and the Prandtl number is 6.0. For these values the present theory predicts nominal critical times of 66, 86, and 102 sec. for a velocity disturbance growth of 1, 2, and 3 orders of magnitude respectively. The critical wavenumber is predicted to be 26, which gives a horizontal wavelength of 2.4 cm. From Spangenberg and Rowland's photographs (however, not at exactly the same conditions as for the above case) the distance between the plunging sheets appears to range from about 1.5 to 3 cm.

Spangenberg and Rowland conclude from their experiments that the number of plunging sheets per unit area increased (thus the wavenumber increased) with increased heat-transfer rate (thus for increased β or Rayleigh number) in agreement with the preditions of the present theory exhibited in Fig. 7. They further observed slow motion of the heavy surface layer starting a short time before "onset of stability." This is also predicted in the way the disturbances start to slowly grow after a critical time is reached as exhibited in Fig. 4. Finally they found that the system was independent of the depth of the fluid layer after a minimum depth for a given heat-transfer rate had been exceeded, in complete agreement with the theory.

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Stability of a Non-Newtonian Liquid Film Flowing Down an Inclined Plane

CHIA-SHUN YIH
The University of Michigan, Ann Arbor, Michigan
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A layer of a non-Newtonian liquid, of which the constitutive equation is triply nonlinear, flows down an inclined plane under the action of gravity. The stability of the flow against wave formation is investigated. With M denoting a parameter involving the first and the second viscosities, the critical Reynolds number is given as a function of M and the slope of the plane, for small values of M. The theory presented here shows how free-surface instability of non-Newtonian fluids can be attacked, and provides a basis for stability experiments with non-Newtonian fluids.

1. INTRODUCTION

THE problem of the stability of a Newtonian liquid film flowing down an inclined plane was formulated by Yih[1] and solved by Benjamin.[2] Further features of the phenomenon were discussed by Yih,[3] who also gave a simple method for dealing with stability problems involving a free surface or an interface. Since free-surface instability of non-Newtonian fluids is of much practical interest, and at present no analysis of it is available, it seems desirable to provide an analysis which illustrates how problems involving such an instability can be attacked, and provides a basis for experiments on the stability of non-Newtonian fluids. The method of analysis is the same as that used in Yih.[3]

The specific problem considered is the stability of a layer of a triply nonlinear isotropic liquid flowing down an inclined plane, with the angle of inclination denoted by β, as shown in Fig. 1. The direction of the gravitational acceleration is vertical.

2. EQUATIONS OF MOTION

The velocity components in the directions of the increasing Cartesian coordinate $x_i (i = 1, 2, \text{and } 3)$ will be denoted by u_i, with i ranging from 1 to 3. With ρ, t, and X_i denoting the density, the time, and the body-force components, respectively, and with τ_{ij} denoting the components of the stress tensor,

[1] C.-S. Yih, in *Proceedings of the Second U. S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1955), pp. 623–628.
[2] T. B. Benjamin, J. Fluid Mech. 2, 554 (1957).
[3] C.-S. Yih, Phys. Fluids 6, pp. 321–334 (1963).

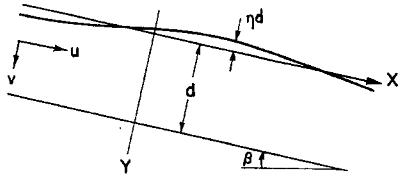

FIG. 1. Definition sketch.

the equations of motion are still

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = \frac{\partial \tau_{\alpha i}}{\partial x_\alpha} + \rho X_i, \tag{1}$$

in which repeated indices in the same term indicate summation. As usual, the second index in τ_{ij} indicates the direction of the corresponding force component, and the first index indicates the axis perpendicular to the plane in which it acts.

