

8. CONCLUDING REMARKS

The theory we have been studying here may be extended in two ways: One may develop a topological generalization or an algebraic one.

(1) In the case of a topological extension, $(E_i)_{i \in I}$ and F are supposed to be topological vector spaces. We define then \mathcal{F} -continuous formal series of $(E_i)_{i \in I}$ into F as being formal series of $(E_i)_{i \in I}$ into F such that their kernels $\varphi_{(n_i)}$ define continuous monomial mappings of $\prod_{i \in I} E_i$ into F . For instance, in quantum field theory the "generating functional"

$$\begin{aligned} T[J] &= \sum_{n \geq 0} \frac{i^n}{n!} \int \tau_n(x_1, \dots, x_n) J(x_1) \cdots J(x_n) \\ &= \sum_{n \geq 0} \frac{i^n}{n!} \tau_n J^n \end{aligned}$$

is expected to be a continuous formal series of $\overline{\mathbb{C}}[\mathcal{S}, [J]]$, where $\overline{\mathbb{C}}$ is the field of complex numbers and \mathcal{S} the space of rapidly decreasing functions.

(2) In the case of an algebraic extension, $(E_i)_{i \in I}$ and F are no longer supposed to be vector spaces, but they are modules over a ring A (or A -modules).

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⁹ For linear and multilinear algebra our notations are those of Ref. 7, Chaps. 2 and 3.

Elastic Radiation in a Half-Space

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A Green's function for the elastic wave equation, which satisfies certain boundary conditions on the surface of a homogeneous half-space, is derived by means of the Fourier transformation. This half-space Green's function is then applied to the computation of radiative effects due to the earth's surface when a radiating source is located on or within that surface. The results obtained are to be taken as an extension of a previous and similar formulation for the infinite medium due to Case and Colwell.

1. INTRODUCTION

A method for computing the elastic radiation from a small source in the earth's interior has been presented by Case and Colwell.¹ This method, which assumes the earth to be an infinite medium, can be modified to include effects due to the earth's surface simply by replacing the (known) infinite-space Green's function, which is used, by an appropriate half-space Green's function. Our purpose here is, first of all, to obtain a representation for the half-space Green's function, and secondly, to demonstrate its applicability in computing corrections to the solution of Case and Colwell.

Our method is straightforward. To obtain the desired Green's function, we formulate the problem in terms of an integral equation, which equation is then solved by Fourier transforms. Our application of the Green's function then proceeds in a manner closely parallel to that of Ref. 1. The only complication lies in the fact that, once we choose a definite orientation for our half-space, the matrices which occur are not tensors, i.e., not rotation covariant. Hence tensor theory arguments, with the computational simplifications they often afford, are not available to us.

The notation to be used here differs in several minor

respects from that of Case and Colwell. In particular, 2-dimensional vectors will be denoted by an arrow:

$$\vec{a} \equiv (a_1, a_2), \quad \mathbf{a} = (a, a_3)$$

and we will use the convenient abbreviation

$$\partial_i \equiv \frac{\partial}{\partial x_i}, \quad \partial'_i \equiv \frac{\partial}{\partial x'_i}, \quad i = 1, 2, 3.$$

The Fourier transforms of a function $f(\mathbf{r})$ with respect to 2- and 3-dimensional space will be denoted by \hat{f} and \check{f} , respectively, with the conventions

$$\hat{f}(\vec{k}, x_3) = \int \exp(i\vec{k} \cdot \vec{r}) f(\mathbf{r}) d\vec{r},$$

$$\check{f}(\mathbf{k}) = \int \exp(i\mathbf{k} \cdot \mathbf{r}) f(\mathbf{r}) d\mathbf{r}.$$

Note that we deal exclusively, and hence tacitly, with the Fourier transforms with respect to time. (It will become evident below that the inverse transform for the time variable is essentially as trivial here as it was in Ref. 1.) The summation convention is used throughout.

2. THE HALF-SPACE GREEN'S FUNCTION

Integral Formulation

We consider first a general region V of \mathbf{r} -space, and seek the solution $f_{ij}(\mathbf{r}, \mathbf{r}_0)$ to

$$-\omega^2 \rho f_{ij}(\mathbf{r}, \mathbf{r}_0) = \partial_k D_{ikm}(\partial) f_{mj}(\mathbf{r}, \mathbf{r}_0) + \delta_{ij} \delta(\mathbf{r} - \mathbf{r}_0),$$

$$\mathbf{r}, \mathbf{r}_0 \in V, \quad (2.1)$$

with the boundary condition

$$n_i D_{ikm}(\partial) f_{mj}(\mathbf{r}_s, \mathbf{r}_0) = 0, \quad \mathbf{r}_s \in S. \quad (2.2)$$

Here, n_i is (the i th component of) the inward normal to the region V with boundary S and

$$D_{ikm}(\partial) \equiv \lambda \delta_{ik} \partial_m + \mu (\delta_{im} \partial_k + \delta_{km} \partial_i). \quad (2.3)$$

