Bubble pulsation and cavitation in viscoelastic liquids

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(Received 21 March 1973)

An analysis is performed to investigate free and forced oscillation of a gas bubble and cavitation in viscoelastic liquids of a three-parameter Oldroyd model. The first-order perturbation method for small amplitudes has been employed to obtain periodic solutions to the bubble dynamics equation. Consideration is given to the influence of surface tension, vapor pressure, and thermodynamic behavior of the gas inside the bubble. It is disclosed that transient behavior of the bubble in viscoelastic liquids is governed by five dimensionless parameters. Conditions for stable bubbles are obtained. Free oscillation consists of a decaying exponential component and a damped sinusoidal oscillation associated with the natural frequency of the bubble-liquid system. The effects of the five governing parameters on the natural frequency are determined. The radial motion of the bubble in an oscillating pressure field includes a sinusoidal oscillation associated with the forcing frequency, a decaying exponential component, and a damped sinusoidal oscillation associated with the natural frequency. Criteria for the onset of incipient cavitation are determined when a bubble is suddenly released into the liquid or is situated in an oscillating pressure field.

I. INTRODUCTION

When a bubble is released into a liquid at constant pressure, its wall will undergo so-called free oscillation, i.e., radial pulsation of natural frequency. In a purely viscous liquid, the pulsation is basically a dampled sinusoidal oscillation with the amplitude decaying by viscous friction.[1] The frequency of pulsation was first derived by Minnaert[2] for the simplest case in which the effects of viscosity, surface tension, compressibility and vapor pressure were all neglected. The expression was subsequently modified by Richardson[3] and Neppiras and NoltingK[4] to include the contribution of surface tension, then by Hirose and Okuyama[5] and Houghton[1] for viscosity and recently by Shima[6] for compressibility. Shima[7] has also derived the natural frequency for a gas bubble in Bingham liquids.

It is generally considered that cavitation bubbles are generated from small gas- and vapor-filled nuclei that can exist at least for short periods of time, in quasi-equilibrium with a liquid. When an oscillating pressure field is turned on, such a gas-vapor nucleus initially at rest in a liquid may be set into various types of motion[1,8]: it may pulsate linearly about its equilibrium radius; it may oscillate in a nonlinear motion; or it may expand to some maximum size and contract so rapidly that its initial motion would resemble that of a collapsing Rayleigh bubble. (A Rayleigh bubble cannot be in equilibrium with the surrounding liquid and must immediately start to collapse.) Linear oscillation is a lower limit to the motion generated by an oscillating pressure field. A bubble that oscillates nonlinearly about its equilibrium radius over relatively long intervals of time is called the stable bubble, whereas a transient bubble would grow to some maximum size—a phenomenon called "cavitation"—and then collapse violently with its initial motion approximately that of a Rayleigh bubble. An oscillating pressure field may generate a range of bubbles between the transient bubble and the stable bubble—two limiting types. In Newtonian liquids, a stable bubble will resonate, being transformed into a transient cavity when the forcing frequency coincides with the natural frequency. It is the transient bubble which is of widest interest since its motion will bring about drastic physical effects of acoustic cavitation.

The dynamic equation of a gas bubble in a viscoelastic liquid of the three-parameter Oldroyd model has been derived in Ref. 9. Bubble pulsation induced by a sudden increase in the system pressure was studied by numerical integration using a digital computer. This paper investigates both free and forced oscillation of a gas bubble in viscoelastic liquids. The bubble dynamics equation is linearized by means of the first-order perturbation method for small amplitudes. The solutions to the linearized equation are obtained which describe the responses of the autonomous (or free) and nonautonomous (or forced) systems. Conditions for the onset of cavitation are determined when a bubble is suddenly released into the liquid or is situated in an oscillating pressure field.

II. ANALYSIS

Consider a bubble containing a mixture of gas and vapor situated in an incompressible viscoelastic liquid, at an initial state when the bubble is at rest in the liquid at the equilibrium temperature T_0 and the ambient equilibrium pressure p_e. The bubble then has an equilibrium radius R_0 determined from

$$p_{g0}+p_v=p_e+2\sigma/R_0, \tag{1}$$

where p_{g0} is the initial pressure of the gas, p_v is the equilibrium vapor pressure at T_0, and σ is the surface tension.