Since the liquid is isotropic, the relationship between the stress components and the rates of deformation e_{ij} is, in the most general form,

$$\tau_{ij} = f_0(I)\delta_{ij} + f_1(I)e_{ij} + f_2(I)e_{i\alpha}e_{\alpha j}, \tag{2}$$

in which

$$e_{ij} = \partial u_j/\partial x_i + \partial u_i/\partial x_j, \tag{3}$$

and I is a collective symbol indicating the three invariants of the tensor e_{ij}, i.e.,

$$I_1 = \partial u_\alpha/\partial x_\alpha, \tag{4}$$

$$I_2 = \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix} + \begin{vmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{vmatrix} + \begin{vmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{vmatrix}, \tag{5}$$

$$I_3 = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix}. \tag{6}$$

The three functions f_0, f_1, and f_3 are functions of I_1, I_2, and I_3. Equation (2) is called the constitutive equation. In the present case, the liquid can be considered incompressible, so that

$$I_1 = 0, \tag{7}$$

which is the equation of continuity of an incompressible fluid. Of course f_0, f_1, and f_2 can be very complicated functions[4] of I_2 and I_3. For the triply nonlinear fluid under consideration,

$$f_0(I) = -p + \lambda_2 I_2 + \lambda_3 I_3, \qquad f_1(I) = \mu - \mu_2 I_2, \qquad f_2(I) = \mu_3, \tag{8}$$

in which p is a function of space (but is no longer the mean pressure, as it would be for a Newtonian fluid), μ is the ordinary viscosity, and μ_2 and μ_3 can be called the second and third viscosities. All the coefficients μ, μ_2, μ_3, λ_2, and λ_3 are assumed to be constant. The coefficients λ_2 and μ_3 have the same dimension M/L, λ_3 and μ_2 have the same dimension MT/L, and μ has the dimension M/TL, with M, L, and T denoting the dimensions of mass, length, and time, respectively.

The equations of motion in terms of velocity components only are obtained by substituting (8) in (2) and then (2) in (1), and are

$$\rho\left(\frac{\partial u_i}{\partial t} + u_\alpha \frac{\partial u_i}{\partial x_\alpha}\right) = -\frac{\partial}{\partial x_i}(p - \lambda_2 I_2 - \lambda_3 I_3) + \rho X_i + \frac{\partial}{\partial x_\alpha}\left[\mu\left(\frac{\partial u_i}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_i}\right) - \mu_2 I_2\left(\frac{\partial u_i}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_\alpha}\left[\mu_3\left(\frac{\partial u_k}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_k}\right)\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right)\right]. \tag{9}$$

3. THE PRIMARY FLOW

The primary flow is unidirectional, with the only nonzero velocity component $\bar{u}_1$ in the direction of x_1, or X as shown in Fig. 1. Further, $\bar{u}_1$ depends only on x_2 or Y, so that I_3 is zero. If the body force is assumed to be conservative,

$$X_i = -\partial\Omega/\partial x_i,$$

in which Ω is the body-force potential. The second and third equations in (9) state, since $\bar{u}_2 = 0 = \bar{u}_3$, that

$$\frac{\partial}{\partial x_1}(p + \rho\Omega - \lambda_2 I_2 - \lambda_3 I_3)$$

cannot be a function of x_2 and x_3. The first equation in (8) then states that

$$\frac{\partial}{\partial x_1}(p + \rho\Omega - \lambda_2 I_2 - \lambda_3 I_3) = -K \text{ (a constant)}, \tag{10}$$

because $\bar{u}_1$ is not a function of x_1. Since

$$I_2 = -(d\bar{u}_1/dY)^2, \quad \text{and} \quad I_3 = 0,$$

the first equation in (9) is

$$0 = K + \frac{d}{dY}\left(\mu\frac{d\bar{u}_1}{dY}\right) + \frac{d}{dY}\left[\mu_2\left(\frac{d\bar{u}_1}{dY}\right)^3\right], \tag{11}$$

integration of which yields

$$\mu\frac{d\bar{u}_1}{dY} + \mu_2\left(\frac{d\bar{u}_1}{dY}\right)^3 = -KY, \tag{12}$$

[4] Non-Newtonian fluids for which the shear stress varies as a power of the shearing rate of deformation are known to exist. For such fluids $f_1(I) = \mu_2 I_2^n$. The method given in this paper applies to such fluids and to a fluid with any constitutive equation.

the constant of integration being zero because the shear stress (τ_{21}) is zero at $Y = 0$. Let

$$U = \mu \bar{u}_1/Kd^2, \qquad x = X/d, \qquad y = Y/d, \tag{13}$$

in which d is the depth of the liquid. Equation (12) then assumes the form

$$U' + M(U')^3 = -y, \tag{14}$$

in which the accent indicates differentiation with respect to y, and

$$M = \mu_2(Kd)^2/\mu^3. \tag{15}$$

The parameter M is a measure of the importance of the first relative to the second viscosity. If M is large, the second viscosity predominates, and the fluid is predominantly non-Newtonian. If M is small, the first viscosity predominates, and the fluid is only slightly non-Newtonian. In this paper M is supposed to be small, say less than 0.2. The sign of M may be positive or negative.

