The Depth Dependence of Earth Conductivity Upon Surface Potential Data

W. G. KECK* AND W. F. COLBY
University of Michigan, Ann Arbor, Michigan
(Received December 16, 1941)

The problem of determining electrical earth conductivity from the surface, in the case where it is a function of the depth only, is solved by a perturbation method, which formally at least, allows the unperturbed functions to be perfectly arbitrary. Numerical work, however, is restricted to those functions which are available as solutions of the so-called "inverse" problem, a group of which is given. Of this group the case in which the unperturbed conductivity varies exponentially is treated in detail and two examples showing the success of the method are presented. A numerical method of solving integral equations of the Laplace type, which occur in the above, is also submitted.

INTRODUCTION

IN the prospecting for shallow geologic structures by electrical resistivity methods, the problem of determining earth conductivity as a function of depth is of the utmost importance. In its simplest form the problem consists of finding the conductivity function from a knowledge of the surface potential around a single point electrode. Since more elaborate electrode arrangements may be dealt with once the method for the point electrode has been worked out, the majority of papers on the subject deal with this simple arrangement. These papers may be divided into two groups, depending upon the manner in which they approach the problem. In the "direct" method the purpose is to determine the form of the conductivity function from a knowledge of the potential produced at the surface. This is inherently the more difficult and has received attention only recently. The so-called "inverse" problem assumes that the conductivity as a function of the depth is known, and has as its objective the computation of the surface potential. Practical application of the inverse method would consist of having available a large number of solutions giving the surface potentials for chosen conductivity distributions and comparing the experimental data to them. The assumed conductivity function whose surface potentials best agree with those obtained experimentally is then taken as the correct solution. The limitations of such a method are obvious.

The "inverse" problem is of the type ordinarily encountered in potential theory. Thus, means for attacking the problem were already available and it is understandable that most of the early literature on the subject should fall into this classification. Hummel[1] supposed the earth to consist of horizontal layers of different thickness and conductivity and obtained a solution by the method of images so familiar in classical electrical theory. Peters and Bardeen,[2] with the same assumption, gave a direct solution of Laplace's equation in terms of Bessel functions. Ehrenburg and Watson[3] modified Hummel's method by assuming equal thickness of strata, thus obtaining superposition of images, in an attempt at solving the many-layered earth. Roman[4] published numerical tables for computing the potential distribution in the case of a two-layered earth. Finally, Muskat[5] made a detailed analysis of the formal solutions for a many-layered earth, obtaining special formulas appropriate for computation at large and at small distances from the electrode.

It is to be noticed that all the above references deal with a layered earth. The problem of a continuously varying conductivity seems to have been neglected until Slichter,[6] although primarily interested in the direct problem, indicated the solutions for a limited number of cases in the form of infinite integrals.

An early attempt at a "direct" solution was

* Now at Michigan State College, East Lansing, Michigan.

[1] J. N. Hummel, Zeits. Geophysik **5**, 89, 228 (1929).
[2] L. J. Peters and J. Bardeen, Physics **2**, 103 (1932).
[3] D. O. Ehrenburg and R. J. Watson, Trans. A. I. M. M. E. **97**, 423 (1932).
[4] I. Roman, U. S. Bur. Mines, Tech. Paper **502** (1931).
[5] M. Muskat, Physics **4**, 129 (1933).
[6] L. B. Slichter, Physics **4**, 307 (1933).

made by Tagg[7] who prepared a set of charts for the two-layered earth, from which he showed that one could determine the depth to the discontinuity, as well as the conductivity of the second layer, from a knowledge of the potential measured at two different distances from the electrode. Pirson[8] extended Tagg's method with slight variation to the case of three layers. Slichter[6] and Langer[9] collaborated on a formal solution of the "direct" problem. They conclude that a knowledge of the surface potentials alone, suffices to determine uniquely the variation of the conductivity with depth. Unfortunately their method demands the representation of the experimental data in the form of a reciprocal power series. Thus it is likely to be limited in its practical application. In fact Stevenson[10] has shown that in certain cases the method must fail since an expansion of the form postulated is not capable of representing the experimental data. Pekeris,[11] on the assumption of a layered earth, modifies Slichter's method to a determination of the depth to the discontinuities involved and the conductivities of the various layers. It is a graphical method designed to eliminate the power series representation mentioned above.

Stevenson[10] has developed a method of successive approximations in which the zero approximation is that of a uniform earth. He then carries out computations for the first approximation, showing that although the results obtained are not in very good agreement with the true solution, he is able to apply the method to certain problems which the more exact method of Slichter and Langer cannot handle at all.

The present paper attempts to supplement the already existent methods. The basis of the solution rests upon a method of perturbations which is distinct from that which Stevenson has used. In Part I we shall develop the formal solution when the unperturbed functions for the potential and conductivity are perfectly arbitrary. In Part II we shall restrict ourselves to a certain class of the unperturbed functions, showing the type of integral equations which they yield. Of this class the type in which the conductivity is assumed to be exponential will be analyzed with the objective in mind of developing a simplified procedure for numerical treatment. In Part III a method is proposed for numerically inverting integral equations of the Laplace type which occur in Part II. Finally, in Part IV, two examples are treated numerically. These show what success to expect from the perturbation method when the actual conductivity is either continuous or discontinuous. The numerical inversion outlined in Part III is applied in these problems.