A spherical coordinate system is chosen with its origin at the center of the bubble. When the equilibrium state is disrupted, the radial motion of the spherical interface is governed by the nonlinear integrodifferential equation[9]

$$\rho_l(\ddot{R}R+\tfrac{3}{2}\dot{R}^2)=p_l(R)-p_\infty(t)+\tau_{rr,l}(\infty)-\tau_{rr,l}(R)+3\int_r^\infty(\tau_{rr,l}/r)dr, \tag{2}$$

where ρ_l is the liquid density; R, $\dot{R}$, and $\ddot{R}$ are the instantaneous bubble radius and its time derivatives, respectively; $p_l(R)$ is the liquid pressure at the cavity wall; p_∞ is the pressure at infinity; $\tau_{rr,l}$ is the component of the deviatric stress tensor in the radial direction r.

Neglecting the radial normal stress due to the gas phase viscosity, the balance of forces at the interface requires that

$$p_l(R)=p_g(R)+p_v-2\sigma/R+\tau_{rr,l}(R), \quad (3)$$

in which p_g is the gas pressure. When the gas undergoes a reversible polytropic process inside the bubble, its pressure can be described by

$$p_g(R)=p_{g0}(R_0/R)^{3\gamma}, \quad (4)$$

wherein γ is the polytropic exponent.

When a bubble is set in motion by a pressure field, both the pressure at the interface and at infinity will vary with time. The pressure at infinity can then be expressed as the equilibrium ambient pressure p_e minus a time-varying pressure field $f(t)$:

$$p_\infty(t)=p_e-f(t). \quad (5)$$

A sign convention is used for convenience so that the radius R initially increases for positive $f(t)$. When a bubble is released into the liquid at constant pressure, i.e., $f(t)=0$, free oscillation takes place. A bubble may collapse by a sudden increase in the system pressure.[9] For such a case, $f(t)$ is a negative constant.

The rheological equation for a viscoelastic liquid by the three-parameter linear Oldroyd model reads

$$\tau_{rr}+\lambda_1(D\tau_{rr}/Dt)=2\eta_0[e_{rr}+\lambda_2(De_{rr}/Dt)], \quad (6)$$

where D/Dt denotes the substantial derivative, λ_1 a characteristic stress-relaxation time, η_0 a shear viscosity; λ_2 a characteristic strain-relaxation time, and e_{rr} the rate of strain tensor. The special case in which $\lambda_1=\lambda_2=0$ corresponds to a Newtonian liquid.

Equations (2)–(6) are combined and rearranged in dimensionless form as

$$\ddot{R}^*R^*+\tfrac{3}{2}(\dot{R}^*)^2 =p_g^*(R^*)^{-3\gamma}+p_v^*-1+F(t^*)-\frac{2\sigma^*}{R^*}-\frac{12\eta^*}{\lambda_1^*}\int_0^{t^*}\exp\left(\frac{\xi^*-t^*}{\lambda_1^*}\right) \times\frac{[R^*(\xi^*)]^2\dot{R}^*(\xi^*)+\lambda_2^*\{[R^*(\xi^*)]^2\ddot{R}^*(\xi^*)+2R^*(\xi^*)[\dot{R}^*(\xi^*)]^2\}}{[R^*(t^*)]^3-[R^*(\xi^*)]^3} \times\ln\frac{R^*(t^*)}{R^*(\xi^*)}\,d\xi^*, \quad (7)$$

where $F(t^*)=f(t)/p_e$. Details of the derivation of the bubble dynamics equation are available in Ref. 9.