The infinite-space Green's function G_{ij} satisfies Eq. (2.1) with V including all space:

$$-\omega^2 \rho G_{ij}(\mathbf{r}, \mathbf{r}') = \partial_k D_{ikm}(\partial) G_{mj}(\mathbf{r}, \mathbf{r}') + \delta_{ij} \delta(\mathbf{r} - \mathbf{r}'),$$

$$\text{all } \mathbf{r}, \mathbf{r}'. \quad (2.4)$$

We now proceed in a standard way to multiply Eq. (2.4) by f_{ij} , Eq. (2.1) by G_{il} , and subtract. Using the easily verified identity

$$G_{il} \partial_k D_{ikm} f_{mj} - f_{ij} \partial_k D_{ikm} G_{ml}$$

$$= \partial_i [G_{kl} D_{ikm} f_{mj} - f_{kj} D_{ikm} G_{ml}], \quad (2.5)$$

we find

$$G_{il}(\mathbf{r}, \mathbf{r}') \delta_{ij} \delta(\mathbf{r} - \mathbf{r}_0) - f_{ij}(\mathbf{r}, \mathbf{r}_0) \delta_{il} \delta(\mathbf{r} - \mathbf{r}')$$

$$+ \partial_i [G_{kl}(\mathbf{r}, \mathbf{r}') D_{ikm}(\partial) f_{mj}(\mathbf{r}, \mathbf{r}_0)$$

$$- f_{kj}(\mathbf{r}, \mathbf{r}_0) D_{ikm}(\partial) G_{ml}(\mathbf{r}, \mathbf{r}')] = 0. \quad (2.6)$$

We now integrate Eq. (2.6) over V , apply the divergence theorem, and note Eq. (2.2). Upon renaming variables (and indices) and using the facts that

$$G_{ij}(\mathbf{r}, \mathbf{r}_0) = G_{ij}(\mathbf{r}_0, \mathbf{r}) \quad (2.7)$$

and

$$D_{ikm}(\partial) G_{mj}(\mathbf{r}, \mathbf{r}') = -D_{ikm}(\partial') G_{mj}(\mathbf{r}', \mathbf{r}), \quad (2.8)$$

we obtain the desired equation

$$f_{ij}(\mathbf{r}, \mathbf{r}_0) = G_{ij}(\mathbf{r}, \mathbf{r}_0) + \int_S d^2 r' n_k(\mathbf{r}') f_{ij}(\mathbf{r}', \mathbf{r}_0)$$

$$\times D_{kilm}(\partial') G_{mi}(\mathbf{r}', \mathbf{r}). \quad (2.9)$$

A method for determining $f_{ij}(\mathbf{r}, \mathbf{r}_0)$ is clear from Eq. (2.9). Indeed, our problem clearly reduces to finding $f_{ij}(\mathbf{r}_s, \mathbf{r}_0)$; and, by taking the limit of Eq. (2.9) as $\mathbf{r} \rightarrow \mathbf{r}_s$, we obtain an integral equation which may be solved for $f_{ij}(\mathbf{r}_s, \mathbf{r}_0)$.

Specializing to the case in which V is the half-space $x_3 \geq 0$, we denote the half-space Green's function by g_{ij} :

$$g_{ij}(\mathbf{r}, \mathbf{r}_0) = G_{ij}(\mathbf{r}, \mathbf{r}_0) + \int d\check{r}' g_{ij}(\check{r}', 0; \mathbf{r}_0)$$

$$\times [D_{3ilm}(\partial') G_{mi}(\mathbf{r}', \mathbf{r})]_{x_3'=0}. \quad (2.10)$$

A certain amount of care is required in taking the limit $x_3 \rightarrow 0$ of Eq. (2.10), since the integrand is singular on S . In fact, if we define

$$\Sigma_{3li}(\mathbf{r}, \mathbf{r}') \equiv D_{3ilm}(\partial) G_{mi}(\mathbf{r}, \mathbf{r}'), \quad (2.11)$$

then it easily follows from Eq. (2.4) that

$$\Sigma_{3li}(\check{r}, 0+; \check{r}', 0) - \Sigma_{3li}(\check{r}, 0-; \check{r}', 0)$$

$$= -\delta_{li} \delta(\check{r} - \check{r}'). \quad (2.12)$$

Thus, we define the "principal value"

$$\Sigma_{3li}^0(\check{r}, \check{r}') \equiv \frac{1}{2} [\Sigma_{3li}(\check{r}, 0+; \check{r}', 0) + \Sigma_{3li}(\check{r}, 0-; \check{r}', 0)]. \quad (2.13)$$

Now let $x_3 \rightarrow 0$ in Eq. (2.10). Using Eqs. (2.12) and (2.13), the result may be written as

$$\frac{1}{2} g_{ij}(\check{r}, 0; \mathbf{r}_0) = G_{ij}(\check{r}, 0; \mathbf{r}_0) + S_{ij}^0(\check{r}, \mathbf{r}_0), \quad (2.14)$$

where

$$S_{ij}^0(\check{r}, \mathbf{r}_0) \equiv \int d\check{r}' g_{ij}(\check{r}', 0; \mathbf{r}_0) \Sigma_{3li}^0(\check{r}', \check{r}). \quad (2.15)$$

Equations (2.10) and (2.14) are the basic relations by means of which our problem is to be solved.