[7] G. F. Tagg, Trans. A. I. M. M. E. **110**, 135 (1934).
[8] S. J. Pirson, Trans. A. I. M. M. E. **110**, 148 (1934).
[9] R. E. Langer, Bull. Am. Math. Soc. **39**, 814 (1933).
[10] A. F. Stevenson, Physics **5**, 114 (1934).
[11] C. L. Pekeris, Geophys. **5**, 31 (1940).

PART I. SOLUTION OF THE GENERAL PROBLEM

We will consider the case of an isotropic, infinitely extendant half-space containing a point source of current at its surface. In this half-space the conductivity is assumed to be a function of the depth beneath the surface only, with the further restriction that it shall not vanish at any finite depth. The symmetry in the potential function resulting from such an assumption makes it convenient to use cylindrical coordinates with origin at the electrode. The condition that the divergence of current be zero everywhere, except at the electrode, yields us the partial differential equation:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\sigma'}{\sigma}\frac{\partial\varphi}{\partial z}+\frac{\partial^2\varphi}{\partial z^2}=0, \qquad (1.1)$$

where it is understood that the potential and conductivity functions are represented, respectively, by:

$$\varphi=\varphi(r, z) \quad \text{and} \quad \sigma=\sigma(z) \quad \text{with} \quad \sigma'=d\sigma/dz.$$

The above equation must be solved, subject to the boundary conditions:

(a) That φ approach zero as either r or z approach infinity. (1.2)

(b) That a disk of small, but finite, radius a be assumed as the electrode and is maintained at constant potential. Then $\partial\varphi/\partial z(r, 0)=0$ if $r>a$; and shall have the same value as would be obtained if the earth were a uniform half-space for $r<a$. This is equivalent to saying that the surface layer is uniform to a depth large compared to a.

(c) That the surface potential $\varphi(r, 0)$ is known everywhere.

One may observe here the distinction between the so-called "inverse" and "direct" problems. The former follows the usual pattern of problems in potential theory. In it the conductivity function is known, which with the differential equation governing and a knowledge of the potential or its normal derivative at the boundaries, enables us to determine the potential everywhere. In our "direct" problem we have both the normal derivative *and* the potential given for all points on the surface. This appears to be an overdetermination until we remember that the conductivity function $\sigma(z)$ is now also unknown. Thus we might suspect that the added boundary condition contains the information necessary to determine $\sigma(z)$. Indeed Slichter has shown that the surface potentials do uniquely determine the conductivity as a function of the depth. A complete solution will then yield us both the potential and the conductivity everywhere. However, the potential at depth has little interest for us and it will be shown that we may solve for the conductivity directly from our knowledge of the surface potential.

Equation (1.1) is separable and yields the two ordinary differential equations:

$$R''+R'/r+\lambda^2R=0, \tag{1.3a}$$

$$Z''+\sigma'Z/\sigma-\lambda^2Z=0. \tag{1.3b}$$

The proper solving function for (1.3a) is the zero-order Bessel function, since the requirement (1.2a) must be met. The solution for (1.3b) is not available until a choice of σ'/σ is made.

We will now make use of a perturbation method in which it is assumed that

$$\varphi=\varphi_0+\varphi_1, \tag{1.4a}$$

$$\sigma'/\sigma=\tau=\tau_0+\tau_1, \tag{1.4b}$$

where φ_0 is the solution of (1.1) corresponding to a choice of the conductivity function such that $\sigma_0'/\sigma_0=\tau_0$. Substitution of (1.4) into (1.1) leaves us with:

$$\frac{\partial^2\varphi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_1}{\partial r}+(\tau_0+\tau_1)\frac{\partial\varphi_1}{\partial z}+\tau_1\frac{\partial\varphi_0}{\partial z}+\frac{\partial^2\varphi_1}{\partial z^2}=0. \tag{1.5}$$

This equation is also separable, yielding the same R part as in (1.3a) and the equation in z:

$$Z_1''+(\tau_0+\tau_1)Z_1'-\lambda^2Z_1=-\tau_1Z_0'. \tag{1.6}$$

We shall now express the complete solutions for φ_0 and φ_1 in the form of Fourier-Bessel integrals,

$$\varphi_0(r,z)=\int_0^\infty A_0(\lambda)\times\frac{\sin(\lambda a)}{\lambda a}J_0(\lambda r)Z_0(\lambda,z)d\lambda, \tag{1.7a}$$

$$\varphi_1(r,z)=\int_0^\infty A_1(\lambda)\times\frac{\sin(\lambda a)}{\lambda a}J_0(\lambda r)Z_1(\lambda,z)d\lambda, \tag{1.7b}$$

where the normalizing factors $A_0(\lambda)$ and $A_1(\lambda)$ have been introduced so that $Z_0(\lambda, 0)$ and $Z_1(\lambda, 0)$ become unity. If we multiply the Z equations (1.3b) and (1.6) by Z_1 and Z_0, respectively, and subtract, we obtain:

