Oblique Incidence of an Electromagnetic Wave on Plasma Half-Space

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(Received 28 April 1966; final manuscript received 10 November 1966)

A radio frequency electromagnetic wave, polarized in the plane of incidence is incident obliquely upon a plasma half-space, where the equilibrium plasma is taken to be homogeneous and isotropic. Employing the specular boundary condition, an exact solution of the coupled Maxwell–Vlasov equations is derived, yielding both transverse and longitudinal waves in the plasma region. The reflection coefficient is derived, and approximately evaluated.

INTRODUCTION

It is well known that transverse and longitudinal waves can propagate in a homogeneous unbounded plasma. It is of interest to know precisely how such waves are generated when vacuum or dielectric radio frequency electromagnetic waves arrive at a bounded plasma. Based upon the coupled Maxwell–Vlasov equations, this problem has been treated by Felderhof[1] for the special case where the electromagnetic wave is incident normally to a half-space filled with a homogeneous isotropic plasma. In this situation only transverse waves are generated. A somewhat similar problem was treated by Shure[2] for the plasma capacitor where only longitudinal waves are generated.

The fundamental boundary condition employed by the above authors was the specular reflection condition, corresponding to the physical situation associated with an infinitesimally thin sheath.[3] In this connection it should be pointed out that the finite sheath case was treated by Pavkovich.[4] Employing the coupled Maxwell–Vlasov equations, he investigated the generation of the electric field component normal to the interface separating the sheath from the semi-infinite plasma half-space, where the sheath potential had parabolic spatial behavior.

The problem that will be treated here is a generalization of Felderhof's case, with the electromagnetic wave incident obliquely upon the interface. In this situation, both transverse and longitudinal waves are generated. The procedure that will be used is based upon Felderhof's[1] analysis as well as Van Kampen's[5] treatment for longitudinal waves, and is outlined as follows. The plane wave will be taken to be polarized in the plane of incidence. The transmitted field in the plasma region will be represented in terms of a linear combination of normal modes, i.e. particular solutions of the coupled Maxwell–Vlasov equations, each of which is associated with a particular value of the propagation constant in a direction normal to the surface. The contribution arising from the continuous portion of the spectrum is obtained in terms of an integral representation containing two unknown functions. To this is added the discrete spectrum contribution. The appropriate conditions are derived governing the existence and number of discrete modes. It is shown that the discrete modes do not exist, provided that the angle of incidence, measured from the normal to the interface, is less than a certain critical angle. For small Debye lengths, this critical angle corresponds to the angle for which complete reflection occurs when a wave is incident upon an optically less dense dielectric medium. The specular boundary condition is employed, and the reduction leads to two coupled integral equations involving the unknown functions associated with the continuous spectrum. Exact solutions are obtained, and explicit expressions are given for the field components. Finally, the reflection coefficient is computed for two cases.

BASIC EQUATIONS

The appropriate equations for the plasma region are Maxwell's equations and Vlasov's linearized equation, with harmonic time dependence $\exp(-i\omega t)$ assumed

$$-i\omega f + \mathbf{v}\cdot\nabla f = (e/m)\mathbf{E}\cdot\nabla_v f_0.$$

The electronic charge is denoted by $-e$, and the unperturbed distribution function f_0 will be taken to be Maxwellian

$$f_0(\mathbf{v}) = n(m/2\pi\kappa T)^{\frac{3}{2}} \exp[-(m/2\kappa T)\mathbf{v}^2].$$

Outside the plasma region, Maxwell's equations hold with $\rho = \mathbf{j} = 0$.

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[2] F. C. Shure, J. Nucl. Energy C6, 1 (1964).
[3] D. Bohm and E. P. Gross, Phys. Rev. 79, 992 (1950).
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A Cartesian coordinate system will be employed such that the $z = 0$ plane represents the interface, and the positive z half-space contains the plasma. An electromagnetic wave, polarized in the plane of incidence, will be incident upon the vacuum side of the interface, producing a reflected electromagnetic wave in the vacuum region, and transmitted waves in the plasma region. The direction of the magnetic vector will be taken to lie parallel to the y axis. The total (incident plus reflected) electric and magnetic field components at the interface will have the form

$$E_x = \cos\alpha(1 - R)(\mu_0/\epsilon_0)^{\frac{1}{2}} \exp(ik_0 \sin\alpha x), \tag{1}$$

$$H_y = (1 + R) \exp(ik_0 \sin\alpha x), \tag{2}$$

where k_0 is the wavenumber in the vacuum half-space. The direction of propagation of the incident wave is given by the vector $(\sin\alpha, 0, \cos\alpha)$, and the voltage reflection coefficient is given by $-R$. For further analyses the positive real quantity k_x, where

$$k_x = k_0 \sin\alpha \tag{3}$$

is used. Thus in the plasma region, the required solutions has the form

$$\mathbf{E} = \exp(ik_x x)[E_x(z), 0, E_z(z)],$$
$$\mathbf{H} = \exp(ik_x x)[0, H_y(z), 0],$$

with similar x dependence for the distribution function. The field will be represented in terms of a linear combination of normal modes where each mode is associated with a z dependence of the form $\exp(ik_z z)$. From Maxwell's equations, it follows that the particular mode, associated with the wavenumber k_z, must satisfy the following two equations:

$$-i\omega\epsilon_0(k_x E_x + k_z E_z) + (k_x j_x + k_z j_z) = 0, \tag{4}$$

$$(k_0^2 - k_x^2 - k_z^2)(k_z E_x - k_x E_z) + i\omega\mu_0(k_z j_x - k_x j_z) = 0, \tag{5}$$

together with Vlasov's equation

$$f(\omega - v_x k_x - v_z k_z) = -(ie/\kappa T)(E_x v_x + E_z v_z) f_0. \tag{6}$$

The modes are restricted by the requirement that only outgoing waves are considered in the plasma domain. This implies that only modes associated with values of k_z such that $0 \le \arg k_z < \pi$ are considered.

CONTINUOUS SPECTRUM

The modes associated with the portion of the continuous spectrum ($0 \le k_z \le \infty$) corresponding to the outgoing wave requirement are considered first. Two classes of modes are considered, transverse waves and longitudinal waves.

