Instability of Three-Layer Viscous Stratified Flow

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(Received 21 January 1969; final manuscript received 22 May 1969)

Instability of plane Couette flow of three superposed layers of fluids of different viscosity between two horizontal planes is investigated. It is found that there are two modes of disturbance dominated, respectively, by the two interfaces. The flow is found to be unstable in the zero-order approximation of wavenumber α for certain values of the depth and viscosity ratios. This is owing to a sort of resonance which prevails when a free wave at the lower interface forces a free wave at the upper interface. As is known from results previously obtained, the existence of a single surface of viscosity discontinuity will cause instability. The presence of an additional surface of discontinuity may or may not be stabilizing, depending upon the values of the depth and viscosity ratios at the additional interface.

I. INTRODUCTION

The linear stability of plane Couette flow has been the subject of several investigations in recent years. All show that the flow with a single fluid is stable no matter how large the Reynolds number. Yih[1] considered the stability of two superposed fluids of different viscosities in plane Couette and Poiseuille flow and found that, for long waves, both plane Couette flow and plane Poiseuille flow can be unstable, however small the Reynolds number. The unstable modes are due to the stratification in the viscosity.

In this investigation, the problem has been extended to plane Couette flow with three superposed fluids of different viscosities and different densities. The effects on the stability or instability owing to the presence of an additional surface of viscosity discontinuity is examined, as is the interaction between the two wave trains, each associated with one interface. It is found that two different modes of waves exist. The two modes of waves correspond to the two surfaces of discontinuity. Because of a sort of resonance, the flow might become unstable in the zero-order approximation for certain values of depth and viscosity ratios. This relatively highly unstable mode might be regarded, as Taylor[2] explained in his problem of superposed of three layers of inviscid fluids, as owing to a free wave at the lower interface forcing a free wave at the upper interface when their velocities in space coincide or nearly coincide.

II. THE PRIMARY FLOW

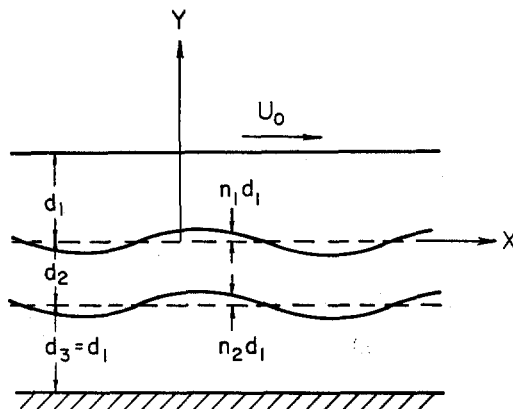Consider the three superposed layers of fluid with the depth d_1, d_2, and $d_3 = d_1$, the viscosity μ_1, μ_2,

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[1] C.-S. Yih, J. Fluid Mech. **27**, 337 (1967).
[2] G. I. Taylor, Proc. Roy. Soc. (London) **A132**, 499 (1931).

FIG. 1. Definition sketch for the three-layer steady flow.

and μ_3, and the density ρ_1, ρ_2, and ρ_3 for the upper, the intermediate, and the lower fluid, respectively. The upper boundary is moving with a constant velocity U_0 while the lower boundary is stationary. The coordinates are chosen as shown in Fig. 1.

We make all quantities nondimensional by using a length scale d_1, a velocity scale U_0, a pressure scale $\rho_1 U_0^2$, and a time scale d_1/U_0. The primary flow is assumed to be steady. In terms of the dimensionless quantities, the equations govern the primary flow are

$$\frac{d^2 U_1}{dy^2} = 2m_a A, \quad \frac{d^2 U_2}{dy^2} = 2A, \quad \text{and} \quad \frac{d^2 U_3}{dy^2} = 2m_b A, \quad (1)$$

for the upper, the intermediate, and the lower fluid, respectively, in which the factor 2 on the right-hand side of (1) is introduced for convenience. U_1, U_2, and U_3 denote, respectively, the mean velocity in those layers of fluid, and

$$A = -\frac{R}{2m_a} K, \quad m_a = \frac{\mu_2}{\mu_1}, \quad m_b = \frac{\mu_2}{\mu_3}, \quad (2)$$

K being the pressure gradient in the horizontal direction which is a constant.