Solution of the Integral Equation

Because of the translation invariance of G_{ij} , $\Sigma_{3li}^0(\check{r}', \check{r})$ depends only on the difference $\check{r}' - \check{r}$. It

follows that the 2-dimensional Fourier transform of Eq. (2.14),

$$\frac{1}{2}\hat{g}_{ij}(\vec{k}, 0; \mathbf{r}_0) = \hat{G}_{ij}(\vec{k}, 0; \mathbf{r}_0) + \hat{S}_{ij}^0(\vec{k}, \mathbf{r}_0), \quad (2.16)$$

reduces to the purely algebraic equations

$$[\frac{1}{2}\delta_{ii} + H_{ii}^0(\vec{k})]\hat{g}_{ij}(\vec{k}, 0; \mathbf{r}_0) = \hat{G}_{ij}(\vec{k}, 0; \mathbf{r}_0). \quad (2.17)$$

Here, we have introduced a less cumbersome notation for $\hat{\Sigma}_{3ii}^0(0; \vec{k})$:

$$H_{ii}^0(\vec{k}) \equiv \frac{1}{2}[H_{ii}(\vec{k}, 0+) + H_{ii}(\vec{k}, 0-)], \quad (2.18)$$

where

$$H_{ii}(\vec{k}, x_3) \equiv \frac{-1}{2\pi} \int_{-\infty}^{\infty} dk_3 e^{-ik_3 x_3} \tilde{\Sigma}_{3ii}(\mathbf{k}; \mathbf{r}' = 0). \quad (2.19)$$

[The minus sign is occasioned by Eq. (2.8).] It is convenient to note here the identity

$$H_{ii}(\vec{k}, 0+) = H_{ii}^0(\vec{k}) - \frac{1}{2}\delta_{ii}, \quad (2.20)$$

which follows from Eqs. (2.12) and (2.13) and which will be useful below.

Equation (2.17) has the solution

$$\hat{g}_{ij}(\vec{k}, 0; \mathbf{r}_0) = \|A^{-1}\|_{ii} \hat{G}_{ij}(\vec{k}, 0; \mathbf{r}_0), \quad (2.21)$$

where $\|A^{-1}\|$ is the matrix inverse to

$$\|A\|_{ii} = \frac{1}{2}\delta_{ii} + H_{ii}^0(\vec{k}). \quad (2.22)$$

We obtain H_{ii}^0 from the known¹ fact that

$$\tilde{G}_{ij}(\vec{k}, \mathbf{r}') = \frac{e^{i\mathbf{k}\cdot\mathbf{r}'}}{\omega^2 \rho} \left(\frac{k_i k_j}{k^2 - k_i^2} + \frac{k_i^2 \delta_{ij} - k_i k_j}{k^2 - k_i^2} \right), \quad (2.23)$$

where

$$k_i^2 = \omega^2 \rho / (\lambda + 2\mu), \quad k_t^2 = \omega^2 \rho / \mu \quad (2.24)$$

and from Eq. (2.13). A straightforward computation yields

$$\begin{aligned} 2\pi i \omega^2 \rho H_{jk} &= \lambda k_i^2 \delta_{3jk} [(k_j - \delta_{3j} k_3) I_{0l} + \delta_{3j} I_{il}] \\ &+ \mu \{ 2(k_i^2 - k_t^2) \\ &\times [(k_k - \delta_{3k} k_3)(k_j - \delta_{3j} k_3) I_2 \\ &+ (k_j - \delta_{3j} k_3) \delta_{3k} I_3 + (k_k - \delta_{3k} k_3) \delta_{3j} I_3 \\ &+ \delta_{3k} \delta_{3j} I_4] + k_i^2 [\delta_{3j} (k_k - \delta_{3k} k_3) I_{0t} \\ &+ \delta_{3k} \delta_{3j} I_{1t} + \delta_{kj} I_{1t}] \}, \end{aligned} \quad (2.25)$$

where²

$$\begin{aligned} I_{0l,t}(\vec{k}, x_3) &= \frac{\pi}{\kappa_{l,t}} e^{-\kappa_{l,t}|x_3|}, \\ I_{1l,t}(k, x_3) &= \mp i \pi e^{-\kappa_{l,t}|x_3|}, \quad x_3 \geq 0, \\ I_2(\vec{k}, x_3) &= [i\pi / (k_t^2 - k_i^2)] (e^{-\kappa_{il}|x_3|} - e^{-\kappa_{it}|x_3|}), \\ I_3(\vec{k}, x_3) &= [\pi / (k_t^2 - k_i^2)] (\kappa_{it} e^{-\kappa_{il}|x_3|} - \kappa_{il} e^{-\kappa_{it}|x_3|}), \\ I_4(\vec{k}, x_3) &= [\mp i\pi / (k_t^2 - k_i^2)] (\kappa_{it}^2 e^{-\kappa_{il}|x_3|} - \kappa_{il}^2 e^{-\kappa_{it}|x_3|}), \\ &x_3 \geq 0. \end{aligned} \quad (2.26)$$

In Eqs. (2.26), we have introduced the abbreviations

$$\kappa_{i,t}^2 = \vec{k}^2 - k_{i,t}^2. \quad (2.27)$$

From Eqs. (2.18), (2.25), and (2.26), we have

$$H_{ij}^0(\vec{k}) = \frac{1}{2} a(\vec{k}^2) \delta_{3j} (k_i - \delta_{3i} k_3) + \frac{1}{2} b(\vec{k}^2) \delta_{3i} (k_j - \delta_{3j} k_3), \quad (2.28)$$

where

$$a(\vec{k}^2) = -i[\lambda \kappa_t - (\lambda + 2\mu) \kappa_i] / (\lambda + 2\mu) \kappa_i (\kappa_i + \kappa_t), \quad (2.29)$$

$$b(\vec{k}^2) = i[\lambda \kappa_i - (\lambda + 2\mu) \kappa_t] / (\lambda + 2\mu) \kappa_t (\kappa_i + \kappa_t), \quad (2.30)$$

whence

$$\|A\| = \frac{1}{2} \begin{bmatrix} 1 & 0 & ak_1 \\ 0 & 1 & ak_2 \\ bk_1 & bk_2 & 1 \end{bmatrix} \quad (2.31)$$

and we easily find

$$\|A^{-1}\| = \frac{2}{1 - ab\vec{k}^2} \begin{bmatrix} 1 - abk_2^2 & abk_1 k_2 & -ak_1 \\ abk_1 k_2 & 1 - abk_1^2 & -ak_2 \\ -bk_1 & -bk_2 & 1 \end{bmatrix}. \quad (2.32)$$