Generalizing the analysis of Felderhof, who treated the special case where $k_x = 0$, the particular transverse modes can be represented in the form

$$E_x^t = (ieu^3/\omega^2\epsilon_0)k_z \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$E_z^t = -(ieu^3/\omega^2\epsilon_0)k_x \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$H_y^t = (ieu/\omega\mu_0\epsilon_0) \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$f^t = \{(e^2u^3/\omega^2\epsilon_0\kappa T)P[f_0(\mathbf{v})v_t/(u - v_p)] + \delta(u - v_p)\Gamma^t(v_t, u)\} \exp(i\mathbf{k}\cdot\mathbf{x}),$$

where $\mathbf{k} = (k_x, 0, k_z)$, and u is related to k_z by the expression

$$k = (k_x^2 + k_z^2)^{\frac{1}{2}} = \omega/u. \tag{7}$$

The velocity components v_t and v_p represent components of velocity transverse and parallel to the direction of propagation, respectively, and are given by the following relations:

$$v_t = (k_z v_x - k_x v_z)/k, \quad v_p = (k_x v_x + k_z v_z)/k. \tag{8}$$

In order for the wave to be transverse, i.e., the total charge is zero, $\Gamma^t(v_t, u)$ must satisfy the relation

$$\int_{-\infty}^{\infty} \Gamma^t(v_t, u)\, dv_t = 0, \tag{9}$$

and the requirement that the mode be a solution of the coupled Maxwell–Vlasov equations, yields the relation

$$\int_{-\infty}^{\infty} v_t\Gamma^t(v_t, u)\, dv_t \equiv \lambda^t(u) = u^2 - c^2 + \frac{\omega_p^2}{\omega^2}u^3 P\int_{-\infty}^{\infty}\frac{F_0(v)}{v - u}\, dv, \tag{10}$$

where the Cauchy principle value is taken. $F_0(v)$ is given by the relation

$$F_0(v) = (m/2\kappa T\pi)^{\frac{1}{2}} \exp[-(m/2\kappa T)v^2], \tag{11}$$

and ω_p is the plasma frequency. $\Gamma^t(v_t, u)$ is unspecified except for relations (9) and (10).

The longitudinal modes have the form (for the continuous spectrum)

$$E_x^l = (ie/\omega^2\epsilon_0)k_x u^2 \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$E_z^l = (ie/\omega^2\epsilon_0)k_z u^2 \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$f^l(\mathbf{x}, \mathbf{v}) = \{(e^2u^2/\epsilon_0\omega^2\kappa T)P[f_0(\mathbf{v})v_p/(u - v_p)] + \delta(u - v_p)\Gamma^l(v_t, u)\} \exp(i\mathbf{k}\cdot\mathbf{x}),$$

where $\Gamma^l(v_t, u)$ has the restrictions

$$\int_{-\infty}^{\infty} v_t \Gamma^l(v_t, u)\, dv_t = 0, \quad (12)$$

$$\int_{-\infty}^{\infty} \Gamma^l(v_t, u)\, dv_t \equiv \lambda^l(u) = 1 + \frac{u^2}{\lambda_D^2 \omega^2} P \int_{-\infty}^{\infty} \frac{F_0(v)v}{v-u}\, dv, \quad (13)$$

with λ_D being the Debye length.

In the analysis that follows, the variable u takes the predominant role in place of k_z. Expressing k_z in terms of u, it follows from Eq. (7) that for u real

$$k_z = (k_x/u)[\Omega^2 - u^2]^{\frac{1}{2}},$$

with

$$\Omega = \omega/k_x, \quad (14)$$

the domains $0 \le k_z \le \infty$ and $-\infty \le k_z \le 0$ correspond, respectively, to the domains $0 \le u \le \Omega$ and $-\Omega \le u \le 0$. The continuous spectrum of the total field is then comprised of a linear combination of transverse and longitudinal modes for u in the range $0 \le u \le \Omega$ and is expressible in the following form as demonstrated for the x component of the electric field:

$$E_x^c = \frac{iek_x}{\omega^2 \epsilon_0} \int_0^{\Omega} \{u^2[\Omega^2 - u^2]^{\frac{1}{2}} A(u) + u^2 B(u)\} \cdot \exp(i\mathbf{k}\cdot\mathbf{x})\, du,$$

where

$$\mathbf{k} = [k_x, 0, k_x/u(\Omega^2 - u^2)^{\frac{1}{2}}].$$

Apart from the factor $\exp(ik_x x)$, the portion of the distribution function arising from the continuous spectrum has the following form at the interface $z = 0$:

$$f^c(v_x, v_z; 0) = \frac{e^2 f_0(\mathbf{v})}{\omega^2 \epsilon_0 \kappa T} \int_0^{\Omega} \frac{u^3 A(u) v_t + u^2 B(u) v_p}{u - v_p}\, du + \int_0^{\Omega} \delta(u - v_p)[A(u)\Gamma^t(v_t, u) + B(u)\Gamma^l(v_t, u)]\, du, \quad (15)$$

with

$$v_p = [uv_x + (\Omega^2 - u^2)^{\frac{1}{2}} v_z]/\Omega, \quad (16)$$

$$v_t = [(\Omega^2 - u^2)^{\frac{1}{2}} v_x - uv_z]/\Omega. \quad (17)$$

The above representation (where the superscript c refers to the contribution from the continuous spectrum) contains two unknown functions $A(u)$ and $B(u)$ associated with the transverse and longitudinal waves, respectively, as well as the functions $\Gamma^t(v_t, u)$ and $\Gamma^l(v_t, u)$ which are only partially determined by the relations (9), (10), (12) and (13). However, it is seen that, on applying the specular boundary condition at the interface, $A(u)$ and $B(u)$ can be completely determined.