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Equations (1) are easily solved with the boundary conditions being that the mean velocity is equal to a specified velocity U_0 at the upper boundary and zero at the lower boundary, and the mean velocity as well as the shear stress being continuous at the interfaces. The solutions are

$$U_1 = m_a A y^2 + a_1 y + b,$$
$$U_2 = A y^2 + a_2 y + b, \qquad (3)$$
$$U_3 = m_b A y^2 + a_3 y + b_1,$$

in which

$$a_2 = (n + ma + m_b)^{-1}$$
$$\cdot [1 - m_a A + n(n + 2m_b)A + m_b A],$$
$$a_1 = m_a a_2, \qquad a_3 = m_b a_2,$$
$$b = 1 - m_a A - a_1,$$
$$b_1 = \frac{m_b(1 + n)}{n + m_a + m_b}$$
$$\cdot [1 - (2m_a + nm_a + nm_b - n)A],$$

where $n = d_2/d_1$ is the depth ratio. The velocity gradients at interfaces are discontinuous for either of two adjacent fluids if $m_a \neq 1$ and $m_b \neq 1$. As pointed out by Yih and again true here, this is the cause of the instability.

III. THE DIFFERENTIAL SYSTEM GOVERNING STABILITY

As is customary in hydrodynamic stability problems, an infinitesimal disturbance is applied to the primary flow. The flow will be unstable if the disturbance grows with time, and is stable if it attenuates with time. Since it has been shown by Squire[3] for homogeneous fluids, and later by Yih[4] for stratified fluids that the stability or instability of three-dimensional disturbance, can be determined from that of the corresponding two-dimensional disturbance, it is sufficient to consider only two-dimensional disturbances.

The equation of continuity for the perturbation flow is

$$\frac{\partial u_i'}{\partial x} + \frac{\partial v_i'}{\partial y} = 0, \qquad (4)$$

in which the subscript i is taken to be 1, 2, and 3 for the upper, intermediate, and lower layer of fluid, respectively. The u_i' and v_i' are the perturbed velocity components in the direction of increasing x and y,

[3] H. B. Squire, Proc. Roy. Soc. (London) A142, 621 (1933).
[4] C.-S. Yih, Phys. Fluids 6, 321 (1963).

respectively. Equation (4) permits the use of a stream function ψ_i, in terms of which

$$u_i' = (\psi_i)_y, \qquad v_i' = -(\psi_i)_x, \qquad (5)$$

the subscripts x and y denoting partial differentiation. The Navier–Stokes equations as well as the boundary conditions governing the perturbation flow admit a solution of the form

$$(\psi_i, p_i') = [\varphi_i(y), f_i(y)] \exp [i\alpha(x - c\tau)], \qquad (6)$$

in which p_i' is the pressure perturbation, α is the dimensionless wavenumber, and $c = c_r + ic_i$ is the complex phase velocity. We wish to determine the sign of c_i, since the flow is stable, unstable, or neutrally stable according to whether c_i is negative, positive, or zero.

It is well known that the perturbation flow in parallel flow is governed by the Orr–Sommerfeld equation. If we write φ for φ_1, χ for φ_2, and θ for φ_3, equations governing the amplitude functions will be

$$\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi$$
$$= i\alpha R [(U_1 - c)(\varphi'' - \alpha^2\varphi) - U_1''\varphi], \qquad (7)$$
$$\chi'''' - 2\alpha^2\chi'' + \alpha^4\chi$$
$$= \frac{i\alpha\gamma_a R}{m_a} [(U_2 - c)(\chi'' - \alpha^2\chi) - U_2''\chi], \qquad (8)$$
$$\theta'''' - 2\alpha^2\theta'' + \alpha^4\theta$$
$$= \frac{i\alpha\gamma_b m_b R}{m_a} [(U_3 - c)(\theta'' - \alpha^2\theta) - U_3''\theta], \qquad (9)$$

in which the primes denote the differentiation with respect to y,

$$\gamma_a = \rho_2/\rho_1, \qquad \gamma_b = \rho_3/\rho_1$$

are the density ratios, and R is the Reynolds number $\rho_1 d_1 U_0/\mu_1$. To solve the three fourth-order differential equations (7), (8), and (9), there must be 12 boundary conditions imposed to specify the mathematical problem completely. The rigid boundaries demand that

$$\varphi(1) = 0, \qquad \varphi'(1) = 0,$$
$$\theta(-1 - n) = 0, \qquad \theta'(-1 - n) = 0. \qquad (10a, b, c, d)$$