If the amplitude of bubble pulsation in the liquid is small, then it is convenient to write

$$R^*=1+\epsilon^*(t^*). \quad (8)$$

Here, $\epsilon(t^*)$ is small compared with unity, i.e., the amplitude of bubble pulsation $R_0\,\epsilon$ is small compared with the equilibrium radius R_0 of the bubble. On substitution of Eq. (8) into Eq. (7), we get the following integrodifferential equation after neglecting terms of order ϵ^2 and higher:

$$\ddot{\epsilon}+\Omega_s\epsilon+\frac{4\eta^*}{\lambda_1^*}\int_0^{t^*}\exp\left(\frac{\xi^*-t^*}{\lambda_1^*}\right)[\dot{\epsilon}(\xi^*)+\lambda_2\ddot{\epsilon}(\xi^*)]d\xi^*=F, \quad (9)$$

where

$$\Omega_s=[3\gamma(1-p_v^*)+2\sigma^*(3\gamma-1)]. \quad (10)$$

Ω_s represents the natural frequency of a bubble in an inviscid liquid.[3,4] Equation (9) is differentiated with respect to t^*. The resulting expression is then combined with Eq. (9) to eliminate the integral terms. It yields the third-order linear differential equation

$$\dddot{\epsilon}+\beta_1\ddot{\epsilon}+\beta_2\dot{\epsilon}+\beta_3\epsilon=F+\dot{F}, \quad (11)$$

where

$$\beta_1=(1+4\eta^*\lambda_2^*)/\lambda_1^*,\quad \beta_2=\Omega_s^2+4\eta^*/\lambda_1^*,\quad \beta_3=\Omega_s^2/\lambda_1^*. \quad (12)$$

A bubble-liquid system described by Eq. (11) is nonautonomous and is called the forced system.
The system becomes autonomous or a free system, when $F=0$ and consequently $\dot{F}=0$.

The characteristic equation of the system is a cubic equation:

$$S^3+\beta_1S^2+\beta_2S+\beta_3=0, \quad (13)$$

where S is the differential operator or Laplace variable. Because an actual system may be subjected to all types and variations of input excitations $F(t^*)$, it becomes impractical to investigate the system response $\epsilon(t^*)$ or $R^*(t)$ for every possible excitation. However, a good measure of the transient behavior may be obtained directly from the roots of the characteristic equation (13):

$$S=-a_0,\ -a_1+b_1i,\ -a_1-b_1i, \quad (14)$$

in which i is $(-1)^{1/2}$ and

$$a_0=\tfrac{1}{3}\beta_1-(A+B),\quad a_1=\tfrac{1}{3}\beta_1+\tfrac{1}{2}(A+B),\quad b_1=3^{1/2}(A-B)/2,$$
$$A=(-\tfrac{1}{2}b+Y^{1/2})^{1/3},\quad B=(-\tfrac{1}{2}b-Y^{1/2})^{1/3},\quad Y=\tfrac{1}{4}b^2+\tfrac{1}{27}a^3,$$
$$a=\tfrac{1}{3}(3\beta_2-\beta_1^2)/3,\quad b=\tfrac{1}{27}(2\beta_1^3-9\beta_1\beta_2+27\beta_3). \quad (15)$$

It is known that the nature of the roots will be different depending upon the value of Y being greater than, equal to, or less than zero. If Y is greater than zero, then a_0, a_1, and b_1 will be real numbers, i.e., one real root and two conjugate complex roots. When Y is zero, b_1 becomes zero and the three roots are reduced to

$$S=-a_2,\ -a_3,\ -a_3, \quad (16)$$

in which

$$a_2=\tfrac{1}{3}\beta_1-2(-\tfrac{1}{2}b)^{1/3},\quad a_3=\tfrac{1}{3}\beta_1+(-\tfrac{1}{2}b)^{1/3}. \quad (17)$$

In the case of $Y<0$, the three roots can be rewritten as

$$S=-a_4,\ -a_5,\ -a_6 \quad (18)$$

where

$$a_4=\tfrac{1}{3}\beta_1-2(-\tfrac{1}{3}a)^{1/2}\cos(\tfrac{1}{3}\theta),$$
$$a_5=\tfrac{1}{3}\beta_1-2(-\tfrac{1}{3}a)^{1/2}\cos(\tfrac{1}{3}\theta+\tfrac{2}{3}\pi),$$
$$a_6=\tfrac{1}{3}\beta_1-2(-\tfrac{1}{3}a)^{1/2}\cos(\tfrac{1}{3}\theta+\tfrac{4}{3}\pi),$$
$$\cos\theta=-b/[2(-\tfrac{1}{27}a^3)]^{1/2}. \quad (19)$$