$$\{Z_0''Z_1-Z_1''Z_0\}+\tau_0\{Z_0'Z_1-Z_1'Z_0\}-\tau_1Z_1'Z_0=\tau_1Z_0'Z_0A_0(\lambda)/A_1(\lambda), \tag{1.8}$$

in which we have used the normalizing functions introduced in (1.7). The term $\tau_1Z_1'Z_0$ is the product of two perturbing functions which we shall neglect in comparison to the other terms in the equation. Equation (1.8) may then be rewritten:

$$\frac{d}{dz}\{Z_0'Z_1-Z_1'Z_0\}+\tau_0\{Z_0'Z_1-Z_1'Z_0\}=\tau_1Z_0'Z_0A_0(\lambda)/A_1(\lambda), \tag{1.9}$$

a differential equation of the first order, which when integrated over all positive z gives us

$$\Big|_0^\infty\sigma_0\{Z_0'Z_1-Z_1'Z_0\}=\frac{A_0(\lambda)}{A_1(\lambda)}\int_0^\infty\sigma_0\tau_1Z_0'Z_0dz. \tag{1.10}$$

From conditions at the infinite boundaries we know that $Z_0(\lambda,\infty)$ and $Z_1(\lambda,\infty)$ are both zero. We may further invoke condition (1.2b) to show that

$$A_0(\lambda)=-\lambda/C\sigma_0(0)Z_0'(\lambda,0)$$

where

$$C=2\pi/I. \tag{1.11}$$

Finally, if $\sigma(0)$ be set equal to $\sigma_0(0)$, which is a convenient choice, we may show that $\partial\varphi_1/\partial z(r, 0)$ must be zero everywhere. This leads immediately to the conclusion that $Z_1'(\lambda, 0)\equiv 0$. The above results are put into (1.10) to obtain:

$$\int_0^\infty \sigma_0\tau_1 Z_0' Z_0 dz = C\sigma_0^2(0)[Z_0'(\lambda, 0)]^2 A_1(\lambda)/\lambda. \quad (1.12)$$

If we look back to (1.7b) we see that the function $A_1(\lambda)$ comprises our known information since that integral may be inverted to yield

$$\frac{A_1(\lambda)}{\lambda}=\frac{\lambda a}{\sin \lambda a}\int_0^\infty \varphi_1(r, 0)J_0(\lambda r)rdr. \quad (1.13)$$

At this point it is convenient to allow a to become vanishingly small, under which condition $\lambda a/\sin(\lambda a)$ approaches unity. By substitution we then have

$$\int_0^\infty \sigma_0\tau_1 Z_0' Z_0 dz = C\sigma_0^2(0)[Z_0'(\lambda, 0)]^2 \times\int_0^\infty \varphi_1(r, 0)J_0(\lambda r)rdr. \quad (1.14)$$

Now remembering that

$$\tau_1=\sigma'/\sigma-\sigma_0'/\sigma_0$$

and making use of a partial integration of (1.14)

$$\int_0^\infty \log\left\{\frac{\sigma}{\sigma_0}\right\}\frac{d}{dz}\{\sigma_0 Z_0' Z_0\}dz = -C\sigma_0^2(0)Z_0'(\lambda, 0)\int_0^\infty \varphi_1(r, 0)J_0(\lambda r)rdr, \quad (1.15)$$

which gives us our solution in the form of an integral equation of the first kind. In this equation $\varphi_1(r, 0)$ represents the experimental data. The indicated integration of the right-hand member may, as is usually necessary in practical applications, be carried out by numerical quadrature and will yield a function of λ only. The unknown conductivity function, which is the objective of our analysis, rests under the left-hand integral in the form $\log \sigma/\sigma_0$. The kernel of the equation, it will be noticed, changes with each new choice of the unperturbed conductivity function. In the event that the inversion of (1.15) is accomplished the required solution for the conductivity is:

$$\sigma(z)=\sigma_0(z)\exp\{\log(\sigma/\sigma_0)\}. \quad (1.16)$$

PART II

A. Some Special Unperturbed Conductivity Functions and the Integral Equations Which They Yield

In determining the kernel of (1.15) it becomes necessary to integrate (1.3b). Unfortunately very few solutions of this differential equation which involve physically occurring conductivities have been reported. Slichter gives a few which occur as special cases under a more general group which we shall now develop. Along with each solution the corresponding integral equation will be given.