Before considering the contribution due to the discrete spectrum, some preliminary analysis are performed which are required later in the application of the specular boundary condition. From relations (16) and (17) it is seen that

$$v_p(u, v_x, -v_z) = -v_p(-u, v_x, v_z),$$

$$v_t(u, v_x, -v_z) = v_t(-u, v_x, v_z);$$

hence it follows that after replacing v_z by $-v_z$ in (15) and changing the variable of integration from u to $-u$,

$$f^c(v_x, -v_z; 0) = \frac{e^2 f_0(\mathbf{v})}{\omega^2 \epsilon_0 \kappa T} \cdot \int_{-\Omega}^{0} \frac{u^3 A(-u) v_t + u^2 B(-u) v_p}{u - v_p}\, du + \int_{-\Omega}^{0} \delta(u - v_p)[A(-u)\Gamma^t(v_t, -u) + B(-u)\Gamma^l(v_t, -u)]\, du. \quad (18)$$

The following functions are now introduced:

$$\psi(u) = uA(u), \quad u > 0,$$
$$\quad\; -uA(-u), \quad u < 0;$$
$$\phi(u) = u(\Omega^2 - u^2)^{\frac{1}{2}} B(u), \quad u > 0,$$
$$\quad\; -u(\Omega^2 - u^2)^{\frac{1}{2}} B(-u), \quad u < 0.$$

For further simplification, the new independent variables s and t, given by the relations

$$s = \Omega v_z [v_z^2 + (\Omega - v_x)^2]^{-\frac{1}{2}}, \quad (19)$$

$$t = \operatorname{sgn}(v_x - \Omega)[v_z^2 + v_x(v_x - \Omega)] \cdot [v_z^2 + (\Omega - v_x)^2]^{-\frac{1}{2}}, \quad (20)$$

are introduced in place of v_x and v_z. For both v_x and v_z ranging in values from $-\infty$ to $+\infty$, s and t vary over the intervals $(-\Omega, \Omega)$ and $(-\infty, \infty)$, respectively.

On combining Eqs. (17) and (18), and introducing the functions $\psi(u)$ and $\phi(u)$ and the new independent variables s and t, it can be shown that

$$\Omega v_z[f^c(v_x, v_z; 0) - f^c(v_x, -v_z; 0)] = \frac{sF_0(s)F_0(t)}{\omega^2 \lambda_D^2} \left[s \int_{-\Omega}^{\Omega} \left(\frac{u\phi(u)}{u - s} - u^2 \psi(u) \right) du + (\Omega^2 - s^2)^{\frac{1}{2}} t \int_{-\Omega}^{\Omega} \frac{u^2 \psi(u)}{u - s}\, du \right] + \operatorname{sgn}[(\Omega^2 - s^2)^{\frac{1}{2}} - t][(\Omega^2 - s^2)^{\frac{1}{2}} \psi(s) \cdot \Gamma^t(t, |s|) + \phi(s)\Gamma^l(t, |s|)]. \quad (21)$$

Expression (21) is used later in connection with the determination of the unknown functions $\psi(u)$ and $\phi(u)$, through the application of the specular boundary condition. The importance of this representation is that, with s fixed, integration with respect to t from $-\infty$ to $+\infty$ replaces the functions $\Gamma^t(t, s)$ and $\Gamma^l(t, s)$ by known quantities.

DISCRETE SPECTRUM

Unlike the continuous spectrum, the modes for the discrete spectrum cannot be split up into longitudinal and transverse waves. Thus the field components corresponding to a particular discrete mode are given by

$$E_x = (ie/\omega^2\epsilon_0)[u^3k_zA + u^2k_xB] \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$E_z = (ie/\omega^2\epsilon_0)[-u^3k_xA + u^2k_zB] \exp(i\mathbf{k}\cdot\mathbf{x}),$$
$$H_y = (ie/\omega\epsilon_0\mu_0)uA \exp(i\mathbf{k}\cdot\mathbf{x}).$$

Extending the domain of variable u defined by (7) into the complex plane, k_z is defined in terms of a function of u which is analytic outside the cut $(-\Omega < \mathrm{Re}\, u < \Omega)$ on the real axis, as follows:

$$k_z = (ik_x/u)(u^2 - \Omega^2)^{\frac{1}{2}},$$

which implies that $0 \le \arg k_z < \pi$, outside the cut. As u approaches the cut from above or below, k_z takes the form

$$k_z(u + i0) = -(k_x/u)(\Omega^2 - u^2)^{\frac{1}{2}},$$
$$k_z(u - i0) = (k_x/u)(\Omega^2 - u^2)^{\frac{1}{2}}.$$

From Eq. (6) it follows that the distribution function is given by

$$f = (e^2/\omega^2\epsilon_0\kappa T)u^2[(Auv_t + Bv_p)/(u - v_p)]f_0(v),$$

where

$$v_p = [v_xu + i(u^2 - \Omega^2)^{\frac{1}{2}}v_z]/\Omega,$$
$$v_t = [i(u^2 - \Omega^2)^{\frac{1}{2}}v_x - uv_z]/\Omega.$$

The constants A and B as well as the particular values of u, are obtained from the requirement that the above representations must satisfy (4) and (5), yielding the two linear homogeneous equations

$$B\left[1 + \frac{e^2u^2}{\omega^2\epsilon_0kT}\int\frac{f_0(v)v_p}{v_p - u}\,d\mathbf{v}\right] + \frac{Ae^2u^3}{\omega^2\epsilon_0\kappa T}\int\frac{f_0(\mathbf{v})v_t}{v_p - u}\,d\mathbf{v} = 0, \quad (22)$$

$$\frac{Be^2u^3}{\omega^2\epsilon_0kT}\int\frac{f_0(\mathbf{v})v_t}{v_p - u}\,d\mathbf{v} + A\left[u^2 - c^2 + \frac{e^2u^3}{\omega^2\epsilon_0\kappa T}\int\frac{f_0(\mathbf{v})v_t^2}{v_p - u}\,d\mathbf{v}\right] = 0. \quad (23)$$