Equations (2.10), (2.21), and (2.32) provide the desired half-space Green's function. It is conveniently written in the form

$$g_{ij}(\mathbf{r}, \mathbf{r}_0) = G_{ij}(\mathbf{r}, \mathbf{r}_0) + S_{ij}(\mathbf{r}, \mathbf{r}_0), \quad (2.33)$$

where

$$\begin{aligned} S_{ij}(\mathbf{r}, \mathbf{r}_0) &= \frac{-1}{(2\pi)^2} \\ &\times \int d\vec{k} \exp(-i\vec{k} \cdot \vec{r}) H_{ij}(\vec{k}, x_3) \\ &\times \|A^{-1}\|_{im} \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}_0). \end{aligned} \quad (2.34)$$

3. APPLICATION OF THE HALF-SPACE GREEN'S FUNCTION

We demonstrate the usefulness of the Green's function obtained above by applying it to the solution of an idealized problem similar to that considered by Case and Colwell. That is, we provide an integral formulation by means of which the radiation field from a small radiating cavity may be computed. As in Ref. 1, the problem is simplified by assuming to be given certain quantities which could in theory be determined (by solving an integral equation). Our refinement here consists, of course, in taking the earth to be a homogeneous half-space, rather than an infinite medium. It is to the effect of this refinement, i.e., to the difference between the half-space solution and the solution of Case and Colwell, that we generally confine our attention.

General Formulation

Let V be the half-space as in Sec. 2, with the exclusion of a small cavity, located on or "above" S , with boundary B . Note that, if $\mathbf{r} = (x_1, x_2, x_3)$ is within B , then $x_3 \geq 0$. We wish to determine the functions $u_i(\mathbf{r})$ which satisfy

$$-\omega^2 \rho u_i(\mathbf{r}) = \partial_k D_{ikm} u_m(\mathbf{r}), \quad \mathbf{r} \in V, \quad (3.1)$$

with the boundary conditions

$$n_j D_{jikm} u_m(\mathbf{r}) = 0, \quad \mathbf{r} \in S, \quad (3.2)$$

$$n_j D_{jikm} u_m(\mathbf{r}) = F_k(\mathbf{r}), \quad \mathbf{r} \in B \text{ (} F_k \text{ given)}. \quad (3.3)$$

That is, the normal component of the stress is to vanish on the surface of the earth and to be prescribed on the surface of the cavity.

Following a procedure closely analogous to that of Eqs. (2.1)–(2.9), it is a simple matter to show that Eqs. (3.1)–(3.3) may be restated in the form

$$u_j(\mathbf{r}) = \int_B d^2 r_B [g_{ij}(\mathbf{r}_B, \mathbf{r}) F_i(\mathbf{r}_B) - u_i(\mathbf{r}_B) n_k D_{kim} g_{mj}(\mathbf{r}_B, \mathbf{r})]. \quad (3.4)$$

Given the geometry of B , we could now, of course, let $\mathbf{r} \rightarrow \mathbf{r}_B$ and attempt to solve the resulting integral equation for $u_i(\mathbf{r}_B)$ (as in Sec. 2). Instead, we take a more practical approach, paralleling that of Case and Colwell, and assume the $u_i(\mathbf{r}_B)$ to be known. But first it is convenient to isolate the effects of the surface S , with which effects we are exclusively concerned below. Let $u_j^{(0)}(\mathbf{r})$ satisfy

$$u_j^{(0)}(\mathbf{r}) = \int_B d^2 r_B [G_{ij}(\mathbf{r}_B, \mathbf{r}) F_i(\mathbf{r}_B) - u_i(\mathbf{r}_B) n_k D_{ikm} G_{mj}(\mathbf{r}_B, \mathbf{r})]. \quad (3.5)$$

$u_j^{(0)}$ is precisely the solution investigated by Case and Colwell. Here, we are interested in the functions

$$v_j(\mathbf{r}) \equiv u_j(\mathbf{r}) - u_j^{(0)}(\mathbf{r}). \quad (3.6)$$

It is clear from Eqs. (2.33) and (3.5) that these are to be determined from

$$v_j(\mathbf{r}) = \int_B d^2 r_B [S_{ij}(\mathbf{r}_B, \mathbf{r}) F_i(\mathbf{r}_B) - u_i(\mathbf{r}_B) n_k D_{kim} S_{mj}(\mathbf{r}_B, \mathbf{r})], \quad (3.7)$$

where S_{ij} is given by Eq. (2.34).

Our problem is solved by Eq. (3.7). The remainder of this paper is concerned with bringing this equation into an explicit form directly suitable for evaluation.

