Research Notes

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Laminar Free-Convection Heat Transfer from a Needle

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The partial differential equations for laminar free convection over a needle are reduced to ordinary differential equations by a similarity analysis, and the values of local skin friction, heat transfer for various needles are obtained.

The boundary-layer equations for free-convection laminar flow over a vertical needle in nondimensional form are as follows:

Continuity

$$\frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0; \tag{1}$$

Momentum

$$v_z \frac{\partial v_z}{\partial z} + v_r \frac{\partial v_z}{\partial r} = \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \theta; \tag{2}$$

Energy

$$v_z \frac{\partial \theta}{\partial z} + v_r \frac{\partial \theta}{\partial r} = \frac{1}{\Pr}\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right), \tag{3}$$

where z, r, v_z, and v_r are the scale variables given by

$$z = z' \frac{u_0^2}{\beta \bar{g}(T_w - T_\infty)L},$$

$$r = (\mathrm{Re})^{1/2} r', \qquad v_r = (\mathrm{Re})^{1/2} v_r',$$

$$v_z = v_z' \frac{u_0^2}{\beta \bar{g}(T_w - T_\infty)L}, \qquad \mathrm{Re} = \frac{u_0 L}{\nu}. \tag{4}$$

Here z' and r' are nondimensional cylindrical coordinates and v_r', v_z' are nondimensional velocity components, which are obtained by dividing the usual dimensional variables by L (reference length) and u_0 (reference velocity), respectively. The nondimensional temperature θ is defined as $\theta = (T - T_\infty)/(T_w - T_\infty)$ where the subscripts ∞ and w denote reference and wall conditions, respectively. ν denotes kinematic viscosity, β coefficient of thermal expansion, and Pr Prandtl number.

The boundary conditions for the surface and for the outer edge of the boundary layer are

$$r = r(z), \quad v_r = v_z = 0, \quad \theta = 1,$$
$$r \to \infty, \quad v_z = 0, \quad \theta = 0. \tag{5}$$

By using the definition of stream function ψ we can write Eqs. (2) and (3) as

$$-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial z\, \partial r} = \frac{\partial^3 \psi}{\partial r^3} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial \psi}{\partial r} + \theta, \tag{6}$$

$$-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial r} + \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z} = \frac{1}{\Pr}\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right). \tag{7}$$

The boundary conditions (5) then become

$$r = r(z), \quad \frac{\partial \psi}{\partial r} = \frac{\partial \psi}{\partial z} = 0, \quad \theta = 1,$$
$$r \to \infty, \quad \frac{1}{r}\frac{\partial \psi}{\partial r} = 0, \quad \theta = 0. \tag{8}$$

To obtain a similarity transformation for the system represented by (6), (7), and (8), consider the linear transformation group[1] defined by

$$z = A^{\alpha_1}\bar{z}, \quad r = A^{\alpha_2}\bar{r}, \quad \psi = A^{\alpha_3}\bar{\psi}, \quad \theta = A^{\alpha_4}\bar{\theta}, \tag{9}$$

where A is a parameter of transformation and α_1, α_2, α_3, and α_4 are constants.

With these transformations and the conditions of invariancy, it follows that

$$\alpha_3 = \alpha_1 \quad \text{and} \quad \alpha_1 - 3\alpha_2 = \alpha_4. \tag{10}$$

The absolute invariants can then be obtained by eliminating the parameter of transformation A and by putting

$$\eta = \left(\frac{r}{z^\alpha}\right)^2, \qquad f(\eta) = \frac{\psi}{z}, \qquad g(\eta) = \frac{\theta}{z^{1 - 3\alpha}}, \tag{11}$$

where

$$\alpha = \frac{\alpha_2}{\alpha_1} \quad \text{and} \quad 1 - 3\alpha = \frac{\alpha_4}{\alpha_1}.$$

With this transformation, the boundary conditions given by Eq. (8) become

$$r(z) = \eta^{1/2} z^\alpha, \qquad f = f' = 0, \qquad z^{1-3\alpha} g = 1,$$
$$\eta \to \infty, \qquad f'(\infty) = 0, \qquad g = 0, \tag{12}$$

which shows that the boundary conditions can be transformed if $\alpha = \frac{1}{3}$; then the equation of the surface is given by

$$\frac{r^2}{z^{2/3}} = r_0 = \text{const.} \tag{13}$$

Hence, by using the transformation given by Eq. (11) and by taking $\alpha = \frac{1}{3}$, the momentum and energy equations for laminar free convection can be transformed into the following ordinary differential equations (primes denote differentiation with respect to η):

$$8\eta f''' + 8f'' + 4ff'' - 2(f')^2 + g = 0, \tag{14}$$

$$g'' + (\tfrac{1}{2}\,\text{Pr}\,f + 1)\eta g' = 0, \tag{15}$$

with the boundary conditions

$$f(r_0) = f'(r_0) = 0, \qquad g(r_0) = 1,$$

$$f'(\infty) = 0, \qquad g(\infty) = 0. \tag{16}$$

The solution of Eqs. (14) and (15) is obtained by the method described in Ref. 2.