A. Conditions for onset of incipient cavitation

According to linear control theory, whether a system

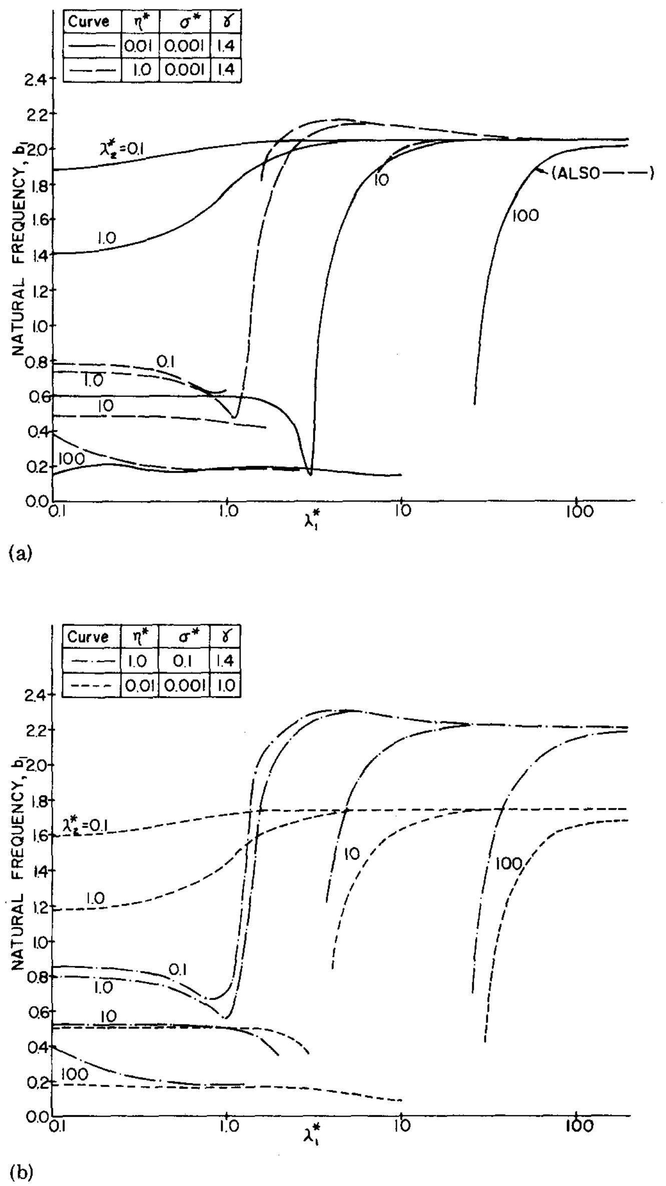


FIG. 1. b_1 vs λ_1^*.

is stable or unstable is a basic property of the system itself, which is described by its characteristic equation (13), and not the particular excitation to the system. If any constant a_n is negative, then the system is basically unstable. In other words, the bubble will grow radially without bound and there will be cavitation. On the other hand, if all a_n's are positive, the bubble will be stable and no cavitation will be induced by free or forced oscillation. Therefore, the criterion for cavitation at small pressure amplitudes is

$$\text{any } a_n < 0 \quad (n = 0, 1, \ldots, 6).$$

An examination of Eq. (15) reveals that when both Y and b_1 are positive, both constants a_0 and a_1 are also positive. When both Y and b_1 are zero, Eqs. (15) and (17) give

$$\beta_1\beta_2 < 3\beta_3 \quad \text{and} \quad 9\beta_1\beta_2 > 2.25\beta_1^3 + 27\beta_3 \tag{20a}$$