Equation (14) can be solved exactly for U', and the result, after expansion in power series of M, can be integrated to obtain U. Alternatively, we can use

$$U' = -y \tag{16}$$

as the first approximation, substitute (16) in the cubic term of (14) and integrate to get a second approximation, and repeat the iteration process as many times as we wish. The result can be integrated to obtain U. In either way we obtain, to the second power in M,

$$U' = -y + My^3 - 3M^2y^5 \tag{17}$$

and

$$U = \tfrac{1}{2}(1 - y^2) - \tfrac{1}{4}M(1 - y^4) + \tfrac{1}{2}M^2(1 - y^6), \tag{18}$$

in which the constant of integration has been determined by the nonslip condition

$$U = 0 \quad \text{at} \quad y = 1. \tag{19}$$

It can be shown that the root for U', of which the one given by (17) is the approximation, is the only real root of (14).

The value of K in (10) is determined from the fact that $\tau_{22} = 0$ along the free surface, so that

$$p - \lambda_2 I_2 - \lambda_3 I_3$$

is independent of x on the free surface. Thus

$$K = -\frac{\partial}{\partial x_1}\rho\Omega = \rho g \sin\beta, \tag{20}$$

in which g is the gravitational acceleration. Since K has been shown to be a constant for the entire field of flow, its value given by (20) is valid for the entire field of flow, although it has been determined from a free-surface condition.

4. DIFFERENTIAL SYSTEM GOVERNING STABILITY

Only two-dimensional flows are considered in this paper. For two-dimensional flows, the general equations of motion given by (9) can be considerably simplified. Utilizing the equation of continuity

$$\partial u_1/\partial X + \partial u_2/\partial Y = 0, \tag{21}$$

and using Δ to denote the Laplacian

$$\partial^2/\partial X^2 + \partial^2/\partial Y^2,$$

we can reduce (9) to the form

$$\begin{aligned}
&\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial X} + u_2\frac{\partial u_1}{\partial Y}\right)\\
&= -\frac{\partial}{\partial X}(p - \lambda_2 I_2 + \mu_3 I_2) + \rho g \sin\beta + \mu\Delta u_1\\
&\quad - 2\mu_2\frac{\partial}{\partial X}\left(I_2\frac{\partial u_1}{\partial X}\right) - \mu_2\frac{\partial}{\partial Y}\left[I_2\left(\frac{\partial u_1}{\partial Y} + \frac{\partial u_2}{\partial X}\right)\right],\\
&\rho\left(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial X} + u_2\frac{\partial u_2}{\partial Y}\right)\\
&= -\frac{\partial}{\partial Y}(p - \lambda_2 I_2 + \mu_3 I_2) + \rho g \cos\beta + \mu\Delta u_2\\
&\quad - \mu_2\frac{\partial}{\partial X}\left[I_2\left(\frac{\partial u_2}{\partial X} + \frac{\partial u_1}{\partial Y}\right)\right] - 2\mu_2\frac{\partial}{\partial Y}\left(I_2\frac{\partial u_2}{\partial Y}\right).
\end{aligned} \tag{22}$$

In addition to those introduced in (13), the following dimensionless variables will be used:

$$(u, v) = (u_1, u_2)\frac{\mu}{Kd^2}, \qquad \tau = \frac{tKd}{\mu},$$

$$P = \frac{1}{\rho}\left(\frac{\mu}{Kd^2}\right)^2(p - \lambda_2 I_2 + \mu_3 I_2).$$