The coefficients of A and B in the above systems of equations can be simplified by changing the variables of integration (v_x, v_z) to θ and r where $v_x = \Omega + r\cos\theta$ and $v_z = r\sin\theta$, with the domain of integration given by $-\pi \le \theta \le \pi$, $0 \le r < \infty$. Retaining only even functions of θ, integrating with respect to r, and then replacing the variable θ by y where $y = \Omega\sin\theta$, one can obtain the following results:

$$\int\frac{f_0(\mathbf{v})v_t}{v_p - u}\,d\mathbf{v} = -i\pi\omega^2\lambda_D^2n_e\beta,$$
$$\int\frac{f_0(\mathbf{v})v_p}{v_p - u}\,d\mathbf{v} = n_e[1 + \pi\omega^2\lambda_D^2uF^*(u)], \quad (24)$$
$$\int\frac{f_0(\mathbf{v})v_t^2}{v_p - u}\,d\mathbf{v} = \pi n_e\omega^2\lambda_D^2\left[\frac{\kappa T}{m}F^*(u) + \beta(u^2 - \Omega^2)^{\frac{1}{2}}\right],$$

where $-\pi\omega^2\lambda_D^2\beta = \exp[-(m/2\kappa T)\Omega^2]$. The function $F^*(u)$ is defined as follows. For complex u outside the cut, $F^*(u)$ is given by

$$F^*(u) = \frac{1}{\pi}\int_{-\Omega}^{\Omega}\frac{f^*(y)}{y - u}\,dy,$$

where

$$f^*(y) = \frac{2}{\omega^2\lambda_D^2}F_0(y)\int_0^{y_0}F_0(t)\,dt, \qquad y_0 = (\Omega^2 - y^2)^{\frac{1}{2}}.$$

For u approaching the cut, the following relations hold:

$$F^*(u + i0) - F^*(u - i0) = 2if^*(u),$$
$$F^*(u + i0) + F^*(u - i0) = \frac{2}{\pi}P\int_{-\Omega}^{\Omega}\frac{f^*(y)}{y - u}\,dy,$$

where the Cauchy principle value of the integral is taken. Further simplification is achieved by the introduction of the following functions:

$$\Lambda^t(u) = u^2 - c^2 + (\kappa T/m)\pi u^3F^*(u),$$
$$\Lambda^l(u) = 1 + (u^2/\lambda_D^2\omega^2) + \pi u^3F^*(u),$$

which are analytic in the cut-u plane. Using the following identity:

$$P\int_{-\Omega}^{\Omega}\frac{f^*(t)}{t - u}\,dt = \frac{1}{\omega^2\lambda_D^2}P\int_{-\infty}^{\infty}\frac{F_0(t)}{t - u}\,dt,$$

which holds for $-\Omega \le u \le \Omega$, it can be shown that the above functions have the following important property on the cut:

$$\Lambda^t(u + i0) + \Lambda^t(u - i0) = 2\lambda^t(u),$$
$$\Lambda^l(u + i0) + \Lambda^l(u - i0) = 2\lambda^l(u).$$

The system of equations (22) and (23) can now be expressed in the following form:

$$B\Lambda^l(u) - i\pi\beta u^3 A = 0,$$

$$-Bi\pi\beta u^3 + A[\Lambda^t(u) + \beta\pi u^3(u^2 - \Omega^2)^{\frac{1}{2}}] = 0.$$

Nontrivial solutions exist, provided that the determinant

$$L(u) = \Lambda^l(u)[\Lambda^t(u) + \beta\pi u^3(u^2 - \Omega^2)^{\frac{1}{2}}] + \pi^2\beta^2u^6$$

vanishes. The values u_i, of the discrete spectrum are thus given by the zeros of the equation

$$L(u_i) = 0.$$

Because of the complicated nature of the function, $L(u)$, the evaluation of the roots u_i, is not carried out. Instead, a precise relation is given, specifying the number of zeros as a function of plasma frequency, operating frequency, debye length, and the incident wavelength component λ_x. The approach is based on the result that the number of zeros of a function which is analytic in a region enclosed by a contour, is equal to the total change in phase of the function taken around the contour, divided by 2π. Since $L(u)$, is analytic in the cut plane, and behaves like u^2 as $|u|$ approaches infinity, then the total number of roots N is given by

$$N = 2 + (1/2\pi)\, \Delta_c \arg L(u),$$

where $\Delta_c \arg L(u)$ is the change in phase of $L(u)$ taken in a clockwise direction around the cut. Using the result

$$L(u + i0) = (\lambda^l + i\pi u^3 f^*)(\lambda^t + i\pi u^3 g) + \pi^2\beta^2u^6, \quad (25)$$

where

$$g(u) = (\kappa T/m)f^*(u) + \beta(\Omega^2 - u^2)^{\frac{1}{2}}, \qquad -\Omega \leq u \leq \Omega,$$

it follows that

$$N = 2 + \frac{2}{\pi}\,\Delta_{(0,\Omega)} \tan^{-1}\left(\frac{\pi u^3(f^*\lambda^t + g\lambda^l)}{\lambda^l\lambda^t - \pi^2u^6f^*g + \pi^2\beta^2u^6}\right). \quad (26)$$

In order to interpret this relation from a physical point of view, the particular case expressed by the inequality

$$\omega_p\lambda_D \ll \omega\lambda_x \quad (27)$$

is considered. This inequality implies that the term containing β^2 in the denominator if expression (26) can be neglected, yielding

$$N = 2 + \frac{2}{\pi}\,\Delta_{(0,\Omega)}\left[\tan^{-1}\left(\frac{\pi u^3 g}{\lambda^t}\right) + \tan^{-1}\left(\frac{\pi u^3 f^*}{\lambda^l}\right)\right].$$

The above inequality (27) also implies that

$$\lambda^l(\Omega) \sim \left(1 - \frac{\omega_p^2}{\omega^2}\right) - \left(\frac{\omega_p^2}{\omega}\right)\frac{3\kappa T}{m\Omega^2}.$$

In addition, $\lambda^t(u)$ can be approximated by the expression

$$\lambda^t(u) \sim u^2\left(1 - \frac{\omega_p^2}{\omega^2}\right) - c^2 - \frac{\omega_p^2}{\omega^2}\frac{\kappa T}{m}O(1), \quad (28)$$

where $O(1)$ is a term the order of unity. Precise analysis of $\lambda^t(u)$, indicates that it has no roots when the plasma frequency is greater than the operating frequency, otherwise it may have one root in the range $0 < u < \Omega$. It is seen from Eq. (28) that if

$$\Omega^2(\omega^2 - \omega_p^2) > \omega^2c^2 \quad (29)$$

then a root does occur in this range. Since $\lambda^l(u)$ is unity for vanishing u, then there will be either no roots, or an even number of roots when the plasma frequency is sufficiently less than the operating frequency, otherwise $\lambda^l(u)$ may have one root in the interval $(0 < u < \Omega)$.