If one transforms (1.3b) by means of the relation

$$Y(\lambda, z)=\sigma_0^{\frac{1}{2}}(z)Z_0(\lambda, z), \quad (2.1)$$

we obtain

$$Y''+\left[\left\{\frac{1}{4}\left(\frac{\sigma_0'}{\sigma_0}\right)^2-\frac{1}{2}\frac{\sigma_0''}{\sigma_0}\right\}-\lambda^2\right]Y=0, \quad (2.2)$$

which under the condition that

$$\tfrac{1}{4}(\sigma_0'/\sigma_0)^2-\tfrac{1}{2}\sigma_0''/\sigma_0=-\alpha^2 \quad (2.3)$$

has the solution in terms of Z_0

$$\sigma_0^{\frac{1}{2}}Z_0=\exp[-(\alpha^2+\lambda^2)^{\frac{1}{2}}z]. \quad (2.4)$$

In the above σ_0 must satisfy (2.3) which may be rewritten to read

$$d^2(\sigma_0^{\frac{1}{2}})/dz^2+\alpha^2\sigma_0^{\frac{1}{2}}=0, \quad (2.5)$$

the solution of which is

$$\sigma_0^{\frac{1}{2}}=Ae^{\alpha z}+Be^{-\alpha z}. \quad (2.6)$$

For this case then, (1.15) becomes

$$\int_0^\infty \log\left(\frac{\sigma}{\sigma_0}\right)\left[\lambda^2+\left\{(\alpha^2+\lambda^2)^{\frac{1}{2}} +\alpha\left(\frac{Ae^{\alpha z}-Be^{-\alpha z}}{Ae^{\alpha z}+Be^{-\alpha z}}\right)\right\}^2\right]e^{-2(\alpha^2+\lambda^2)^{\frac{1}{2}}z}dz$$

$$= -C(A+B)^2\{(\alpha^2+\lambda^2)^{\frac{1}{2}}(A+B)$$

$$+\alpha(A-B)\}^2\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr. \quad (2.7)$$

One may specialize (2.6) and obtain considerably simpler expressions for the above. Consider:

Conductivity Uniform

Taking $\alpha=0$ and $\sigma_0(0)$ as unity,

$$\int_0^\infty \log(\sigma/\sigma_0)e^{-2\lambda z}dz$$

$$= -\frac{1}{2}C\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr. \quad (2.8)$$

It is interesting to note that Stevenson's method, which employs this approximation, leads him to an integral equation which cannot be solved by ordinary means. He then transforms this equation into the somewhat more tractable equation of the Laplace type. If one then performs a partial integration of Stevenson's final equation the result is our (2.8). Thus it appears that the present method, in addition to providing more latitude in the choice of unperturbed functions, leads more directly to equations which may be treated numerically.

Conductivity Exponential

With $\sigma_0(z)=e^{2\alpha z}$

$$\int_0^\infty \log(\sigma/\sigma_0)e^{-2(\alpha^2+\lambda^2)^{\frac{1}{2}}z}dz=$$

$$-\frac{1}{2}C\left\{\frac{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}{(\alpha^2+\lambda^2)^{\frac{1}{2}}}\right\}\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr. \quad (2.9)$$

Conductivity Hyperbolic

With $\sigma_0^{\frac{1}{2}}(z)=2\cosh(\alpha z)$

$$\int_0^\infty \log(\sigma/\sigma_0)[\lambda^2+\{(\alpha^2+\lambda^2)^{\frac{1}{2}}$$

$$+\alpha\tanh(\alpha z)\}^2]e^{-2(\alpha^2+\lambda^2)^{\frac{1}{2}}z}dz$$

$$= -C(\alpha^2+\lambda^2)\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr. \quad (2.10)$$

If we go back to (2.5) setting

$$\frac{d^2}{dz^2}(\sigma_0^{\frac{1}{2}})=0 \quad \text{we obtain} \quad \sigma_0^{\frac{1}{2}}=az+b. \quad (2.11)$$

Then our integral equation becomes

$$\int_0^\infty \log(\sigma/\sigma_0)\left[\lambda^2+\left\{\lambda+\frac{a}{(az+b)^2}\right\}^2\right]e^{-2\lambda z}dz$$

$$= -C(b^2\lambda^2+2ab\lambda+a^2)$$

$$\times\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr. \quad (2.12)$$

All of the above integral equations are closely related to the Laplace type. The inversion of the Laplace integral has been accomplished when the integral is known as a function of a complex variable. This is of no help in the present situation however, since we know the (λ) function only along the real axis. In Part III we shall propose a numerical inversion, not only for the equation which is strictly of the Laplace type, but also for integrals like the above which are closely related.

B. Application of the Method when the Conductivity is Assumed to be Exponential

It will facilitate numerical computations considerably if we can find a function $\sigma_0(z)$ such that by simply varying one of its parameters it can be made to approximate any conductivities encountered in practice with sufficient closeness to make our perturbation method effective. Since Stevenson has had some success with an approximation in which the earth is considered of uniform conductivity we have reason to expect that

$$\sigma_0(z)=\sigma_0(0)e^{2\alpha z} \quad (2.13)$$

will answer our needs. It has the desirable feature of being able to approximate conductivities either increasing or decreasing with depth as well as the situation of a uniform earth.