The Case of a Source on the Surface

We consider first a situation in which the complexities of Eqs. (3.7) and (2.34) are considerably reduced: When the source cavity B is on the earth's surface S . That is,

$$\mathbf{r}_B = (x_{1B}, x_{2B}, 0+). \quad (3.8)$$

[The + sign is necessary because of the discontinuity of the integrand in Eq. (3.7) on S —cf. Eqs. (2.26). Note that our original differential Eq. (3.1) holds only for $x_3 \geq 0$.] Recalling the identity (2.20) and Eq. (2.22), we see that the integrand of Eq. (2.34) here takes the form

$$H_{il}(\vec{k}, 0+) \|A^{-1}\|_{lm} \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}) = [\|A\|_{il} - \delta_{il}] \|A^{-1}\|_{lm} \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}) \quad (3.9)$$

$$= -B_{im}(\vec{k}) \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}), \quad (3.10)$$

where we have introduced the quantities

$$B_{im} = \|A^{-1}\|_{im} - \delta_{im}. \quad (3.11)$$

The B_{ij} are most conveniently given in matrix form:

$$\|B\| = \frac{2}{1 - ab\vec{k}^2} \begin{bmatrix} -abk_2^2 & abk_1k_2 & -ak_1 \\ abk_1k_2 & -abk_1^2 & -ak_2 \\ -bk_1 & -bk_2 & 0 \end{bmatrix}. \quad (3.12)$$

The "surface-effects Green's function" is now given by

$$S_{ij}(\mathbf{r}_B, \mathbf{r}) = \frac{1}{(2\pi)^2} \times \int d\vec{k} \exp(-i\vec{k} \cdot \vec{r}_B) B_{im}(\vec{k}) \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}). \quad (3.13)$$

Heretofore, we have been dealing with the total elastic disturbance. We now wish to compute the effects of the radiation field only. This may be accomplished,¹ assuming the origin of coordinates to be within B , by replacing in our formulation the exact infinite-space Green's function

$$G_{ij}(\mathbf{r}', \mathbf{r}) = \frac{-1}{4\pi\omega^2\rho} \left(\partial_i \partial_j \frac{e^{ik_i|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} - (k_i^2 \delta_{ij} + \partial_i \partial_j) \frac{e^{ik_i|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \right) \quad (3.14)$$

by its asymptotic form for $|\mathbf{r}| \gg |\mathbf{r}'|$:

$$G_{ij}(\mathbf{r}', \mathbf{r}) = \frac{-1}{4\pi\omega^2\rho} \frac{1}{r} [\exp(ik_i \hat{\mathbf{r}} \cdot \mathbf{r}') \partial_i \partial_j e^{ik_i r} - \exp(-ik_i \hat{\mathbf{r}} \cdot \mathbf{r}') (k_i^2 \delta_{ij} + \partial_i \partial_j) e^{ik_i r}] + O\left(\frac{r'^2}{r^2}\right), \quad (3.15)$$

where $\hat{\mathbf{r}} \equiv \mathbf{r}/|\mathbf{r}|$. Because of the very simple dependence upon \mathbf{r}' in Eq. (3.15), there is no difficulty in obtaining the Fourier transform needed for Eq. (3.13):

$$\hat{G}_{ij}(\vec{k}, 0; \mathbf{r}) = \frac{-(2\pi)^2}{4\pi\omega^2\rho} \frac{1}{r} \left[\delta\left(\vec{k} - k_i \frac{\vec{r}}{r}\right) \partial_i \partial_j e^{ik_i r} - \delta\left(\vec{k} - k_i \frac{\vec{r}}{r}\right) (k_i^2 \delta_{ij} + \partial_i \partial_j) e^{ik_i r} \right]. \tag{3.16}$$

Here, of course,

$$\delta(\vec{k} - k_{i,t}(\vec{r}/r)) \equiv \delta(k_1 - k_{i,t}(x_1/r)) \delta(k_2 - k_{i,t}(x_2/r)) \tag{3.17}$$

and, since Eq. (3.16) is exact with regard to the radiation field, we have omitted the error term.

From Eq. (3.16), we see that the integral of Eq. (3.13) is entirely trivial. The result is conveniently written as

$$S_{ij}(\mathbf{r}', \mathbf{r}) = S_{ij}^l(\mathbf{r}) \exp(-ik_i \hat{\mathbf{r}} \cdot \mathbf{r}') + S_{ij}^t(\mathbf{r}) \exp(-ik_i \hat{\mathbf{r}} \cdot \mathbf{r}'), \tag{3.18}$$

where

$$S_{ij}^l(\mathbf{r}) \equiv B_{im} \left(k_i \frac{\vec{r}}{r} \right) \left(\frac{-1}{4\pi\omega^2\rho} \frac{1}{r} \partial_m \partial_j e^{ik_i r} \right) \tag{3.19}$$

$$= \frac{k_i^2}{4\pi\omega^2\rho} \frac{e^{ik_i r}}{r} B_{im} \left(k_i \frac{\vec{r}}{r} \right) \hat{\mathbf{r}}_m \hat{\mathbf{r}}_j \tag{3.20}$$

and, similarly,

$$S_{ij}^t(\mathbf{r}) = \frac{k_i^2}{4\pi\omega^2\rho} \frac{e^{ik_i r}}{r} B_{im} \left(k_i \frac{\vec{r}}{r} \right) (\delta_{mj} - \hat{\mathbf{r}}_m \hat{\mathbf{r}}_j). \tag{3.21}$$