The shear stress τ on the needle is defined as

$$\tau_w = \mu\left(\frac{\partial \bar{v}_z}{\partial \bar{r}}\right)_w, \tag{17}$$

where μ is the dynamic viscosity. In terms of similarity variables, Eq. (17) can be written as

$$\frac{\tau_w}{\frac{1}{2}\rho u_0^2}\frac{\text{Re}^{5/2}}{Gr} = 4(r_0)^{1/2}f''(r_0), \tag{18}$$

where

$$Gr = \frac{\beta\bar{g}(T_w - T_\infty)L^3}{\nu^2}.$$

Thus, the shear parameter $f''(r_0)$ indicates the skin friction on the surface of the needle. Figure 1

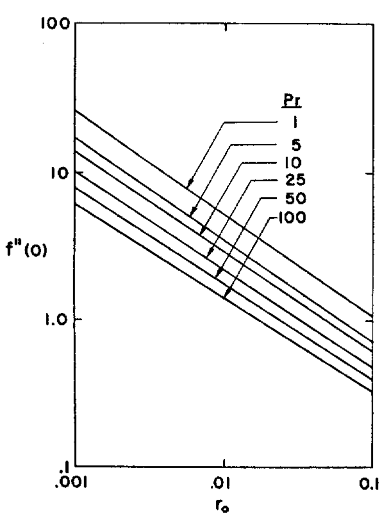

FIG. 1. Skin friction on various needles at different Pr.

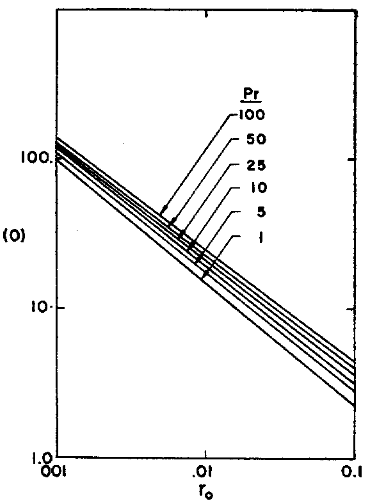

FIG. 2. Heat transfer from various needles at different Pr.

shows the values of $f''(r_0)$ as a function of r_0 and Pr. The results show that the skin friction increases as r_0 is decreased, as it does in forced flow. The dependence of the skin friction on the Prandtl number is such that increasing the Prandtl number decreases the skin friction.

The heat transfer q_w is defined as

$$q_w = -k\left(\frac{\partial T}{\partial \bar{r}}\right)_w, \tag{19}$$

where k is the thermal conductivity. In terms of similarity variables, Eq. (19) becomes

$$q_w = -2\,\frac{k(T_w - T_\infty)}{L}\,(\text{Re}\,r_0)^{1/2}\,\frac{g'(r_0)}{z^{1/3}}. \tag{20}$$

The average heat transfer is, therefore,

$$\bar{q}_w = \frac{1}{z_L}\int_0^{z_L} q_w(z)\,dz = -\frac{3k(T_w - T_\infty)(\text{Re}\,r_0)^{1/2}g'(r_0)}{z_L^{1/3}},$$

where z_L is obtained from Eq. (4) by putting $\bar{z} = L$ (i.e., $z' = 1$), that is,

$$z_L = \frac{u_0^2}{\beta\bar{g}(T_w - T_\infty)L}. \tag{21}$$

The average Nusselt number $\overline{Nu}$ is defined as

$$\overline{Nu} = \frac{\bar{h}L}{k}, \tag{22}$$

where $\bar{h}$ is the average heat transfer coefficient. From Eq. (20) we get

$$\overline{Nu}\,\frac{\text{Re}^{1/6}}{Gr^{1/3}} = -3(r_0)^{1/2}g'(r_0). \tag{23}$$

Thus, the slope of g, $g'(r_0)$, indicates the heat transfer. Figure 2 shows the values of $g'(r_0)$ as a function of r_0 and Pr. Again, decreasing r_0 has the effect of increasing the heat transfer. The dependence of the heat transfer on the Prandtl number is similar to the dependence of the heat transfer in free convection over a plate, in that it increases with increasing Prandtl numbers.