In what follows it will be convenient to choose $\sigma_0(0)$ as our unit for measurement of conductivity. Then according to (2.9)

$$\int_0^\infty g(z)e^{-2(\alpha^2+\lambda^2)^{\frac{1}{2}}z}dz=G(\lambda), \quad (2.14)$$

where

$$g(z)=\log \sigma/\sigma_0$$

and

$$G(\lambda)=-\frac{1}{2}C\left\{\frac{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}{(\alpha^2+\lambda^2)^{\frac{1}{2}}}\right\}\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr.$$

The function $G(\lambda)$ may be determined from our knowledge of $\varphi_1(r, 0)$ by quadrature. It now becomes necessary to establish some criterion for the selection of α. It must be remembered that (2.14) is an approximation which is best when $\sigma_0(z)$ is most nearly like $\sigma(z)$. To make these two functions coincide for all values of z with a single choice of α will, in general, not be possible. However we may so choose α that the assumed fits the actual conductivity well in that range on z in which we are most interested. Correspondingly then our solution will be most reliable in the same range. However, we have no previous knowledge of the actual conductivity so we shall have to study $G(\lambda)$ for this information. Inspection of (2.14) shows that λ controls the interval over which the integration is effective. Thus for large λ information concerning $g(z)$ is restricted to shallow values of depth, while as λ becomes smaller the range of integration becomes deeper. Thus the reciprocal correspondence between the metrics λ and z is evident and in the numerical inversion process described in Part III it is possible to obtain the approximate depth to which the integration is effective for any value of λ. Suppose then that we have decided upon the depth to which we wish to determine the conductivity. From it we obtain the corresponding value of λ_m and so select α that $G(\lambda_m)=0$. This means then that contributions to the integral where $g(z)$ is too small have been offset by contributions where it is too large. Due allowance must be made of course for the fact that the weighting factor in the integral has exponential form. Similar reasoning may be applied to show that the point or points at which the curve for the assumed crosses the true conductivity will be revealed by the values of λ at which $\partial G/\partial\lambda=0$, since if extension of the effective range of integration has not changed the value of the integral, we must conclude that $g(z)$ is zero at the extremity of this range.

Applications of the above criteria in the selection of α, since they are somewhat a matter of judgment, are facilitated considerably by experience. A set of curves showing $G(\lambda)$ against λ for various choices of α is obtained with little additional labor and is very helpful. Generally it is readily apparent which values of α are entirely outside the range of consideration. Experience has shown that a considerable variation of α near the correct choice does not seriously affect the result.

PART III. THE NUMERICAL INVERSION OF INTEGRAL EQUATIONS OF THE LAPLACE AND RELATED TYPES

Consider the integral equation

$$\int_0^\infty g(z)e^{-\beta z}dz=G(\beta). \tag{3.1}$$

This equation may be inverted under suitable restrictions to give

$$g(z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(\beta)e^{\beta z}d\beta. \tag{3.2}$$

A rather complete discussion of the inversion of this integral is given by Pipes[12] in a paper on the operational calculus. Unfortunately the method requires a knowledge of $G(\beta)$ as a function of a complex variable whereas in our problem we know it for real values of β only. An inversion which can be employed however, assumes that

$$g(z)=\sum_{n=0}^{\infty} b_n z^n,\quad G(\beta)=\sum_{n=1}^{\infty} b_n\frac{n!}{\beta^{n+1}} \tag{3.3}$$

and if one can expand $G(\beta)$ in reciprocal powers of β it is merely necessary to identify coefficients in the two series. This method has the limitation that regions of rapidly varying $g(z)$ cannot be well represented by a power series and that a large number of coefficients will have to be evaluated if any detail in the function sought is to be obtained. As an alternative the following numerical method of inversion is submitted.

Assume that an approximate solution of (3.1) is available. If we call this solution $g_0(z)$ we may

[12] Louis A. Pipes, J. App. Phys. 10, 3–5 (1939).

write

$$\int_0^\infty \{g(z)-g_0(z)\}e^{-\beta z}dz$$
$$=G(\beta)-G_0(\beta)=G_1(\beta) \quad (3.4)$$

or

$$\int_0^\infty \bar{g}_1(z)e^{-\beta z}dz=G_1(\beta), \quad (3.5)$$

which is a new integral equation of the same form as (3.1). Obviously we may operate upon it as we did upon (3.1) to obtain its approximate solution $g_1(z)$ and a new residual function $G_2(\beta)$. This process is to be repeated until the residual function $G_n(\beta)$ has been reduced to zero everywhere, within the experimental uncertainty of determining $G(\beta)$ originally. The solution may then be written

$$g(z)=g_0(z)+g_1(z)+\cdots g_n(z). \quad (3.6)$$

The number of terms one will have to determine in this series depends upon the closeness of the approximate solution.