If we also define the (known) vector quantities

$$f_{i,l}^{t,l}(\hat{\mathbf{r}}) \equiv \int_B d^2r' F_i(\mathbf{r}') \exp(-ik_{i,l} \hat{\mathbf{r}} \cdot \mathbf{r}') \tag{3.22}$$

and the symmetrized tensors

$$T_{ij}^{t,l} \equiv \frac{1}{2} \int_B d^2r' [u_i(\mathbf{r}') n_j(\mathbf{r}') + n_i(\mathbf{r}') u_j(\mathbf{r}')] \times \exp(-ik_{i,l} \hat{\mathbf{r}} \cdot \mathbf{r}'), \tag{3.23}$$

then it requires only some elementary manipulations to write Eq. (3.7) in the form

$$v_j(\mathbf{r}) = S_{ij}^l(\mathbf{r}) f_i^l(\hat{\mathbf{r}}) + ik_i [\lambda \hat{\mathbf{r}}_m S_{mj}^l(\mathbf{r}) T_{nn}^l(\hat{\mathbf{r}}) + 2\mu \hat{\mathbf{r}}_i T_{ik}^l(\hat{\mathbf{r}}) S_{kj}^l(\hat{\mathbf{r}})] + S_{ij}^t(\mathbf{r}) f_i^t(\hat{\mathbf{r}}) + ik_i [\lambda \hat{\mathbf{r}}_m S_{mj}^t(\mathbf{r}) T_{nn}^t(\hat{\mathbf{r}}) + 2\mu \hat{\mathbf{r}}_i T_{ik}^t(\hat{\mathbf{r}}) S_{kj}^t(\mathbf{r})]. \tag{3.24}$$

Note, from Eqs. (3.20), (3.21), and (3.24) (which are exact for the radiation field), that the transverse and longitudinal waves are unmixed when the source is on the surface.

We could now proceed, as in Ref. 1, to make the small source approximation ($\mathbf{r}' \in B \Rightarrow k_{i,l} r' \ll 1$) and expand the exponentials in Eqs. (3.22) and (3.23):

$$\exp(-ik_{i,l} \hat{\mathbf{r}} \cdot \mathbf{r}') = 1 - ik_{i,l} \hat{\mathbf{r}} \cdot \mathbf{r}' + \dots \tag{3.25}$$

Since this calculation would proceed exactly as in Ref. 1, we omit it, and instead consider the explicit form of the matrices $S_{ij}^{t,l}$.

Consider first the longitudinal terms. According to Eq. (3.20), our first task is to substitute $k_i \vec{r}/r$ for \vec{k} in the matrix B_{im} . This entails the substitution [cf. Eqs. (2.27)]

$$\kappa_l = ik_l \cos \theta, \quad \kappa_t = i(k_l^2 - k_i^2 \sin^2 \theta)^{\frac{1}{2}}, \tag{3.26}$$

where θ is the angle between \mathbf{r} and the positive z axis, and we have chosen the signs in Eqs. (3.26) essentially by means of a radiation condition (that we have chosen them correctly will become clear below). Using Eqs. (3.26) we compute

$$a_i \equiv a \left(k_i^2 \frac{\vec{r}^2}{r^2} \right) = - \frac{C_t^2 [(C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}} - C_t \cos \theta] - 2C_i^2 (C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}}}{\omega C_l \cos \theta [C_t \cos \theta + (C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}}]}, \tag{3.27}$$

where the C 's are the velocities of the longitudinal and transverse modes:

$$C_l^2 = (\lambda + 2\mu)/\rho, \quad C_t^2 = \mu/\rho. \tag{3.28}$$

Similarly,

$$b_l \equiv b \left(k_i^2 \frac{\vec{r}^2}{r^2} \right) = \frac{C_i^2 C_t [(C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}} - C_t \cos \theta] - 2C_i^2 (C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}}}{\omega C_l (C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}} [(C_i^2 - C_t^2 \sin^2 \theta)^{\frac{1}{2}} + C_t \cos \theta]}. \tag{3.29}$$

The matrix product in Eq. (3.20) is now easily computed and we find

$$\|S^t(\mathbf{r})\| = \frac{k_t^2}{4\pi\omega^2\rho} \frac{e^{ik_t r}}{r} \cdot \left(\frac{-2k_t}{1 - a_t b_t k_t^2 \sin^2 \theta} \right) \begin{pmatrix} a_t \hat{f}_1^2 \hat{f}_3 & a_t \hat{f}_1 \hat{f}_2 \hat{f}_3 & a_t \hat{f}_1 \hat{f}_3^2 \\ a_t \hat{f}_1 \hat{f}_2 \hat{f}_3 & a_t \hat{f}_2^2 \hat{f}_3 & a_t \hat{f}_2 \hat{f}_3^2 \\ b_t \hat{f}_1 \sin^2 \theta & b_t \hat{f}_2 \sin^2 \theta & b_t \hat{f}_3 \sin^2 \theta \end{pmatrix}. \quad (3.30)$$