as the conditions for negative a_3 (if $b > 0$) and a_2 (if $b < 0$), respectively. For $Y < 0$, the condition for the onset of cavitation is found from Eqs. (15) and (19) to be

$$4\beta_2 < \beta_1^2. \tag{20b}$$

B. Stable bubbles in free systems

Pulsation or transient of the bubble-liquid system may be started by giving the bubble wall an initial radial velocity such that $\dot{R}^*(0) = 1$. Such an initial condition simulates the release of a bubble from a jet underneath a liquid where the impulse created by suddenly closing the interface may be interpreted in terms of an initial radial velocity. In addition to the initial conditions

$$R^*(0) = 1 \quad \text{and} \quad \dot{R}^*(0) = 1, \tag{21}$$

the third initial condition may be obtained from Eq. (7) with the collaboration of Eq. (1) as

$$\ddot{R}^*(0) = 1 - 2\sigma^*. \tag{22}$$

Equations (21) and (22) can be rewritten for ϵ and its derivatives as

$$\epsilon(0) = 0, \quad \dot{\epsilon}(0) = 1, \quad \ddot{\epsilon}(0) = 1 - 2\sigma^*. \tag{23}$$

For Y and b_1 both greater than zero, Eq. (11) with $F = \dot{F} = 0$ is integrated subject to the initial conditions (23). The solution is

$$\epsilon(t^*) = A_0 \exp(-a_0 t^*) + [(A_1^2 + B_1^2)^{1/2}/b_1 \exp(-a_0 t^*) \sin(b_1 t^* + \phi_1), \tag{24}$$

in which

$$A_0 = (1 - \sigma^* - a_0 + \beta_1)/B_0, \quad B_0 = (a_0 - a_1)^2 + b_1^2,$$

$$A_1 = [(1 - \sigma^* - a_1 + \beta_1)(a_0 - a_1) + b_1^2]/B_0, \quad B_1 = -A_0, \tag{25}$$

$$\phi_1 = \tan^{-1}(B_1/A_1).$$

It is self-evident that b_1 represents the natural or resonance frequency of the bubble-liquid system. Pulsation will cease altogether when $b_1 = 0$ or equivalently $Y = 0$. Since both a_0 and a_1 are also positive, pulsation of the bubble consists of a decaying exponential component and a damped sinusoidal oscillation of natural frequency b_1. The amplitude of pulsation decays with a time constant $1/a_1$.

When both Y and b_1 are zero, Eqs. (15) and (17) yield

$$2.25\beta_1^3 + 27\beta_3 \geq 9\beta_1\beta_2 \geq 27\beta_3 \tag{26}$$

as the conditions for both a_2 and a_3 to be positive. The radial motion of the bubble then includes three decaying exponential components $\exp(-a_2 t^*)$, $\exp(-a_3 t^*)$, and $t^* \exp(-a_3 t^*)$. In the case of $Y < 0$, conditions for a_4, a_5, and a_6 to be all positive are obtained from Eqs. (15) and (19) to be

$$4\beta_2 \geq \beta_1^2. \tag{27}$$

Then the bubble motion consists of three decaying exponential components $\exp(-a_4 t^*)$, $\exp(-a_5 t^*)$, and $\exp(-a_6 t^*)$.

C. Stable bubbles in forced systems

Consideration is given to the behavior of a single bubble in an oscillating pressure field

TABLE I. Natural frequency of a gas bubble ($p_v^*=0$) in viscous and inviscid liquids ($\lambda_1^*=\lambda_2^*=0$).

η^*	σ^*	γ	Viscous Ω_v	Inviscid Ω_s	b_1 for $\lambda_1^*\to\infty$, any λ_2^*
0.01	0.001	1.4	2.048	2.051	2.047
1.0	0.1	1.4	0.9165	2.200	2.196
1.0	0.001	1.4	0.454	2.051	2.048
0.01	0.001	1.0	1.641	1.733	1.733

$$f(t)=p_a \sin\omega t \quad \text{or} \quad F(t^*)=p_a^* \sin\Omega t^*, \tag{28}$$

where p_a is the amplitude of the stationary pressure wave, ω is the circular frequency, $p_a^*=p_a/p_e$, and $\Omega=\omega R_0(\rho_1/p_e)^{1/2}$.