Here the primes indicate differentiation with respect to y. The continuity of v' at the interfaces demands that

$$\varphi(0) = \chi(0), \qquad (10e, f)$$
$$\chi(-n) = \theta(-n).$$

If η represents the deviation of interface from its

mean position, the kinematic condition at the interface is

$$\left(\frac{\partial}{\partial \tau} + U \frac{\partial}{\partial x}\right)\eta = v', \qquad (11)$$

in which U is the mean velocity at the interface. Applying Eq. (11) to both upper and lower interfaces, we obtain

$$\eta_1 = \frac{\varphi(0)}{c'} \exp\left[i\alpha(x - c\tau)\right], \qquad (12)$$

$$\eta_2 = \frac{\chi(-n)}{c''} \exp\left[i\alpha(x - c\tau)\right], \qquad (13)$$

respectively, where

$$c' = c - U_1(0) \quad \text{and} \quad c'' = c - U_2(-n). \qquad (14)$$

The η_1 and η_2 are the deviation of the upper and lower interfaces from their mean positions. The continuity of $U + u'$ should be imposed at the actual positions of interface, i.e., at $y = \eta_1$ and $y = -n + \eta_2$. Within the framework of the linear theory, this condition is represented by the continuity of $u' + (\partial U/\partial y)\eta$ at the mean position of interface, that is, at $y = 0$ and $y = -n$. These give

$$\varphi'(0) - \chi'(0) = \frac{\varphi(0)}{c'} (a_2 - a_1), \qquad (10g)$$

and

$$\chi'(-n) - \theta'(-n)$$
$$= \frac{\chi(-n)}{c''} [2n(1 - m_b)A + a_3 - a_2]. \qquad (10h)$$

In regard to the continuity of shear stress at the interfaces, we first note that the gradient of the mean shear stress is the same for either of two adjacent layers of fluid, so that we can apply the continuity of shear stress at the mean position of each of the interfaces. The results are

$$\varphi''(0) + \alpha^2\varphi(0) = m_a[\chi''(0) + \alpha^2\chi(0)], \qquad (10i)$$

$$\chi''(-n) + \alpha^2\chi(-n) = m_b^{-1}[\theta''(-n) + \alpha^2\theta(-n)]. \qquad (10j)$$

Finally, the continuity of the normal stress at the interfaces demands that the difference of the quantity

$$\rho g \, d_1 \eta - p'\rho_1 U_0^2 + \frac{2\mu U_0}{d_1} \frac{\partial v'}{\partial y}, \qquad (15)$$

evaluated for the upper fluid from its valve for the lower fluid at either interface should be equal to

$$-\frac{T}{d_1} \frac{\partial^2 \eta}{\partial x^2},$$

in which T is the surface tension. T_1 and T_2 will be used to denote the surface tension at the upper and lower interfaces, respectively. In (15), the first term takes care of the contribution of hydrostatic pressure to the actual pressure at the interface. The last two terms in (15) are evaluated at the mean positions of interface. Evaluate p' for each fluid from the x component of the Navier–Stokes equation governing the perturbation flow. With the results so obtained and with η_1 and η_2 evaluated from Eqs. (12) and (13), we get

$$-i\alpha R \ (c'\varphi' + a_1\varphi) - (\varphi''' - \alpha^2\varphi')$$
$$+ 2\alpha^2\varphi' + i\gamma_a\alpha R \ (c'\chi' + a_2\chi)$$
$$+ m_a(\chi''' - \alpha^2\chi') - 2\alpha^2 m_a\chi'$$
$$= i\alpha R \ (F_a^{-2} + \alpha^2 S_a) \frac{\varphi}{c'}, \qquad (10k)$$

for the upper interface, in which all variables are evaluated at $y = 0$, and

$$-i\alpha R \ \gamma_a(c''\chi' - 2nA\chi + a_2\chi)$$
$$- m_a(\chi''' - \alpha^2\chi') + 2m_a\alpha^2\chi'$$
$$+ i\gamma_b\alpha R \ (c''\theta' - 2m_anA\theta + a_3\theta)$$
$$+ \frac{m_a}{m_b} (\theta''' - \alpha^2\theta') - 2\alpha^2 \frac{m_a}{m_b} \theta'$$
$$= i\alpha R \ (F_b^{-2} + \alpha^2 S_b) \frac{\chi}{c''}, \qquad (10l)$$