The equations of motion (22) can then be written as

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{\sin\beta}{F^2} + \frac{1}{R}\Delta u\\
&\quad - \frac{M}{R}\left\{\frac{\partial}{\partial x}\left(2J_2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[J_2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\right\},\\
&\frac{\partial v}{\partial \tau} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} + \frac{\cos\beta}{F^2} + \frac{1}{R}\Delta v\\
&\quad - \frac{M}{R}\left\{\frac{\partial}{\partial x}\left[J_2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left(2J_2\frac{\partial v}{\partial y}\right)\right\},
\end{aligned} \tag{23}$$

in which Δ now stands for the Laplacian in terms of x and y, and J_2 is the dimensionless form of I_2. The symbols R and F represent the Reynolds number and the Froude number based on the reference velocity

$$V = \frac{Kd^2}{\mu} = \frac{gd^2 \sin \beta}{\nu}, \qquad (24)$$

ν being the kinematic viscosity μ/ρ. Thus

$$R = \frac{Vd}{\nu} = \frac{gd^3 \sin \beta}{\nu^2} \qquad F^2 = \frac{V^2}{gd} = \frac{gd^3 \sin^2 \beta}{\nu^2}, \qquad (25)$$

so that

$$F^2 = R \sin \beta.$$

As usual, the variables u, v, and P will each be resolved into a part representing the primary flow and a part representing the perturbation, or

$$u = U + u',\ v = v',\ P = \Pi + p', \qquad (26)$$

in which U and Π are for the primary flow, and the accented quantities are perturbation quantities. Substituting (26) into (23), cancelling out the terms corresponding entirely to the primary flow, and neglecting quadratic terms in the perturbation quantities, we obtain

$$u'_\tau + Uu'_x + U_y v' = -p'_x + \frac{1}{R}\Delta u' - \frac{M}{R} \cdot [U_y^2 u'_{xx} - 3U_y^2 u'_{yy} - 6U_y U_{yy}(v'_x + u'_y)], \qquad (27)$$

$$v'_\tau + Uv'_x = -p'_y + \frac{1}{R}\Delta v' - \frac{M}{R} \cdot (-U_y^2 u'_{yx} - 3U_y^2 v'_{xx} - 4U_y U_{yy} v'_y), \qquad (28)$$

in which the subscripts indicate partial differentiation.

The equation of continuity now takes the form

$$\partial u'/\partial x + \partial v'/\partial y = 0,$$

which permits the use of the stream function ψ, in terms of which

$$u' = \psi_y, \qquad v' = -\psi_x. \qquad (29)$$

Assuming

$$\psi = \phi(y) \exp i\alpha(x - c\tau), \qquad (30)$$

substituting (30) into (29) and the result into (27) and (28), and eliminating p' in (27) and (28) by cross differentiation, we obtain finally

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi + 3M(U'^2\phi'')'' + 2M\alpha^2(U'^2\phi'' + 2U'U''\phi' + 3U''^2\phi + 3U'U'''\phi) + 3M\alpha^4U'^2\phi = i\alpha R[(U - c)(\phi'' - \alpha^2\phi) - U''\phi], \qquad (31)$$

in which the primes now indicate differentiation with respect to y. In (30) and (31), α is the wavenumber $2\pi d/\lambda$, λ being the wavelength, and

$$c = c_r + ic_i,$$

c_r being the wave velocity and αc_i the rate of amplification. If c_i is positive, the waves grow exponentially. If it is negative, they are damped out exponentially. The chief concern of this paper is to determine the sign of c_i for various values of R, M, and β, and for small values of α.

Equation (31) is the differential equation governing the perturbation motion. The boundary conditions at the bottom of the layer are

$$u' = 0 = v' \quad \text{at} \quad y = 1,$$

or

$$\phi'(1) = 0 \quad \text{and} \quad \phi(1) = 0. \qquad (32)$$

The surface conditions are more complicated, since they must be applied on the free surface and not merely at $y = 0$. Let ηd be the (dimensional) deviation of the free surface from its mean position, so that

$$\eta_\tau + U(0)\eta_x = v' = -i\alpha\phi(0) \exp i\alpha(x - c\tau),$$

or

$$\eta = [\phi(0)/c'] [\exp i\alpha(x - c\tau)],\ c' = c - U(0). \qquad (33)$$

The mean shear $\bar{\tau}_{21}$ in the primary flow is given by the equation of equilibrium (since there is no acceleration for that flow):

$$(d/dy)\bar{\tau}'_{21} = -\rho g d \sin \beta,$$

in which the sign is negative because of the convention that the direction of the shear force on a surface with its *outward* normal in the *positive* y direction determines its sign. The shear at the free surface due to the perturbation motion is

$$\tau'_{21} = \frac{\mu V}{d}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) + \frac{3\mu_2 V^3}{d^3} U'^2\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) - (\rho g d \sin \beta)\eta, \qquad (34)$$

the last term in which gives the effect of variation in mean shear as the free surface deviates from its mean position. The first two terms on the right-hand side of (34) are evaluated at $y = 0$. Since $U(0) = 0$, (34) can be simplified. With η given by (33), V by (24), and u' and v' by (29) and (30), the condition $\tau'_{21} = 0$ on the free surface becomes, finally,

$$\phi''(0) + (\alpha^2 - 1/c')\phi(0) = 0, \qquad (35)$$

which is the same as that for a Newtonian fluid.