Since $g(u)$ is a positive real even function, vanishing at $u = \Omega$, and since $\lambda^t(0) = -c^2$, then it follows that

$$\Delta_{(0,\Omega)} \tan^{-1}\left(\frac{\pi u^3 g}{\lambda^t}\right) = \begin{cases} 0 & \text{for} \quad \Omega^2(\omega^2 - \omega_p^2) < \omega^2c^2, \\ -\pi & \text{for} \quad \Omega^2(\omega^2 - \omega_p^2) > \omega^2c^2, \end{cases}$$

and similarly

$$\Delta_{(0,\Omega)} \tan^{-1}\left(\frac{\pi u^3 f^*}{\lambda^l}\right) = \begin{cases} 0 & \text{for} \quad \omega_p < \omega, \\ \pi & \text{for} \quad \omega_p \geq \omega. \end{cases}$$

From this it is seen that $L(u) = 0$ will have no roots, implying that there are no discrete modes, when the operating frequency is sufficiently greater than the plasma frequency such that inequality (29) holds. Otherwise, the discrete modes exist, and there are either two or four roots of $L(u) = 0$ according as $\omega_p < \omega$ or $\omega_p \geq \omega$. The inequality $\Omega^2(\omega^2 - \omega_p^2) < c^2\omega^2$ governing the existence of the discrete modes can be rewritten in the following form:

$$\sin\alpha > \epsilon,$$

where α is the angle of incidence, and $\epsilon = (1 - \omega_p^2/\omega^2)^{\frac{1}{2}}$ is the relative permittivity of the cold plasma. This inequality corresponds to the case of oblique incidence upon a surface bounding an optically less dense medium, wherein complete reflection occurs.

Since $L(u)$ is an even function of u, implying that both u_i and $-u_i$ are roots, and since k_z is also an even function of u, the total number of distinct discrete modes is either one or two according as

$\omega_p < \omega$ or $\omega_p \geq \omega$. However, in the expression for the portion of the field components arising from the discrete position, the summation will be taken over all values of u_i. The component E^d_x is given by the relation

$$E^d_x = \frac{iek_x}{\omega^2\epsilon_0} \sum [i(u_i^2 - \Omega^2)^{\frac{1}{2}} u_i^2 A_i + u_i^2 B_i] \exp(i\mathbf{k}_i\cdot\mathbf{x})$$

with the coefficients A_i and B_i related as follows:

$$B_i\Lambda^l(u_i) = i\pi\beta u_i^3 A_i.$$

In the application of the specular boundary condition the following relation is required:

$$\begin{aligned}\Omega v_z[f^d(v_x, v_z, 0) - f^d(v_x, -v_z, 0)] &= \frac{s^2 F_0(s) F_0(t)}{\omega^2\lambda_D^2} \sum_i \frac{u_i^3}{(u_i^2 - s^2)} \{2A_i[s^2 - u_i^2 \\ &\quad + (\Omega^2 - s^2)^{\frac{1}{2}} t] + 2iB_i(u_i^2 - \Omega^2)^{\frac{1}{2}}\},\end{aligned} \tag{30}$$

where s and t are specified by Eqs. (19) and (20). In the above expression the factor $\exp(ik_x x)$ has been dropped.

EVALUATION OF THE UNKNOWN QUANTITIES

The unknown functions $A(u)$ and $B(u)$ will now be determined explicitly through the application of the specular boundary condition $f(v_x, v_z, 0) = f(v_x, -v_z, 0)$, upon the total distribution function. This condition can be expressed in the form

$$\begin{aligned}v_z\Omega[f^c(v_x, v_z, 0) - f^c(v_x, -v_z, 0)] \\ + \Omega v_z[f^d(u_x, v_z, 0) - f^d(v_x, -v_z, 0)] = 0,\end{aligned} \tag{31}$$

where the subscripts c and d refer to the components arising from the continuous and discrete spectrums, respectively. The explicit representations (21) and (30) can now be used, where the variables v_x and v_z are replaced by the new variables s and t. Since Eq. (31) must hold for all real values of v_x and v_z, it must hold for all values of s and t such that $-\Omega \leq s \leq \Omega$, and $-\infty \leq t \leq \infty$. Further reduction of Eq. (31) is achieved by multiplication with respect to sgn $[(\Omega^2 - s^2)^{\frac{1}{2}} - t]$, followed by integration with respect to t over its complete domain. Using relations (9) and (11) the resulting equation is obtained

$$\begin{aligned}&\phi(s)\lambda^l(s) + s^3 f^*(s)\left(\int_{-\Omega}^{\Omega} \frac{\phi(u)}{u - s}\,du - \frac{b}{s}\right. \\ &\left. + s\sum \frac{2iu_i(u_i^2 - \Omega^2)^{\frac{1}{2}}}{u_i^2 - s^2} B_i\right) + \beta s^3(\Omega^2 - s^2)^{\frac{1}{2}} \\ &\cdot\left(\int_{-\Omega}^{\Omega} \frac{\psi(u)}{u - s}\,du + \frac{a}{s} + s\sum \frac{2u_i A_i}{u_i^2 - s^2}\right) = 0.\end{aligned} \tag{32}$$