for the lower interface, in which all variables are evaluated at $y = -n$. In (10k and l),

$$F_a^{-2} = \frac{\rho_2 - \rho_1}{\rho_1} \frac{gd_1}{U_0^2}, \qquad F_b^{-2} = \frac{\rho_3 - \rho_2}{\rho_1} \frac{g(d_1 + d_2)}{U_0^2},$$

$$S_a = \frac{T_1}{\rho_1 d_1 U_0^2}, \quad \text{and} \quad S_b = \frac{T_2}{\rho_1 d_1 U_0^2},$$

and F_a and F_b are the modified Froude numbers.

IV. THE SOLUTIONS OF THE EIGENVALUE PROBLEMS

The method of regular perturbation of Yih[5] for long waves ($\alpha \ll 1$) will be adopted to solve this eigenvalue problem. The functions φ, χ, θ, and c can be expanded in a power series in α:

$$\varphi = \sum_{i=0}^{\infty} \alpha^i\varphi_i, \qquad \chi = \sum_{i=0}^{\infty} \alpha^i\chi_i,$$

$$\theta = \sum_{i=0}^{\infty} \alpha^i\theta_i, \qquad c = \sum_{i=0}^{\infty} \alpha^i c_i.$$

[5] C.-S. Yih, Quart. Appl. Math. **12**, 434 (1955).

The quantity αR is assumed to be small compared with 1. Thus, however large the Reynolds number R, there exists a range of small α for which the perturbation technique is valid. Upon substituting these expansions into (7), (8), and (9) as well as the boundary conditions, and collecting terms, involving α to the same power and setting them equal to zero, we have the governing equations and the boundary conditions for each order of approximation.

A. The Zero-Order Approximation

The equations corresponding to α^0 are

$$\varphi_0^{iv} = 0, \qquad \chi_0^{iv} = 0, \qquad \theta_0^{iv} = 0. \qquad (16)$$

The boundary conditions (10a)–(10h) corresponding to α^0 remain the same except that the variables all have subscripts zero, whereas (10i)–(10l) corresponding to α^0 become

$$\varphi_0''(0) - m_a\chi_0''(0) = 0,$$
$$m_b\chi_0''(-n) - \theta_0''(-n) = 0,$$
$$\varphi_0'''(0) - m_a\chi_0'''(0) = 0, \qquad (17)$$
$$m_b\chi_0'''(-n) - \theta_0'''(-n) = 0.$$

Equations (16) can be integrated at once subject to the pertinent boundary conditions. The solutions are

$$\varphi_0 = 1 + B_1y + C_1y^2 + D_1y^3,$$
$$\chi_0 = 1 + B_2y + C_2y^2 + D_2y^3, \qquad (18)$$
$$\theta_0 = E_3 + B_3y + C_3y^2 + D_3y^3,$$

in which

$$D_2 = \frac{Q_b \pm (Q_b^2 + 4Q_aQ_c)^{1/2}}{2Q_a},$$

where Q_a, Q_b, and Q_c are given in the Appendix;

$$B_2 = \frac{1}{m_an}(n^2 + m_a - m_b)$$
$$- \left(n^2 + 2n - 3m_b - \frac{4m_b}{n}\right)D_2,$$

$$C_2 = \frac{1}{m_a} - 2D_2,$$

$$B_3 = 2(1 + n)\frac{m_b}{m_a} - m_b(7 + 10n + 3n^2)D_2,$$

$$D_1 = m_aD_2, \quad \text{and} \quad C_1 = m_aC_2.$$

The eigenvalue $c_0'(= c_0 - b)$ is given by

$$c_0' = -\frac{a_2 - a_1}{2C_1 + 3D_1 + B_2}. \qquad (19)$$

There are two roots for D_2 hence two possible eigenvalues can be found to satisfy the governing equations and the boundary conditions. These two phase velocities give two possible modes of wave motion. This is true since there are two interfaces which admit two degrees of freedom in disturbances. The two modes have equal importance in stability considerations.

