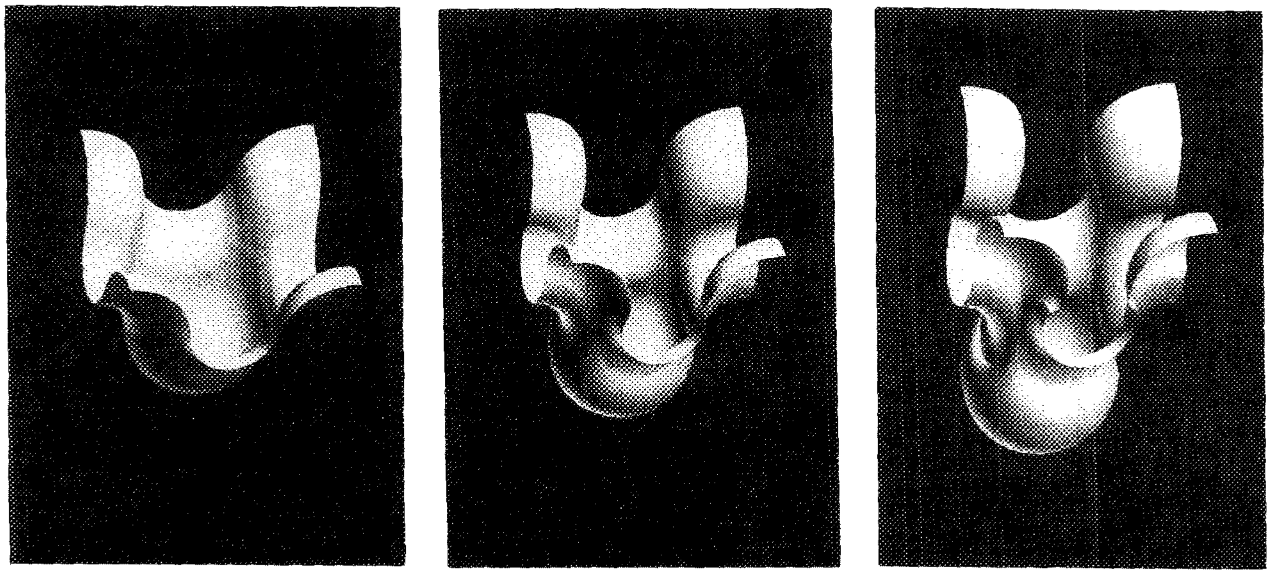

FIG. 1. The evolution of an interface disturbed by two modes. The nondimensional times are 2.0, 2.5, and 3.0.

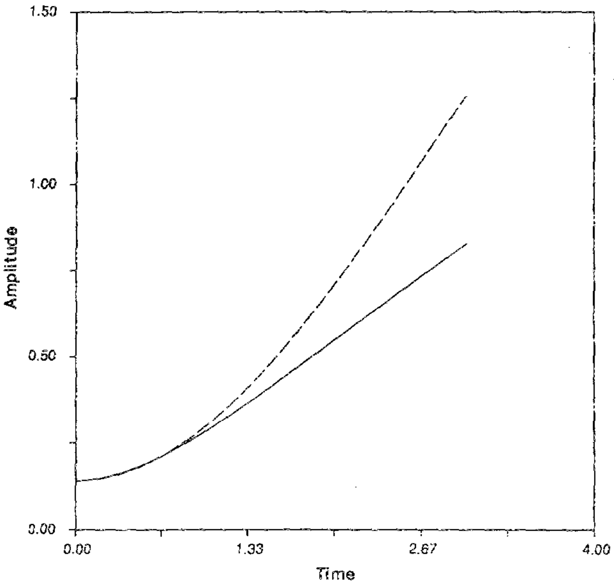

FIG. 3. The amplitude versus time for the runs in Figs. 1 and 2. Two-dimensional disturbance: —; three-dimensional disturbance:- - -.