Similarly, by multiplying Eq. (31) with respect to t sgn $[(\Omega^2 - s^2)^{\frac{1}{2}} - t]$ and then integrating with respect to t, one obtains

$$\begin{aligned}&\psi(s)\lambda^t(s) + s^3 g(s) \\ &\cdot\left(\int_{-\Omega}^{\Omega} \frac{\psi(u)}{u - s}\,du + \frac{a}{s} + s\sum \frac{2u_i A_i}{u_i^2 - s^2}\right) \\ &+ \frac{s^3\beta}{(\Omega^2 - s^2)^{\frac{1}{2}}}\left(\int_{-\Omega}^{\Omega} \frac{\phi(u)}{u - s}\,du - \frac{b}{s}\right. \\ &\left. + s\sum \frac{2iu_i(u_1^2 - \Omega^2)^{\frac{1}{2}} B_i}{u_1^2 - s^2}\right) = 0.\end{aligned} \tag{33}$$

The constants a and b employed above are defined as follows:

$$a = \int_{-\Omega}^{\Omega} \psi(u)\,du + 2\sum u_i A_i = \frac{-i2\omega}{ec^2} H_y(0, 0),$$

$$\begin{aligned}b = \int_{-\Omega}^{\Omega} [u^2\psi(u) - \phi(u)]\,du \\ + 2\sum [u_i^3 A_i - iu_i(u_i^2 - \Omega^2)^{\frac{1}{2}} B_i] = c^2 a.\end{aligned}$$

The problem has been reduced to a set of coupled singular integral equations, which is easily solved employing known techniques.[6] Continuation of the unknown functions $\phi(u)$ and $\psi(u)$ is extended into the complex u plane by the introduction of the following functions:

$$\Phi(u) = \frac{1}{\pi}\int_{-\Omega}^{\Omega} \frac{\phi(y)}{y - u}\,dy, \qquad \Psi(u) = \frac{1}{\pi}\int_{-\Omega}^{\Omega} \frac{\psi(y)}{y - u}\,dy,$$

which are analytic outside of the cut, vanish at infinity, and satisfy relations of the type

$$\Phi(u + i0) - \Phi(u - i0) = 2i\phi(u),$$

$$\Phi(u + i0) + \Phi(u - i0) = \frac{2}{\pi} P\int_{-\Omega}^{\Omega} \frac{\phi(y)}{y - u}\,dy$$

for $-\Omega \leq u \leq \Omega$. Using the similar relations developed in the previous section for $F^*(u)$, $\Lambda^l(u)$, and $\Lambda^t(u)$, Eqs. (32) and (33) can be written in form

$$P(u + i0) = P(u - i0), \tag{34}$$

$$R(u + i0) = R(u - i0), \tag{35}$$

where

$$P(u) = \Psi^*(u)\pi\beta u^3(u^2 - \Omega^2)^{\frac{1}{2}} + \Lambda^l(u)\Phi^*(u) + \frac{b}{u\pi},$$

$$\begin{aligned}R(u) = \Psi^*(u)[\Lambda^t(u) + \pi\beta u^3(u^2 - \Omega^2)^{\frac{1}{2}}] \\ - \frac{\beta\pi u^3}{(u^2 - \Omega^2)^{\frac{1}{2}}}\Phi^*(u) + \frac{c^2 a}{u\pi},\end{aligned}$$

[6] N. I. Muskhelishvili, *Singular Integral Equations* (Stechert-Hafner Service Agency, Inc., New York, 1953).

with

$$\Psi^*(u) = \Psi(u) + \frac{a}{u\pi} + \frac{2u}{\pi} i \sum \frac{u_i A_i}{u_i^2 - u^2},$$

$$\Phi^*(u) = \Phi(u) - \frac{b}{u\pi} + \frac{2iu}{\pi} \sum \frac{u_i(u_i^2 - \Omega^2)^{\frac{1}{2}} B_i}{u_i^2 - u^2}.$$

Since both $P(u)$ and $R(u)$ vanish at infinity, and are analytic in the cut plane, the required solutions of Eqs. (34) and (35) are given by $P(u) = R(u) = 0$, from which the following expressions are obtained:

$$\Psi^*(u) = [-c^2 a/u\pi L(u)] \cdot [\Lambda^l(u) + \pi u^3 \beta(u^2 - \Omega^2)^{-\frac{1}{2}}], \quad (36)$$

$$\Phi^*(u) = [-b\Lambda^t(u)/u\pi L(u)]. \quad (37)$$

It can be shown that both $\Psi(u)$ and $\Phi(u)$ vanish at infinity, and are analytic outside of the cut, provided that the constants B_i are chosen such that there are no poles at the roots u_i of $L(u)$. This requirement yields the following:

$$(u_i^2 - \Omega^2)^{\frac{1}{2}} B_i = [-ib\Lambda^t(u_i)/2u_i^2 L'(u_i)].$$

Everything has been completely determined. Employing relations of the type

$$2i\phi(u) = \Phi^*(u + i0) - \Phi^*(u - i0) = 2i \operatorname{Im} \Phi^*(u + i0)$$

the continuous spectrum portion of the x component of the electric field can be placed in the form

$$E_x^c = \frac{iek_x}{\omega^2 \epsilon_0} \int_0^\Omega \exp(i\mathbf{k}\cdot\mathbf{x}) u \operatorname{Im}\left[(\Omega^2 - u^2)^{\frac{1}{2}} \Psi^*(u + i0) + \frac{1}{(\Omega^2 - u^2)^{\frac{1}{2}}} \Phi^*(u + i0)\right] du. \quad (38)$$

The discrete spectrum portion can be expressed in the compact form

$$E_x^d = \frac{ek_x \pi}{\omega^2 \epsilon_0 2} \sum_i \exp(i\mathbf{k}_i\cdot\mathbf{x}) u_i \lim_{u\to u_i}\left[(u - u_i) \cdot\left((u^2 - \Omega^2)^{\frac{1}{2}} \Psi^*(u) - \frac{1}{(u^2 - \Omega^2)^{\frac{1}{2}}} \Phi^*(u)\right)\right], \quad (39)$$

where the functions $\Psi^*(u)$ and $\Phi^*(u)$ are given by Eqs. (36) and (37). Similar results hold for the other field components.