B. The First-Order Approximation

The equations obtained from the coefficients of α^1 are

$$\varphi_1'''' = iR[(U_1 - c_0)\varphi_0'' - 2m_aA\varphi_0],$$

$$\chi_1'''' = \frac{i\gamma_aR}{m_a}[(U_2 - c_0)\chi_0'' - 2A\chi_0], \qquad (20)$$

$$\theta_1'''' = \frac{i\gamma_am_bR}{m_a}[(U - c)\theta_0'' - 2m_bA\theta_0],$$

in which φ_0, χ_0, θ_0, and c_0 are given by Eqs. (18) and (19).

The solutions for (20) are

$$\varphi_1 = \Delta B_1y + \Delta C_1y^2 + \Delta D_1y^3 + iR\,h_1(y), \qquad (21)$$

$$\chi_1 = \Delta B_2y + \Delta C_2y^2 + \Delta D_2y^3 + iR\,\frac{\gamma_a}{m_a}h_2(y), \qquad (22)$$

$$\theta_1 = \Delta E_3 + \Delta B_3y + \Delta C_3y^2 + \Delta D_3y^3$$
$$+ iR\,\frac{\gamma_bm_b}{m_a}h_3(y). \qquad (23)$$

where

$$h_1(y) = \frac{m_aAD_1}{210}y^7 + \frac{a_1D_1}{60}y^6$$
$$+ \frac{a_1C_1 - 3c_0'D_1 - m_aAB_1}{60}y^5$$
$$- \frac{c_0'C_1 + m_aA}{12}y^4,$$

$$h_2(y) = \frac{AD_2}{210}y^7 + \frac{a_2D_2}{60}y^6$$
$$+ \frac{a_2C_2 - 3c_0'D_2 - AB_2}{60}y^5 - \frac{c_0'C_2 + A}{12}y^4,$$

$$h_3(y) = \frac{m_bAD_3}{210}y^7 + \frac{a_3D_3}{60}y^6$$
$$+ \frac{a_3C_3 - 3D_3(c_0' + b - b_1) - m_bAB_3}{60}y^5$$
$$- \frac{(c_0 - b_1)C_3 + m_bAE_3}{12}y^4.$$

The first three terms in each of Eqs. (21), (22), and (23) are the complementary solution while the last

term is the particular solution. The term of zero power in y for φ_1 and χ_1 are taken to be zero. The argument is that the solution of the eigenvalue problem is unique only up to a constant multiplier. The constant terms of φ and χ have been taken to be 1. We can and will keep them at that value once and for all. That this does not remove the possibility of satisfying the boundary condition is obvious, since c_1 is yet to be determined. The constants of integration and the eigenvalue will be determined by applying the boundary conditions (10) for terms with the factor α.

As a result of many lengthy mathematical manipulations, the boundary conditions can be solved for

$$\Delta B_2 - \Delta B_1 = \frac{iR[(H_1 - m_b n)G_3 - (G_1 + G_2)H_2]}{m_a n G_1 + H_1 G_2},$$

(24)

in which G_1, G_2, G_3, H_1, and H_2 are given in the Appendix. For details, the reader may consult Ref. 6. The boundary condition (10g) for the first-order approximation reads

$$\Delta B_1 - \Delta B_2 = -(c_1/c_0'^2)(a_2 - a_1).$$

(25)

Separating c_1 into real and imaginary parts

$$c_1 = (c_1)_r + i(c_1)_i,$$

from Eq. (25) we obtain

$$(c_1)_r = 0,$$

$$i(c_1)_i = \frac{\Delta B_2 - \Delta B_1}{a_2 - a_1} c_0'^2.$$

or

$$(c_1)_i = R \, K(n, m_a, m_b, \gamma_a, \gamma_b, F_a, F_b, s_a, s_b),$$

(26)

with

$$K = \frac{c_0'^2[(G_1 + G_2)H_2 - (H_1 - m_a n)G_3]}{(a_2 - a_1)(-m_a n G_1 - H_1 G_2)}.$$

(27)

V. NUMERICAL CALCULATIONS

The eigenvalues of the zeroth-order approximation and first-order approximation expressed in terms of constant parameters such as n, m_a, m_b, $\cdots$, etc., are given, respectively, by Eqs. (19) and (26). To examine the stability or instability of the flow, to understand the influence on the stability or instability due to the variations of n, m_a, and m_b, and to see the interaction between the waves associated with each interface, numerical calculations are carried out for the special case $\gamma_a = 1$, $\gamma_b = 1$, and $A = 0$. Thus, the three layers have the same density,