Under the conditions of $Y>0$ and $b_1>0$, the solution of Eq. (12) subject to the initial conditions

$$\epsilon(0)=\dot\epsilon(0)=\ddot\epsilon(0)=0 \tag{29}$$

is found to be

$$\epsilon(t^*)/p_a^*=A_2\exp(-a_0t^*)+(A_3^2+B_3^2)^{1/2}(\Omega/b_1)\exp(-a_1t^*)\times\sin(b_1t^*+\phi_2)-(A_4^2+B_4^2)^{1/2}\sin(\Omega t^*+\phi_3), \tag{30}$$

wherein

$$\begin{aligned}
&A_2=\Omega(1-a_0)/[B_0(a_0^2+\Omega^2)], \quad B_2=Q_1^2+Q_2^2,\\
&A_3=[(1-a_1)Q_1-b_1Q_2]/B_2, \quad B_3=[b_1Q_1+(1-a_1)Q_2]/B_2,\\
&\phi_2=\tan^{-1}(B_3/A_3), \quad Q_1=(a_0-a_1)(a_1^2-b_1^2+\Omega^2)+2a_1b_1^2,\\
&Q_2=b_1(2a_0a_1-3a_1^2+b_1^2-\Omega^2), \quad A_4=(Q_3-\Omega Q_4)/B_5,\\
&B_4=(\Omega Q_3+Q_4)/B_5, \quad B_5=Q_3^2+Q_4^2, \quad \phi_3=\tan^{-1}(B_4/A_4),\\
&Q_3=-a_0(a_1^2+b_1^2-\Omega^2)+2a_1\Omega^2,\\
&Q_4=\Omega(2a_0a_1+a_1^2+b_1^2-\Omega^2).
\end{aligned} \tag{31}$$

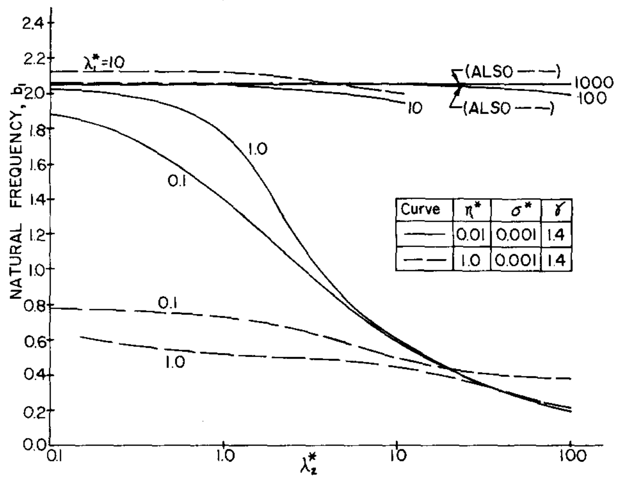
The solution (30) has a form that includes two components called the free oscillation associated with the decaying time constants $1/a_0$ and $1/a_1$ and the natural frequency b_1 and a component called the forced oscillation associated with the forcing frequency Ω. In a linear system, the free oscillation components appear only as a transient and are eventually damped out since there is no coupling between the free and forced oscillations. The bubble then becomes stable and oscillates about its equilibrium radius. This is also true for both $Y=0$ and $Y<0$ cases: Under the conditions of Eqs. (26) and (27), their solutions include the three decaying exponential components, $\exp(-a_2t^*)$, $\exp(-a_3t^*)$, and $t^*\exp(-a_3t^*)$ for the former case and $\exp(-a_4t^*)$, $\exp(-a_5t^*)$, and $\exp(-a_6t^*)$ for the latter case which are associated with the free system response and a sinusoidal oscillating component $\sin(\Omega t^*+\phi)$ which corresponds to the forced system response.