Alternative representations can be given for the field components. This involves representing the components of the discrete spectrum in terms of a contour integral around the cut, through the use of the calculus of residues. In this way it can be shown in particular that the discrete spectrum portion of E_x is given by

$$E_x^d = \frac{ek_x}{2\omega^2 \epsilon_0} \int_{-\Omega}^{\Omega} \left(u(\Omega^2 - u^2)^{\frac{1}{2}} \Psi^*(u - i0) + \frac{u}{(\Omega^2 - u^2)^{\frac{1}{2}}} \Phi^*(u - i0)\right) \cdot \exp\left[\frac{ik_x}{u}(\Omega^2 - u^2)^{\frac{1}{2}} z + ik_x x\right] du.$$

On combining this with the continuous spectrum portion the following final expression is derived:

$$E_x = \frac{ek_x}{\omega^2 \epsilon_0} \exp ik_x x \int_0^\Omega \left(u(\Omega^2 - u^2)^{\frac{1}{2}} \Psi^*(u + i0) + \frac{u}{(\Omega^2 - u^2)^{\frac{1}{2}}} \Phi^*(u + i0)\right) \cdot \cos\left(\frac{k_x}{u}(\Omega^2 - u^2)^{\frac{1}{2}} z\right) du, \quad (40)$$

where

$$\Phi^*(u + i0) = \frac{-b}{u\pi} \frac{\lambda^t(u) + \frac{\kappa T}{m} i\pi u^3 f^*(u)}{L(u + i0)},$$

$$\Psi^*(u + i0) = \frac{-b}{u\pi} \frac{\lambda^l(u) + i\pi u^3 f^*(u) - i\pi u^3 \beta(\Omega^2 - u^2)^{-\frac{1}{2}}}{L(u + i0)},$$

with $L(u + i0)$ specified by Eq. (25) and the constant b related to the magnetic field component at the interface,

$$eb = -i2\omega H_y(0, 0).$$

In the limiting case when the incident wave is normal to the interface, expression (40) reduces to the form

$$E_x(\alpha = 0) = \frac{e}{\omega\epsilon_0} \int_0^\infty u\Psi^*(u + i0) \cos\left(\frac{\omega z}{u}\right) du,$$

where

$$\Psi^*(u + i0) = -(b/u\pi)[\lambda^l(u) + i\pi(\omega_p^2/\omega^2) u^3 F_0(u)]^{-1}.$$

The above expression can be placed in the form

$$E_x(\alpha = 0) = \frac{e}{2\omega\epsilon_0}\left[\int_0^\infty u\Psi^*(u + i0) \exp\left(\frac{i\omega z}{u}\right) du + \int_{-\infty}^0 u\Psi^*(u - i0) \exp\left(\frac{i\omega z}{u}\right) du\right]$$

$$= \frac{i\omega H_y(0)}{\pi\epsilon_0} \int_{-\infty}^{\infty} \left(\omega^2 - k^2c^2 - \omega_p^2 \omega \cdot \int \frac{F_0(v)}{\omega - kv + i\epsilon} dv\right)^{-1} \exp(ikz)\, dk,$$

which agrees with Felderhof's[1] results for normal incidence.

To derive physical insight into the nature of the solution it would be useful to make the approximation

$$(\kappa T/m\Omega^2) = (\kappa T/mc^2)\sin^2\alpha \ll 1,$$

in which case the terms containing the factor β [given by expression (24)] can be dropped. The following is thus obtained:

$$\Phi^*(u+i0) \sim -(b/u\pi)[\lambda^l(u) + i\pi u^3 f^*(u)]^{-1}, \quad (41)$$

$$\Psi^*(u+i0) \sim -(b/u\pi)[\lambda^t(u) + (\kappa T/m)i\pi u^3 f^*(u)]^{-1}, \quad (42)$$

$$L(u) \sim \Lambda^l(u)\Lambda^t(u).$$

The effect of the above approximation is to decouple the modes. The discrete modes are separated into logitudinal waves given by the roots of $\Lambda^l(u) = 0$, and transverse waves given by the roots of $\Lambda^t(u) = 0$.

The physical nature of the solution given by the sum of Eqs. (38) and (39) can now be given. For $\omega > \omega_p$ and $\sin\alpha < (1 - \omega_p^2/\omega^2)^{\frac{1}{2}}$ there are no discrete modes, and hence no attenuated waves. The dominant contribution to the continuous spectrum portion is a transverse wave and arises from the neighborhood of the point $\lambda^t(u_0) = 0$. This dominant contribution is thus expressed as follows:

$$2\sin\alpha\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}} H_v(0)\,\frac{(\Omega^2 - u_0^2)^{\frac{1}{2}}}{\lambda^t(u_0)} \cdot \exp\left\{ik_x\left[x + \left(\frac{\Omega^2}{u_0^2} - 1\right)^{\frac{1}{2}} z\right]\right\}.$$

For the operating frequency still greater than the plasma frequency, but such that the sine of the angle of incidence is greater than $(1 - \omega_p^2/\omega^2)^{\frac{1}{2}}$, the above dominant contribution to the discrete spectrum does not exist. However, this wave appears as the evanescent transverse mode given by the root of $\Lambda^t(u) = 0$. This is the dominant contribution for finite values of z.

As the operating frequency approaches the plasma frequency from above, an important contribution arises from the continuous spectrum. This additional contribution is the longitudinal wave given by the root of $\lambda^l(u_1) = 0$ and is given by

$$2\sin\alpha\left(\frac{\mu_0}{\epsilon_0}\right)^{\frac{1}{2}}[(\Omega^2 - u_1^2)^{\frac{1}{2}}\lambda^l(u_1)']^{-1} \cdot \exp\left\{ik_x\left[x + \left(\frac{\Omega^2}{u_1^2} - 1\right)^{\frac{1}{2}} z\right]\right\} H_v(0).$$

When the operating frequency passes the plasma frequency or more precisely when $\omega^2/\omega_p^2 < 1 + (3\kappa T/mc^2)\sin^2\alpha$, this dominant contribution does not appear in the continuous portion of the spectrum, but appears as the discrete evanescent wave given by the root of $\Lambda^l(u) = 0$.