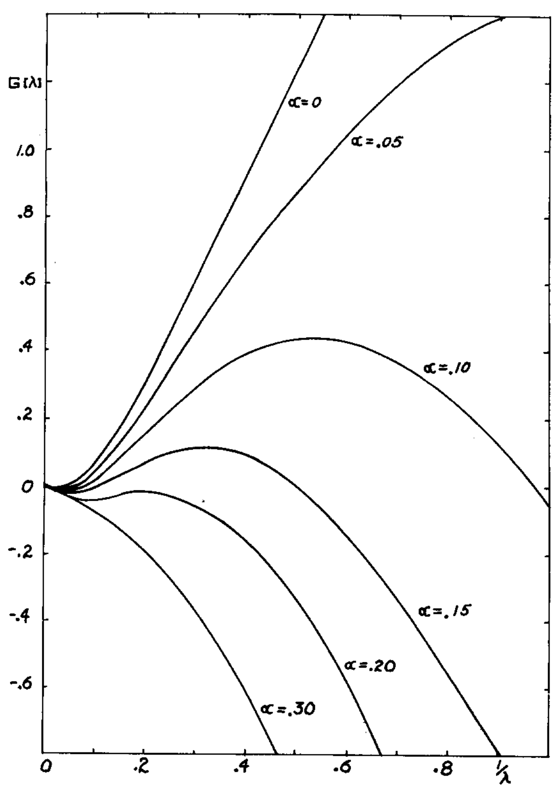

FIG. 1. The two-layered earth. Showing a set of curves useful in determining the most favorable value of the parameter α in the unperturbed conductivity function $e^{2\alpha z}$. Note that certain of the curves show two points at which the slope is zero meaning that there are two places where the unperturbed conductivity curve crosses the actual.

The Approximate Solution

Consider Eq. (3.1) to be rewritten to read:

$$\sum_{k=1}^{m} a_k g(z_k)e^{-\beta_i z_k}+g(z_m)\int_{z_m}^\infty e^{-\beta_i z}dz=G(\beta_i), \quad (3.7)$$

where the subscripts k designate the position of the ordinates at which the value of $g(z)$ is desired. This represents a numerical integration in the interval $0<z<z_m$ from which point on, $g(z)$ is considered constant and the integral evaluated directly. The coefficients a_k will be determined by the particular scheme of numerical quadrature employed. The expression (3.7) is then a set of simultaneous equations, equal in number to the number of points at which we wish to determine $g(z)$. A numerical inspection of these equations reveals that the choice of β, because of the exponential decrement, influences each in such fashion that it may be terminated at any desired value of k. Thus an approximate solution suggests itself, for with some large value of β, say β_1, we might solve (3.7) for the constant value of $g(z)$ which would yield $G(\beta_1)$ and ascribe it to $g(z_1)$ since this ordinate surely had greatest weight in determining it. Proceeding to some smaller value of β we use the previously determined $g(z_1)$ and solve for the constant value, $g(z_2)$, which would yield $G(\beta_2)$ if it were effective in the remaining interval. This process is repeated until we have determined all the $g(z)s$ to $g(z_m)$. The set so determined we shall call the function $g_0(z)$. Insertion of it back into Eq. (3.6) enables us to compute the function $G_0(\beta)$ of (3.4) which when subtracted from $G(\beta)$ leaves the residual function $G_1(\beta)$ and the new integral Eq. (3.5). The procedure from this point on is evident, the process being terminated when $G_n(\beta)$ is everywhere less than the uncertainty in the original function.

It should be pointed out that care in the selection of the β's and the position of the ordinates of $g(z)$ will influence the success of the approximate solution, reducing the labor in the whole inversion process. In the examples pre-

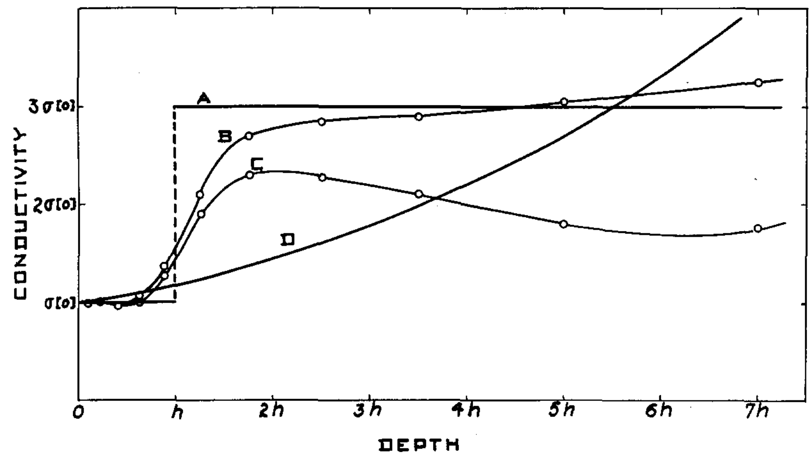

FIG. 2. The two-layered earth. Curve A represents the true conductivity. Curve B is the solution obtained by the present perturbation method for which curve D is the unperturbed conductivity. Curve C is the solution by Stevenson's method in which the unperturbed conductivity is assumed to be uniform.

sented in Part IV four terms in the series (3.6) were sufficient to represent $g(z)$ adequately.

PART IV. NUMERICAL EXAMPLES

In the preceding sections it has been convenient to suppose that $G(\lambda)$ is to be obtained from an evaluation of the integral over $\varphi_1(r, 0)$. However we may obtain it more directly by noting that, utilizing an inversion of (1.7)

$$\int_0^\infty \varphi_1(r,0)J_0(\lambda r)rdr = \int_0^\infty \varphi(r,0)J_0(\lambda r)rdr - A_0(\lambda)/\lambda. \quad (4.1)$$

This step is often, as is the case for σ_0 exponential, a necessity since we cannot obtain $\varphi_0(r, 0)$ except by mechanical quadrature of (1.7). The numerical integration over $\varphi(r, 0)$ may of course be facilitated by taking out the part due to a uniform earth and integrating separately, as is shown in our example.