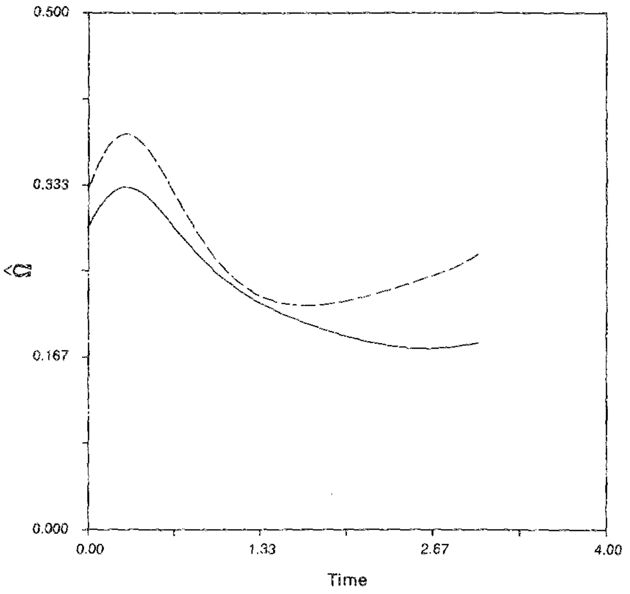

FIG. 4. The second moment of the vorticity, normalized as described in the text, for the runs in Figs. 1 and 2. Two-dimensional disturbance: —; three-dimensional disturbance: - - -.

growth of the three-dimensional disturbance is considerably larger than that for the single wave disturbance.

The major difference between the two- and three-dimensional evolution is due to the quite different vortex structure of the large-amplitude stage for different initial perturbations. When there is only one wave, the evolution is two dimensional, and the vorticity field consists of two counterrotating vortices that are constrained by symmetry to remain at the original interface. For the three-dimensional disturbance, the large-amplitude solution consists of vortex rings that propagate away from the original interface. Since the major portion of the vorticity now advances with the blob, in addition to the curvature effect of a ring vortex, the blob propagates considerably faster into the other fluid than in the single mode case where the vorticity remains behind, confined to straight vortex filaments. To bring this point out more clearly, in Fig. 4 we have plotted the second moment of the enstrophy (vorticity squared), normalized by the total enstrophy times the square of the maximum amplitude, or

$$\hat{\Omega} = \frac{\int_V (z - z_0)^2 \omega^2 \, dv}{(z - z_0)^2_{\max} \int_V \omega^2 \, dv}$$

for the run in Fig. 1 (fully three-dimensional disturbances) as well as the run in Fig. 2 (two-dimensional disturbances). Here, z_0 is the mean elevation of the original interface. Note that $\hat{\Omega}$ would be unity if all the vorticity is located at $z_{\max}$, and its smallness is a measure of how close the vorticity stays to the symmetry line. Obviously, this plot confirms that vorticity is located farther from the symmetry line for the fully three-dimensional case than in the two-dimensional run.

The evolution in Fig. 1 is greatly affected by viscosity; in particular, little roll-up has taken place at the last time shown. To address the effect of viscosity, we repeated this run with ten times smaller viscosity, and the results are shown in Fig. 5, for nondimensional time 2.5. The amplitude is considerably larger than in the more viscous case, and the interface now folds over on the side of the upward and downward moving blobs, suggesting the beginning of the formation of rolled-up vortices.

The results presented here suggest that, at least for small stratification, the evolution of the large-amplitude stage may differ somewhat from that predicted by a two-dimensional simulation with simple initial conditions. However, as emphasized by several investigators[7,10,13] the single wave initial condition may only tell part of the story. At larger amplitudes the interaction of the initial "structures" can modify the long-time evolution substantially.

This brief study is only a first step in a more comprehensive investigation of the large-amplitude, three-dimensional Rayleigh–Taylor instability. The interaction of more modes, in particular, subharmonic instabilities, as well as the effect of finite density stratification, are currently being investigated and will be reported in a later publication.

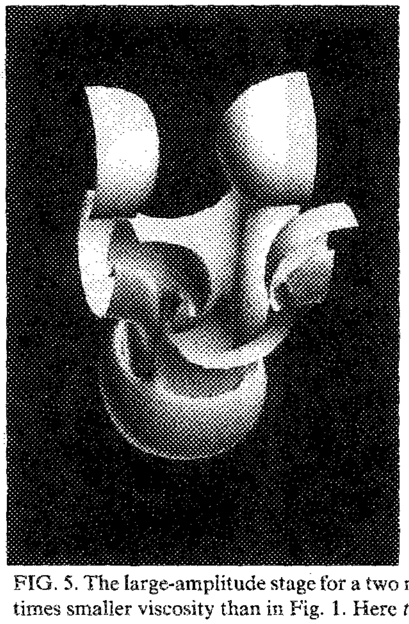

FIG. 5. The large-amplitude stage for a two mode initial condition, and ten times smaller viscosity than in Fig. 1. Here $t = 2.5$.

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