[1] A. G. Hansen, *Similarity Analyses of Boundary Value Problems in Engineering* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964).
[2] A. M. O. Smith and T. Cebeci, Douglas Aircraft Company Report DAC 33735 (1967).

Flow in the Entrance Region of Ducts

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A technique for the solution of an eigenvalue problem arising in entrance region flow is presented along with a simplification of the nonlinear transformation used with such problems. From this, certain other results are obtained.

The purpose of this Note is to point out certain amplifications and simplifications to the results of a paper published in this journal (Ref. 1) regarding flow in the entrance region of ducts. A technique for the solution of the eigenvalue problem posed therein for two dimensional flows will be given as well as a remarkable simplification of the nonlinear transformation and certain results obtained from this simplification.

We refer to Ref. 1 for all definitions, notation, and the problem history. Briefly, the problem is to analyze the incompressible laminar flow of a fluid with constant properties in the entrance region of a straight duct with arbitrary but unchanging cross section. The axis is taken along the positive z direction while x and y are the cross-sectional coordinates. The flow is governed by the momentum and continuity equations with the velocity being zero on the duct walls and equal to the average velocity at the duct entrance. The method employed in Ref. 1 involves a linearization of the inertia terms of the equation of motion by introducing a stretched coordinate in the flow direction. The linearized momentum equation is

$$\epsilon(z)U\frac{\partial u}{\partial z} = \nu\nabla^2 u - \frac{\nu}{A}\oint_c \frac{\partial u}{\partial n}\,ds, \qquad (1)$$

where U is the average velocity, ν is the kinematic viscosity, A is the cross sections area, ∇^2 is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\epsilon(z)$ is an unknown function to be determined. Making the (nonlinear) transformation defined by

$$\frac{dz}{dz^*} = \epsilon \qquad (2)$$

transforms (1) to

$$U\frac{\partial u}{\partial z^*} + \frac{\nu}{A}\oint_c \frac{\partial u}{\partial n}\,ds = \nu\,\nabla^2 u. \qquad (3)$$

The method of solution is to write

$$u = u_e + u_{fd}, \qquad (4)$$

separate (3) into two equations

$$\nabla^2 u_{fd} = \frac{1}{A}\oint_c \frac{\partial u_{fd}}{\partial n}\,ds, \qquad (5)$$

$$\nabla^2 u_e = \frac{U}{\nu}\frac{\partial u_e}{\partial z^*} + \frac{1}{A}\oint \frac{\partial u_e}{\partial n}\,ds \qquad (6)$$

and seek a solution for u_e as

$$u_e = \sum_{i=1}^{\infty} c_i g_i \exp(-\alpha_i^2 z^*). \qquad (7)$$

This reduces the problem to one of solving

$$\nabla^2 g_i + \frac{\alpha_i^2}{\nu} g_i = \frac{1}{A}\oint_c \frac{dg_i}{dn}\,ds \qquad (8)$$

with $g_i = 0$ on C, the boundary of the duct. For a one-dimensional problem (as solved in Ref. 1) this is not a difficult problem. For two-dimensional problems it is considerably more involved. The following is an outline of a technique for the solution of (8) for any cross section which can be mapped onto a rectangle in such a way that the boundary conditions separate, e.g., a rectangle, a circular sector, an annulus.

Let f be a transformation with Jacobian AJ which maps the cross section onto a rectangle and simultaneously makes the coordinates nondimensional. (We use $L = A^{\frac{1}{2}}$ as the characteristic length.)

The problem then becomes

$$\nabla^2 g_i + \alpha_i^2 g_i = \oint_c \frac{dg_i}{dn}\,ds \qquad (9)$$

with $g_i(0, \eta) = g_i(R, \eta) = g_i(\xi, 0) = g_i(\xi, S) = 0$' where $\xi = x/L$, $\eta = y/L$, $R = \xi_{\max}/L$, and $S = \eta_{\max}/L$. We first observe that the right-hand side of (9) is a constant for each g_i and write

$$\oint_c \frac{dg_i}{dn}\,ds \equiv M_i = \sum_{k=1,3,\cdots} \frac{4M_i}{\pi k}\sin\frac{k\pi\xi}{R}. \qquad (10)$$

Assuming

$$g_i(\xi, \eta) = \sum_{k=1,3,\cdots} C_{ik}(\eta)\sin\frac{k\pi\xi}{R} \qquad (11)$$

and using (10) and (11) in (9) gives a second-order ordinary differential equation for $C_{ik}(\eta)$,