The Two-Layered Earth

The methods of Tagg or Pekeris could solve this problem with less labor than our own, since they have been designed especially for it. In our own method the requirement is that $\sigma(z)$ be everywhere continuous. In applying it to a discontinuous case we presume that the potential produced is not far different from that for some continuous $\sigma(z)$ which is everywhere equivalent except in the neighborhood of the discontinuity, at which place it has merely rapidly changing values of conductivity. We should like to check how well this assumption is borne out. In the event it is substantiated we shall feel free to apply the method to more practical problems, complicated by having both continuous and discontinuous combinations.

By method of images we know that the potential due to a two-layered earth is

$$\varphi(r,0)=\frac{1}{C\sigma(0)}\left\{\frac{1}{r}+2\sum_{n=1}^{\infty}\frac{k^n}{(r^2+4n^2h^2)^{\frac{1}{2}}}\right\}, \quad (4.2)$$

where the reflection coefficient $k=(\sigma_1-\sigma_2)/(\sigma_1+\sigma_2)$ and the depth to the second layer is h It may then be verified that

$$G(\lambda)=\frac{1}{2}\left\{\frac{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}{(\alpha^2+\lambda^2)^{\frac{1}{2}}}\right\} \times\left\{\frac{1}{\lambda}+\frac{2}{\lambda}\sum_{n=1}^{\infty}k^n e^{-2nh\lambda}-\frac{1}{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}\right\}. \quad (4.3)$$

In the example at hand we have taken $k=-1/2$ which means that the conductivity of the lower layer is three times that of the upper, while h is the unit of measurement of depth. A series of curves for $G(\lambda)$ plotted against $1/\lambda$ appear in Fig. 1 from which the selection $\alpha=0.1$ was made. The inversion of (2.14) was then accomplished by the method outlined in Part III, a system of twelve ordinates being used. The final solution for $\sigma(z)$ appears in Fig. 2 as curve B. For purposes of comparison the same figure contains the actual conductivity (curve A), the unperturbed conductivity dictated by our selection of α (curve D), and the solution obtained by an application of Stevenson's method (curve C).

Conductivity a Continuous Function of Depth

We shall now consider the case where the actual conductivity obeys the equation

$$\sigma^{\frac{1}{2}}(z)=Ae^{az}+Be^{-az}. \qquad (4.4)$$

By a proper choice of constants this function may be made to have a minimum at any desired depth. Now if this problem were given us as an unknown, we should be free to select a function $A_0(\lambda)$ as close as possible to the actual $A(\lambda)$ observed and thus be led to a $\sigma_0(z)$ of the same form as (4.4). The application of our method would then be quite pointless. As an alternative we will again take $\sigma_0=e^{2\alpha z}$ but apply it somewhat differently from what was done in our first example. The variation we are about to describe will greatly enlarge the field of application of this particular function. In fact it is possible that, for most practical purposes, it will yield results sufficiently accurate so that no other function need be considered. This will be an advantage since a single, rather simple, set of equations for the inversion process will suffice regardless of the nature of the true conductivity.

Proceeding as in the first example it may be verified that

$$G(\lambda)=\frac{1}{2}\left\{\frac{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}{(\alpha^2+\lambda^2)^{\frac{1}{2}}}\right\}$$
$$\times\left\{\frac{1}{(\alpha^2+\lambda^2)^{\frac{1}{2}}+a(A-B)}-\frac{1}{(\alpha^2+\lambda^2)^{\frac{1}{2}}+\alpha}\right\}. \qquad (4.5)$$

The constants $A=0.2$, $B=0.8$ and $a=0.3$ were taken so that the minimum in $\sigma(z)$ would occur at a convenient depth. Again the function $G(\lambda)$ was plotted for various values of α. These are shown in Fig. 3. We notice that there is no α which makes $G(\lambda)$ small over any considerable range in λ. That this should be the case is not surprising, since we are attempting to approximate a function which has a minimum by means of one which has none. However, we notice that successively increasing α moves the point at which $\partial G/\partial\lambda$ is zero to smaller values of λ, meaning that the point at which the assumed and true conductivities coincide is moving to greater depths. The possibility thus exists that one might obtain a number of solutions, each utilizing a different α, then build $g(z)$ as a composite of these individual solutions properly weighting each for the range in which it is most reliable. However this would require a great amount of labor. Approximately the same result may be achieved by preparing a composite $G(\lambda)$ before performing the inversion. This will consist of using a different α for each of the equations (3.7) used in the inversion process. As explained in Part III, these equations represent integrations to successively greater depths, depending upon the value of λ_i assigned to each. If, having predetermined these values of λ_i we force α_i to