[See Yih[3], Eq. (24), in which the factor 3 arose from a different choice of the reference velocity.]

To evaluate the normal stress on the free surface, we need p' at $y = 0$, which can be obtained from (27), and the variation of the normal stress in the primary flow with y, which is given by

$$(d/dy)\bar{\tau}_{22} = -\rho g \cos\beta. \tag{36}$$

A development similar to that leading to (35) then gives the fourth boundary condition

$$\frac{\alpha(\cot\beta + \alpha^2 SR)\phi(0)}{c'} + \alpha(Rc' + 3\alpha i)\phi'(0) - i\phi'''(0) = 0, \tag{37}$$

in which

$$S = T/\rho V^2 d, \tag{38}$$

T being the surface tension, and V being given by (24). Condition (37) is the same as for a Newtonian fluid, because $U'(0) = 0$. [See Yih,[3] Eq. (25), in which the factor 3 before $\cot\beta$ arose from a different choice of the reference velocity.]

Equations (31), (32), (35), and (37) define an eigenvalue problem. Given β, α, S, R, and M, there is a value of c. We shall adopt the approach in Yih,[3] and first consider very long waves ($\alpha = 0$). With the eigenvalue c so obtained, we can then proceed to find the change in c as α is increased. It will be seen that c is real for $\alpha = 0$, and as α is increased the change in c is purely imaginary, giving a c_i positive or negative according as the R is greater or less than a critical value.

5. FIRST APPROXIMATION

For the first approximation, all terms containing α in (31) and the boundary conditions are dropped. The differential equation to be solved is then

$$\phi_0'''' + 3M(U'^2\phi_0'')'' = 0. \tag{39}$$

The boundary conditions (32) stand unmodified. But (35) and (37) are replaced by

$$\phi_0''(0) - (1/c_0')\phi_0(0) = 0 \tag{40}$$

and

$$\phi_0'''(0) = 0, \tag{41}$$

in which, as subsequently, the subscript zero is used to indicate the first approximation.

The solution of (39) is accomplished by four quadratures, the first two of which give

$$\phi_0'' = (Ay + B)/(1 + 3MU'^2). \tag{42}$$

After (42) has been expanded in a power series in M, two more quadratures produce, to the order of M^2,

$$\phi_0 = A\phi_{01} + B\phi_{02} + Cy + D, \tag{43}$$

in which

$$\phi_{01} = \frac{1}{6}y^3 - \frac{3M}{20}y^5 + \frac{5M^2}{14}y^7,\qquad \phi_{02} = \frac{1}{2}y^2 - \frac{M}{4}y^4 + \frac{M^2}{2}y^6. \tag{44}$$

Equation (41) demands that $A = 0$, and we can take $B = 1$ once and for all. Equations (32), now for ϕ_0, determine C and D to be

$$C = -\phi_{02}'(1) \quad\text{and}\quad D = \phi_{02}'(1) - \phi_{02}(1).$$

Equation (40) then gives

$$c_0' = \tfrac{1}{2} - \tfrac{3}{4}M + \tfrac{5}{2}M^2, \tag{45}$$

so that

$$c_0 = c_0' + U(0) = 1 - M + 3M^2. \tag{46}$$

Incidentally, $c_0' = \phi_0(0) = D$. The eigenfunction ϕ_0 is

$$\phi_0 = \tfrac{1}{2}y^2 - y + \tfrac{1}{2} + M(-\tfrac{1}{4}y^4 + y - \tfrac{3}{4}) + M^2(\tfrac{1}{2}y^6 - 3y + \tfrac{5}{2}). \tag{47}$$

For the second approximation, the equation to be solved is

$$\phi_1'''' + 3M(U'^2\phi_1'')'' = i\alpha R[(U - c_0)\phi_0'' - U''\phi_0]. \tag{48}$$