In exactly the same way, for the transverse terms, we find

$$\|S^t(\mathbf{r})\| = \frac{k_t^2}{4\pi\omega^2\rho} \frac{e^{ik_t r}}{r} \left(\frac{-2k_t}{1 - a_t b_t k_t^2 \sin^2 \theta} \right) \begin{pmatrix} a_t b_t k_t \hat{f}_2^2 - a_t \hat{f}_1^2 \hat{f}_3 & -a_t (b_t k_t \hat{f}_1 \hat{f}_2 + \hat{f}_1 \hat{f}_2 \hat{f}_3) & a_t (\hat{f}_1 - \hat{f}_1 \hat{f}_3^2) \\ -a_t (b_t k_t \hat{f}_1 \hat{f}_2 + \hat{f}_1 \hat{f}_2 \hat{f}_3) & a_t (b_t \hat{f}_1^2 - \hat{f}_2^2 \hat{f}_3) & a_t (\hat{f}_2^2 - \hat{f}_2 \hat{f}_3^2) \\ b_t \hat{f}_1 \hat{f}_3^2 & b_t \hat{f}_2 \hat{f}_3^2 & -b_t \hat{f}_3 \sin^2 \theta \end{pmatrix}, \quad (3.31)$$

where

$$a_t = \frac{C_t C_i [(C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} - C_i \cos \theta] + 2C_i^3 \cos \theta}{\omega (C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} [(C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} + C_i \cos \theta]}, \quad (3.32)$$

$$b_t = -\frac{C_t C_i [(C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} - C_i \cos \theta] + 2C_i^3 \cos \theta}{\omega C_i \cos \theta [(C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} + C_i \cos \theta]}. \quad (3.33)$$

Actually, Eqs. (3.32) and (3.33) are correct only for $\theta < \theta_c$, where θ_c is the angle at which the radicand vanishes:

$$\sin^2 \theta_c \equiv C_i^2 / C_t^2 = \mu / (\lambda + 2\mu). \quad (3.34)$$

For $\theta > \theta_c$, we must modify (3.32) and (3.33) by the replacement

$$(C_i^2 - C_i^2 \sin^2 \theta)^{\frac{1}{2}} \rightarrow i(C_i^2 \sin^2 \theta - C_i^2)^{\frac{1}{2}}. \quad (3.35)$$

Equations (3.24) and (3.27)–(3.35) provide an exact description of the surface corrections at any point \mathbf{r} to the radiation field from a surface source. We observe from (3.27)–(3.35) that

(i) the matrices $\|B(k_{i,j}, \hat{r}/r)\|$ are independent of ω ; thus, the inverse Fourier transform with respect to time is essentially as trivial here as it was¹ in the infinite medium case;

(ii) no radiation due to the surface appears (i.e., the radiation field of u_j coincides with that of $u_j^{(0)}$) on the axis $\hat{r} = 0$;

(iii) there may also occur, depending upon the relative magnitudes of λ and μ , cone-shaped regions on which either the longitudinal or the transverse parts of the radiation field due to the surface vanish. Specifically, we find from Eqs. (3.27) and (3.29) that $\|S^t\|$ is zero on the cone (which may or may not be physical)

$$\cos \theta = \lambda / 2\mu \quad (3.36)$$

and, similarly, from Eqs. (3.32) and (3.33), that the transverse terms do not appear on

$$\cos \theta = (\lambda + 2\mu) / 4\mu. \quad (3.37)$$

We conclude our discussion of the case of a source on the earth's surface by considering the effects

observed at a point \mathbf{r} which is also on that surface. (The result here is atypically simple.) Thus, we set

$$\mathbf{r} = (x_1, x_2, 0), \quad \theta = \frac{1}{2}\pi \quad (3.38)$$

and find, from Eqs. (3.27)–(3.33) that a_t and b_t are infinite. It follows immediately that

$$\|S^t\| = 0, \quad x_3 = 0, \quad (3.39)$$

and

$$\|S^t\| = \frac{2k_t^2}{4\pi\omega^2\rho} \frac{e^{ik_t r}}{r} \begin{pmatrix} \hat{f}_2^2 & -\hat{f}_1 \hat{f}_2 & 0 \\ -\hat{f}_1 \hat{f}_2 & \hat{f}_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.40)$$

Thus, the longitudinal waves on the surface are just as in the infinite medium case, while the transverse modes are modified by the correction term

$$v_j(x_1, x_2, 0) = \frac{k_t^2}{4\pi\omega^2\rho} \frac{e^{ik_t r}}{r} (2\delta_{kj} - \hat{f}_k \hat{f}_j) \times \{f_k^t(\hat{\mathbf{f}}) + ik_t [\lambda \hat{f}_k T_{nn}^t(\hat{\mathbf{f}}) + 2\mu \hat{f}_n T_{nk}^t(\hat{\mathbf{f}})]\}, \quad (3.41)$$

where $i, j, k = 1, 2$ only and $v_3 = 0$.

The Case of a Source in the Earth's Interior

While no serious difficulties occur in using Eq. (3.7) to determine the surface effects due to a source buried within the earth, it is not possible in the case to write the equation in any substantially simplified form. [This is because the identity (2.20) is no longer applicable.] Therefore, we confine our attention here mainly to isolating the radiation field.