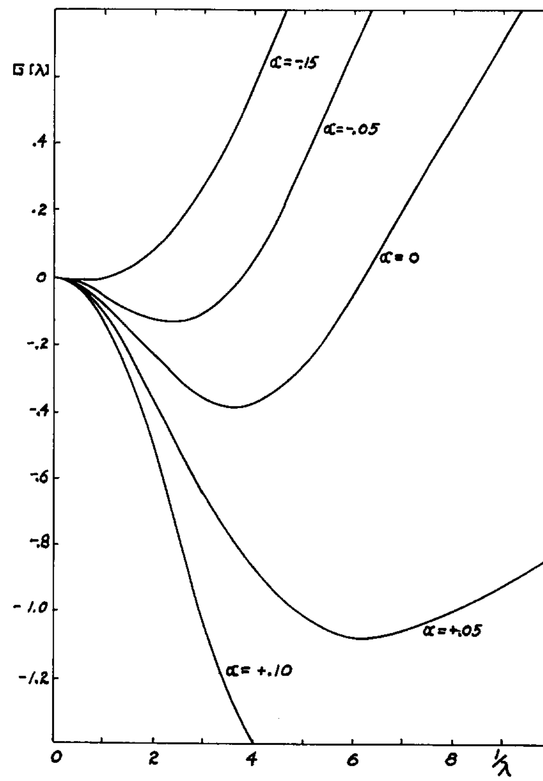

FIG. 3. Curves for determining the most favorable α for the continuous conductivity case. Note that no curve shows more than one place where the slope is zero.

meet the requirement that $\partial G(\lambda_i)/\partial\lambda=0$ it implies coincidence of the assumed and actual conductivities at the lower extremity of the range of integration for each equation. These equations are then solved simultaneously. Thus, since in this process the successive ordinates of $g(z)$ are determined from successively deeper integrals, we have achieved the situation that each ordinate is determined predominently from an equation utilizing its own most favorable unperturbed function.

In Fig. 4 we have plotted the solution for $\sigma(z)$

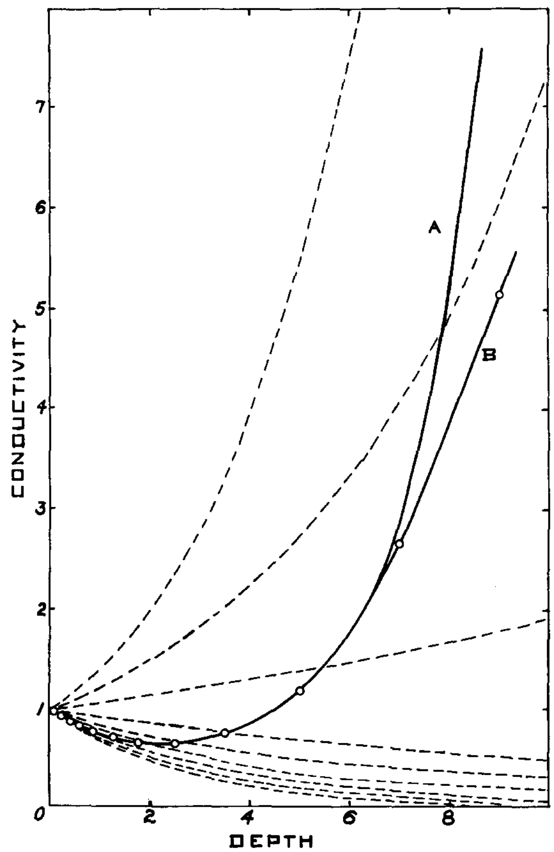

FIG. 4. Curve A is the actual conductivity for the case $\sigma^{\frac{1}{2}} = Ae^{\alpha z} + Be^{-\alpha z}$ while curve B shows the solution obtained by the present method. The dotted curves show the successive unperturbed conductivity functions employed in building the composite function $G(\lambda)$.

so obtained (curve B), comparing it with the true conductivity (curve A). The dotted curves represent $\sigma_0(z)$ for the various α's employed, showing the characteristic predicted that the successive approximations are in better agreement to the true conductivity at greater and greater depths.

CONCLUDING REMARKS

The perturbation method here described, though in principle capable of being repeated as often as desired until convergence to the actual conductivity is attained, is in actual numerical computation restricted to the first approximation. This is so because one knows only a limited number of solutions of the inverse problem. This limitation has made it important that $\sigma_0(z)$ be of such form that, by merely changing a parameter, a wide variety of actual earth conductivity distributions may be approximated. The reasons for selecting an unperturbed conductivity of exponential form have been given in Part IIB.

It is quite in order, then, that one ask concerning the magnitude of the rejected term in Eq. (1.9). From the steps immediately following Eq. (1.9) it may be shown that what one rejects is

$$\int_0^\infty \sigma_0 \tau_1 Z_1' Z_0 dz$$

in comparison with

$$\int_0^\infty \sigma_0 \tau_1 Z_0' Z_0 dz$$

which makes it clear that the success of the approximation depends upon the interval on λ, or what is equivalent, the interval on z in which we are interested. Now we cannot evaluate the above integral directly because of our inability to compute $Z_1(\lambda, z)$. However in Part IIB we have a criterion for selecting the parameter in $\sigma_0(z)$ so that the integral above be small or zero for any particular value of λ, an extension of which results in the method of the second example where continuously shifting the parameter makes the approximation good over a considerable range in depth.

ACKNOWLEDGMENT

The authors wish to express their appreciation to Professor Otto Laporte of the Physics Department, University of Michigan, for much help, especially in obtaining the solution to the general problem of Part I.