III. RESULTS

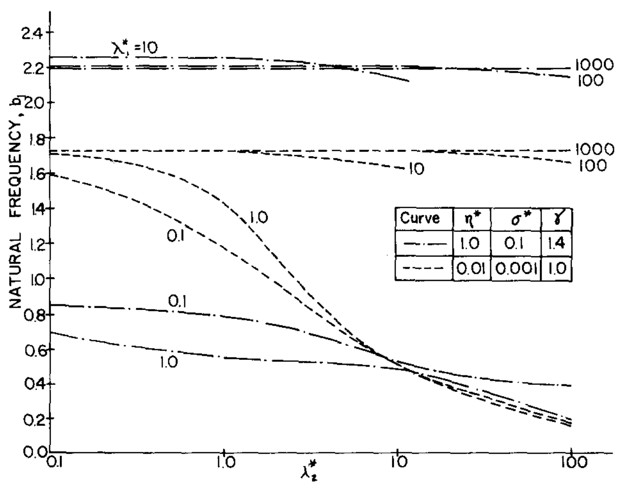
The reciprocals of the constants a_n's are the time constants of the bubble-liquid system, while the parameter b_1 defined in Eq. (15) is the natural frequency of the bubble-liquid system. They are all functions of the five dimensionless parameters: η^*, λ_1^*, and λ_2^* representing the rheological properties of the viscoelastic liquid, surface tension σ^*, and polytropic exponent γ. The polytropic exponent may take a value in two limiting cases: If heat is not transferred across the interface of a pulsating bubble $\gamma=k$, where k is the ratio of specific heats of the gas. Then the contents in the bubble undergo a reversible adiabatic process. In the other extreme case, the gas undergoes a reversible isothermal process. Heat is instantaneously transferred into or out of the bubble. For such an isothermal motion, the value of γ is unity. In reality, heat is transferred at a finite rate during the motion of a bubble. Therefore, γ takes a value between these two extreme values.

The parameters η^*, λ_1^*, λ_2^*, σ^*, and γ are varied over a certain range in order to determine their effects on the natural frequency b_1. Some results are presented graphically in Figs. 1 and 2.

Figure 1 is a plot of b_1 versus λ_1^* with λ_2^* as parameter for four combinations of η^*, σ^*, and γ. Some curves for large values of λ_2^* are discontinuous in certain ranges of λ_1^* where the condition of $Y>0$ is not met. Only if $Y>0$ the characteristic equation (13) would have two conjugate imaginary roots and one real root so that



(a)



(b)

FIG. 2. b_1 vs λ_2^*.

a bubble will oscillate with natural frequency as indicated by Eq. (24). Table I presents the natural frequency of a gas bubble in viscous and inviscid liquids for which both λ_1^* and λ_2^* are zero. It is seen in Fig. 1 that in the low λ_1^* region, the natural frequency b_1 decreases as the value asymptotically approach the limiting ones listed in the last column in Table I. These limiting values of b_1 should be Ω_s (within the accuracy of computations) since $\lambda_1^*=\infty$ as well as $\eta^*=0$ correspond to an inviscid liquid. b_1 increases with λ_1^* in the intermediate λ_1^* region, some curves moderately while others quite drastically following an S-shape change. The effect of the shear viscosity η^* on the natural frequency can be visualized by comparing the families of curves 1 and 3. When both λ_1^* and λ_2^* are small, an increase in η^* results in a substantial decrease in b_1. For large values of λ_1^* and/or γ_2^*, b_1 decreases slightly with an increase in η^*. By comparing curves 2 and 3, it is found that b_1 is increased by increasing surface tension σ^*. A comparison exponent γ results in an increase in b_1. The effect of γ on b_1 is more important in the low λ_2^* region.

Figure 1 is replotted for b_1 versus λ_2^* with λ_1^* as parameter in Fig. 2. Each curve shows a decrease in b_1 as λ_2^* is increased. The effect of λ_2^* on b_1 is most significant when both λ_1^* and η^* are small.