In this regard, a minor difficulty is seen to occur. Equation (3.15), which we previously used to determine the radiation part of the field, is true for $r \gg r'$,

but no such inequality holds if the source is deeply buried and if, as we have stipulated in Sec. 2, our origin of coordinates is on the surface S . Hence, first of all, we affect on Eq. (3.7) a displacement of the origin by a distance R in the x_3 direction. Since the only relevant quantity which is not invariant under such a displacement is $H_{ij}(\vec{k}, x_3)$, this modification is easily accomplished. We let [cf. Eq. (2.34)]

$$R_{ij}(\mathbf{r}', \mathbf{r}) \equiv \frac{-1}{(2\pi)^2} \int d\vec{k} \exp(-i\vec{k} \cdot \vec{r}') \times H_{ij}(\vec{k}, x'_3 + R) \|A^{-1}\|_{lm} \hat{G}_{mj}(\vec{k}, 0; \mathbf{r}). \quad (3.42)$$

Then we can conveniently compute the radiation due to the surface S from Eq. (3.7) with S_{ij} replaced by R_{ij} :

$$v_j(\mathbf{r}) = \int_B d^2r' [R_{ij}(\mathbf{r}', \mathbf{r}) F_i(\mathbf{r}') - u_i(\mathbf{r}') n_k D_{kijm}(\partial') R_{mj}(\mathbf{r}', \mathbf{r})]. \quad (3.43)$$

Equation (3.16) is now directly applicable (i.e., we may again assume the origin is within B), and we find that for the radiation field, Eq. (3.42) reduces to

$$R_{ij}(\mathbf{r}', \mathbf{r}) = \frac{-k_i^2}{4\pi\omega^2\rho} \frac{e^{ik_l r}}{r} \exp\left(-ik_l \frac{\vec{r} \cdot \vec{r}'}{r}\right) \hat{\mathbf{f}}_m \hat{\mathbf{f}}_j \times H_{il}\left(k_i \frac{\vec{r}}{r}, x'_3 + R\right) \|A^{-1}(\vec{k} = k_i \frac{\vec{r}}{r})\|_{lm} - \frac{k_i^2}{4\pi\omega^2\rho} \frac{e^{ik_l r}}{r} \exp\left(-ik_l \frac{\vec{r} \cdot \vec{r}'}{r}\right) (\delta_{mj} - \hat{\mathbf{f}}_m \hat{\mathbf{f}}_j) \times H_{il}\left(k_i \frac{\vec{r}}{r}, x'_3 + R\right) \|A^{-1}(\vec{k} = k_i \frac{\vec{r}}{r})\|_{lm}. \quad (3.44)$$

Here we recall

$$\|A^{-1}(\vec{k} = k_{l,i}(\vec{r}/r))\|_{lm} = \delta_{lm} - \|B(k_{l,i}(\vec{r}/r))\|_{lm}, \quad (3.45)$$

where the $\|B\|$ matrices are given by Eq. (3.12), with Eqs. (3.27), (3.29), (3.32), and (3.33). The quantities H_{il} are obtained by making the appropriate substitution ($\vec{k} \rightarrow k_{l,i}(\vec{r}/r)$) in Eq. (2.25); for example, a fairly typical longitudinal matrix element is readily found to be

$$H_{11}\left(k_i \frac{\vec{r}}{r}, x'_3 + R\right) = \frac{-\mu}{\omega^2\rho} \left(k_i^2 \left\{ \exp\left[-ik_l \frac{x_3}{r} (x'_3 + R)\right] \right. \right.$$

$$- \exp\left[-i\left(k_i^2 - k_i^2 \frac{\vec{r}^2}{r^2}\right)^{\frac{1}{2}} (x'_3 + R)\right] \left. \right\} \frac{x_3^2}{r^2} + \frac{k_i^2}{2} \exp\left[-i\left(k_i^2 - k_i^2 \frac{\vec{r}^2}{r^2}\right)^{\frac{1}{2}} (x'_3 + R)\right]. \quad (3.46)$$

The other elements H_{ij} are similarly trivial to determine but lengthy to reproduce and it seems hardly worthwhile to exhibit them here. We leave our formulation of Eqs. (3.43)–(3.46) with the following remarks:

(i) for a small, deep source ($R \gg x'_3$), we might approximate by setting x'_3 equal to zero in Eq. (3.46). Unfortunately, this does not yield any major computational simplification. In fact, the only apparent situation in which the complexities of R_{ij} are drastically reduced is that for which $\vec{r} = 0$, i.e., when the observation point is directly above the source (it is easily verified that, in this case, most of the quantities R_{ij} vanish);

(ii) unlike the case of a surface source, the surface correction to the radiation from an interior source cannot be separated into terms propagating purely at the velocities of Eqs. (3.28) [cf. the square-root exponents in Eq. (3.46)];

(iii) finally, it is clear from Eq. (3.46) that we chose the signs properly in Eqs. (3.26).

4. CONCLUSION

Equation (3.7) gives a prescription, based on the half-space Green's function presented by Eqs. (2.33) and (2.34), for calculating the elastic disturbance due to the earth's surface when an arbitrary source is embedded on or within that surface. (The earth is idealized as a homogeneous half-space.) We have examined the consequences of Eq. (3.7) in some detail, especially in the case of a source on the surface, and shown in general how the radiation part of the field is to be distinguished. In the combination of our results with those of Ref. 1, a fairly complete prescription for the elastic half-space problem is obtained.

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¹ K. M. Case and J. F. Colwell, *Geophysics* 32, 52 (1967). This reference includes a fairly extensive bibliography.

In addition, the authors are indebted to the referee for pointing out the following useful reference: C. C. Chao, *J. Appl. Mech.* 3, 559 (1960).

² The correspondence between the (\mp) signs and the (l, i) subscripts in Eqs. (2.26) is such that the l goes with the upper ($-$) sign. Note also that the subscripts on k_l^2 and k_i^2 are not coordinate subscripts [cf. Eq. (2.24)] and therefore are not summed over.