REFLECTION COEFFICIENT

Since the charge at the surface is finite, the tangential components E_x and H_v will be continuous across the surface. The voltage reflection coefficient can then be derived from the formula

$$(1 - R)/(1 + R) = S/\cos\alpha,$$

where

$$S = (\epsilon_0/\mu_0)^{\frac{1}{2}}(E_x/H_v)_{z=0}.$$

In computing the value of S, the surface impedance, the representation of E_x given by Eq. (40) is used. For computational purposes the approximation given by Eqs. (41) and (42) is employed. This results in the following expression:

$$S \sim \frac{i2\sin\alpha}{\pi}\int_0^{\Omega}\left[(\Omega^2 - u^2)^{\frac{1}{2}}\left(\lambda^t + i\pi u^3\frac{\kappa T}{m}f^*\right)^{-1} + \frac{1}{(\Omega^2 - u^2)^{\frac{1}{2}}}(\lambda^l + i\pi u^3 f^*)^{-1}\right]du.$$

Both $\lambda^t(u)$ and $\lambda^l(u)$ can be expressed in terms of known functions as follows:

$$\lambda^t(u) = u^2\left(1 - \frac{\omega_p^2}{\omega^2}\right) - c^2 - \frac{\omega_p^2}{\omega^2}\frac{\kappa T}{m}v^2 Z'(v),$$

$$\lambda^l(u) = 1 - \frac{\omega_p^2}{\omega^2}v^2 Z'(v),$$

where the dimensionless quantity v is given by

$$u = (2\kappa T/m)^{\frac{1}{2}}v.$$

Tables of the function $Z(v)$, where

$$Z(v) = \frac{1}{\pi^{\frac{1}{2}}}P\int_{-\infty}^{\infty}\frac{e^{-t^2}}{t - v}\,dt,$$

are given by Fried and Conte.[7] The quantity $f^*(u)$ is approximated by the relation

$$f^*(u) \sim (2/\pi^{\frac{1}{2}})(\omega_p/\omega)^2(m/2\kappa T)^{\frac{3}{2}}\exp[-(m/2\kappa T)u^2],$$

which is quite accurate except in the immediate vicinity of $u = \Omega$. It can be shown that the integral

$$i\frac{2\sin\alpha}{\pi}\int_0^{\Omega}(\Omega^2 - u^2)^{\frac{1}{2}}\left(\lambda^t + i\pi u^3\frac{\kappa T}{m}f^*(u)\right)^{-1}du$$
$$= \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{-i\sin\alpha + \left[\left(1 - \frac{\omega_p^2}{\omega^2}\right)\cdot\left(1 - \frac{\omega_p^2}{\omega^2}\frac{\kappa T}{mc^2}\right) - \sin^2\alpha\right]^{\frac{1}{2}}\right\} + O\left(\frac{(\kappa T/m)^{\frac{1}{2}}}{c}\right)^3,$$

[7] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

where the argument of the square root quantity is to be taken as π when that quantity becomes negative.

An approximate evaluation of the remaining integral

$$\frac{i2\sin\alpha}{\pi}\int_0^{\Omega}[(\Omega^2-u^2)^{\frac{1}{2}}(\lambda^l+i\pi u^3 f^*)]^{-1}\,du, \qquad (43)$$

arising from the contribution of the longitudinal waves is not as easily obtained. It can be shown that when the operating frequency is not too close to the plasma frequency, the above integral is

$$\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{\mathrm{i}\sin\alpha+O\left[\frac{\omega_p^2}{\omega^2}\left(\frac{\kappa T}{m}\right)^{\frac{1}{2}}\frac{1}{c}\right]\right\}.$$

When the operating frequency is very close to the plasma frequency, the dominant part of the integral arises from the range of integration $\delta(2kT/m)^{\frac{1}{2}} < u < \Omega$, where δ is the order of 3, in which case $\lambda^l(u)$ can be approximated by

$$\left(1-\frac{\omega_p^2}{\omega^2}\right)-\frac{3\kappa T}{m}\frac{1}{u^2}\frac{\omega_p^2}{\omega^2}.$$

Employing this approximation, expression (43) becomes

$$\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{i\sin\alpha+\sin^2\alpha\left[\left(\frac{\omega^2}{\omega_p^2}-1\right)\frac{c^2m}{3\kappa T}-\sin^2\alpha\right]^{-\frac{1}{2}}\right\}.$$

Thus when the operating frequency is in the immediate vicinity of the plasma frequency, the surface impedance is given by

$$S(\omega)\sim\left(1-\frac{\omega_p^2}{\omega^2}\right)^{-1}\left\{i\left[\sin^2\alpha-1+\frac{\omega_p^2}{\omega^2}\right]^{\frac{1}{2}}+\sin^2\alpha\left[\left(\frac{\omega^2}{\omega_p^2}-1\right)\frac{c^2m}{3\kappa T}-\sin^2\alpha\right]^{-\frac{1}{2}}\right\}.$$

At the plasma frequency this reduces to

$$S(\omega_p)\sim -i[(c^2m/6\kappa T)]/\sin\alpha.$$

The critical frequency occurs when $\omega^2/\omega_p^2=1+\sin^2\alpha\,3\kappa T/(c^2m)$ in which case S becomes infinite.

To obtain more precise information on the reflection coefficient, the integral (43) would have to be evaluated numerically using the known values of the function $Z(v)$.

CONCLUSION

Based upon the coupled Maxwell–Vlasov equations, the exact solutions of the field generated in the semi-infinite plasma region by a plane wave incident obliquely to the interface, have been obtained. The component of electric intensity parallel to the interface is prescribed by Eqs. (38) and (39) or by Eq. (40), from which the reflection coefficient is derived. The above analysis was based upon the assumption that all the electrons arriving at the interface were scattered specularly. In would be of interest to extend the results to include a more general boundary condition, such as one based upon the assumption that a fraction p of the electrons arriving at the surface is scattered specularly, while the rest are scattered diffusely. Reuter and Sondheimer[8] employed this boundary condition in their treatment of the skin effect of metals, and obtained explicit results for the two cases of specular scattering $p = 1$, and complete diffuse scattering $p = 0$.

[8] G. E. Reuter and E. H. Sondheimer, Proc. Roy. Soc. **A195**, 336 (1948).