It is interesting to compare the present results with those for viscous and inviscid liquids. In the case of viscous liquids, Eq. (11) is reduced to

$$\ddot{\epsilon}+4\eta^*\dot{\epsilon}+\Omega_s^2\epsilon=F. \tag{32}$$

The solution of Eq. (24) with $F=0$ subject to the initial conditions $\epsilon(0)=0$ and $\dot{\epsilon}(0)=1$ is obtained as

$$\epsilon(t^*)=\exp(-2\eta^*t^*)\sin(\Omega_v t^*)/\Omega_v, \tag{33}$$

where

$$\Omega_v=\{3\gamma[1-p_v^*+2(1-\tfrac{1}{3}\gamma)\sigma^*]-4(\eta^*)^2\}^{1/2}. \tag{34}$$

It is evident that the pulsation of a cavity is basically a damped sinusoidal oscillation of natural frequency Ω_v with its amplitude decaying by viscous friction with a time constant of $1/2\eta^*$. The solution of Eq. (32) subject to the initial conditions $\epsilon(0)=\dot{\epsilon}(0)=0$ is

$$\epsilon(t^*)=p_a^*\{(\Omega/\Omega_v)[8\eta^{*2}+(\Omega^2-\Omega_s^2)^2+(4\eta^*\Omega_v)^2]^{1/2}\exp(-2\eta^*t^*)$$
$$\times\sin(\Omega_v t^*+\phi_4)-[(\Omega^2-\Omega_s^2)^2+(4\eta^*\Omega)^2]^{1/2}$$
$$\times\sin(\Omega t^*+\phi_5)\}/[(4\eta^*\Omega)^2+(\Omega^2-\Omega_s^2)^2], \tag{35}$$

where

$$\phi_4=\tan^{-1}[4\eta^*\Omega_v/(8\eta^{*2}+\Omega^2-\Omega_v^2)],$$
$$\phi_5=\tan^{-1}[4\eta^*\Omega_v/(\Omega^2-\Omega_v^2)]. \tag{36}$$

The response of cavity in a periodic pressure field consists of two component, damped and pure sinusoidal oscillations both with phase shifts. For inviscid liquids, the above equations can be further reduced to

$$\epsilon(t^*)=(1/\Omega_s)\sin\Omega_s t^* \tag{37}$$

and

$$\epsilon(t^*)=p_a^*[(\Omega/\Omega_s)\sin\Omega t^*]/(\Omega^2-\Omega_s^2),\quad \Omega\neq\Omega_s \tag{38a}$$

$$\epsilon(t^*)=p_a^*(\sin\Omega_s t^*-\Omega_s t^*\cos\Omega_s t^*)/2\Omega_s,\quad \Omega=\Omega_s \tag{38b}$$

respectively. When the forcing frequency coincides with the natural frequency Ω_s, the cavity will oscillate with an amplitude that increases linearly with time and becomes unstable.

IV. CONCLUDING REMARKS

The application of the first-order perturbation method on the bubble dynamics equation yields a remarkably simple and convenient method of determining whether a bubble will be stable or transient in nature. The solutions for the motions of the stable bubble in both free and forced systems are obtained. It is disclosed that the solutions for viscoelastic liquids have an additional term representing a decaying exponential component compared with that for pure viscous liquids. In this connection, Tanasawa and Yang[9] have pointed out that due to the presence of the elasticity, the viscous damping effect on the collapse of a bubble is less in viscoelastic liquids than in pure viscous liquids. For small pulsation amplitudes, a more general solution [than Eq. (8)] can be found in series form including the first, second, and other higher-order components. It is anticipated that the second-order perturbation solution will demonstrate the possible existence of integral overtones 2_{b_1}, 3_{b_1}, ... of the fundamental. However, that bubble pulsations exhibit a relatively strong and pure tone[9] probably suggests that the overtones are damped out very rapidly and are of lower intensity than the fundamental.

References 1 [in which p_e was treated as $p_1(\infty)$] and 8 as well as Eq. (35) show that in inviscid liquids a bubble will resonate and there will be cavitation when the forcing frequency coincides with the natural frequency or its harmonics or subharmonics. This applies to the bubble-viscoelastic liquid system as well.

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