The boundary conditions for ϕ_1 at $y = 1$ are

$$\phi_1(1) = 0, \qquad \phi_1'(1) = 0. \tag{32a}$$

Since only terms of first order in α are retained in the differential system, (37) becomes

$$(\alpha\cot\beta/c_0')\phi_0(0) + \alpha Rc_0'\phi_0'(0) - i\phi_1'''(0) = 0. \tag{49}$$

As to (35), care must be taken that c' suffers a change in the second approximation, so that the proper form of (35) is now

$$\phi_1''(0) - \frac{1}{c_0'}\phi_1(0) + \frac{\Delta c}{c_0'^2}\phi_0(0) = 0, \tag{50}$$

since

$$\frac{d}{dc'}\left(\frac{1}{c'}\right) = -\frac{1}{(c')^2}, \quad\text{and}\quad \Delta c' = \Delta c.$$

The quantity Δc in (49) denotes the change in c.

Substitution of (47) in (48) and solving for ϕ_1, we have

$$\phi_1(y) = \Delta A\phi_{01} + \Delta Cy + \Delta D + i\alpha Rf(y), \qquad (51)$$

with

$$f(y) = -\frac{1}{120}y^5 + M\left(\frac{13}{840}y^7 + \frac{1}{120}y^5\right) - M^2\left(\frac{463}{10080}y^9 + \frac{13}{840}y^7 + \frac{1}{40}y^5\right). \qquad (52)$$

In (51), the first three terms can be considered either as the complementary solution of (48), or as the correction of ϕ_0 necessitated by $i\alpha Rf(y)$. The term $\Delta B\phi_{02}$ has been suppressed because we have taken B to be unity once and for all. The result is not at all affected by this suppression. For detailed arguments, see Yih.[3]

Since $\phi_0(0) = c_0'$, (49) becomes

$$\Delta A = -i[\alpha \cot\beta + \alpha Rc_0'\phi_0'(0)].$$

Since, furthermore, $\phi_1(0) = 0$, (50) reduces to

$$\Delta c = \Delta D.$$

From the boundary conditions (32a) we can compute ΔD and obtain Δc. The result is

$$\Delta c = i\alpha R[f'(1) - f(1)] - i[\phi_{01}'(1) - \phi_{01}(1)][\alpha\cot\beta + \alpha Rc_0'\phi_0'(0)]. \qquad (53)$$

After (52), (44), and (47) have been substituted in (53) and only terms up to M^2 are included, it becomes

$$\Delta c = i\alpha\left[R\left(\frac{2}{15} - \frac{62M}{105} + \frac{128M^2}{45}\right) + \left(-\frac{1}{3} + \frac{3M}{5} - \frac{15M^2}{7}\right)\cot\beta\right]. \qquad (54)$$

The critical Reynolds number is

$$R_{cr} = \left(\frac{1}{3} - \frac{3M}{5} + \frac{15M^2}{7}\right)\cdot\left(\frac{2}{15} - \frac{62M}{105} + \frac{128M^2}{45}\right)^{-1}\cot\beta, \qquad (55a)$$

which can be simplified to

$$R_{cr} = \left(\frac{5}{2} + \frac{46}{7}M - \frac{2399}{294}M^2\right)\cot\beta. \qquad (55b)$$

If $R > R_{cr}$, c_i is positive, and the flow is unstable. If $R < R_{cr}$, c_i is negative, and the flow is stable. For $M = 0$,

$$R_{cr} = \tfrac{5}{2}\cot\beta,$$

which can be compared to Benjamin's (1957)

$$R_{cr} = \tfrac{5}{4}\cot\beta.$$

and Yih's (1963)

$$R_{cr} = \tfrac{5}{6}\cot\beta.$$

The differences have arisen from the choice of the reference velocity. Benjamin chose it to be the surface velocity, Yih[3] chose it to be the average velocity, and in the present paper it has been chosen, for simplicity, to be Kd^2/μ, which for $M = 0$ is exactly twice the surface velocity or three times the average velocity of the primary flow. There is therefore agreement for the case $M = 0$ with existing results.

From (2), (5), (8), and (15) it can be seen that if M (or μ_2) is positive the second viscosity μ_2 stiffens the fluid, and that if M is negative the second viscosity reduces the shear stress and "softens" the fluid. It is therefore reasonable that for small M the critical Reynolds number is increased if M is positive and reduced if M is negative. Thus the second viscosity stabilizes or destabilizes according as it is positive or negative.

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