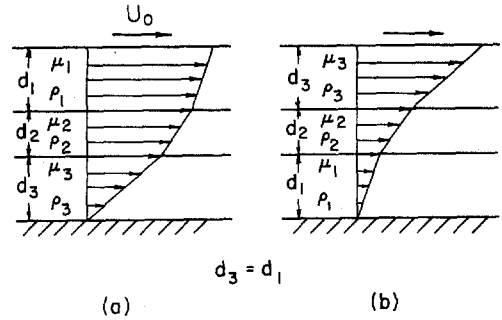⁶ C.-H. Li, Ph.D. thesis, University of Michigan.

FIG. 2. Sketch showing the interchange of the upper and lower fluids: (a) before interchange; (b) after interchange.

and the mean flow is the plane Couette flow, since there is no longitudinal pressure gradient.

As we have previously pointed out, there are two modes of wave motion corresponding to the two interfaces. Let us define the amplitude ratio ζ to be

$$\zeta = \frac{\text{amplitude of upper interface wave}}{\text{amplitude of lower interface wave}}.$$

(28)

Then it can be shown that

$$\zeta = \frac{c_0' - n^2 A + na_2}{c_0'(1 - nB_2 + n^2 C_2 - n^3 D_2)}.$$

Since this investigation is restricted to equal depths in the upper and lower layers, if the fluids in the upper and in the lower layers are interchanged (the fluid in the intermediate layer remains the same), that is, if m_a and m_b are interchanged, we know from a physical point of view that

$$c_0^{\text{I}}(n, m_a, m_b) + c_0^{\text{II}}(n, m_b, m_a) = 1,$$

(29)

in which $c_0^{\text{I}}(n, m_a, m_b)$ represents the phase velocity for the first mode before the interchange, and $c_0^{\text{II}}(n, m_b, m_a)$ represents the phase velocity for the second mode after the interchange. (Figure 2 shows the configuration of the system before and after interchange of the upper and the lower fluids.) Henceforth, the superscripts I and II indicate the quantities related to the first mode and the second mode. The results of numerical computation using the IBM 360 computer withstands the tests of (29). This result confirms the accuracy of both the equations and the numerical results up to the zeroth-order approximation. The calculations were only done for the special case $\gamma_a = 1 = \gamma_b$, corresponding to the case of uniform density in all layers; gravity has no effect on the stability aside from imparting a hydrostatic part to the pressure. With this idea in mind, we know the disturbances before and after interchange of the upper fluid and the lower fluid should
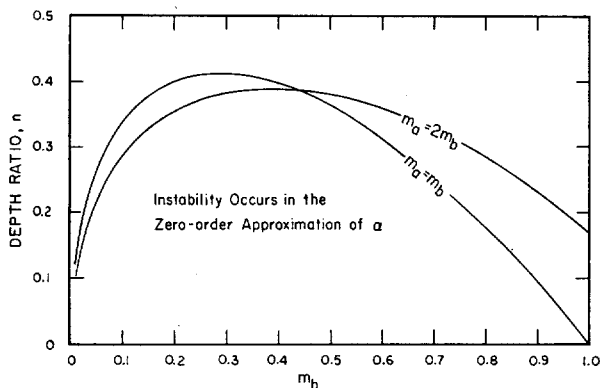
FIG. 3. Prediction of the region of resonance in $n - m_a$ diagram for the cases $m_a = m_b$ and $m_a = 2m_b$, $\gamma_a = \gamma_b = 1$.

have the same amplification or damping rate in the case of plane Couette flow. This leads to

$$(c_1)_i^{\mathrm{I}}(n, m_a, m_b) = (c_1)_i^{\mathrm{II}}(n, m_b, m_a),$$

or

$$K^{\mathrm{I}}(n, m_a, m_b) = \frac{m_a}{m_b} K^{\mathrm{II}}(n, m_b, m_a), \qquad (30)$$

in which $(c_1)_i$ is the imaginary part of the eigenvalue in the first-order approximation which is given by Eq. (26), and K is given by Eq. (27). The factor m_a/m_b in front of K^{II} in (30) has arisen because the definition of Reynolds number before and after interchange is retained. The results of numerical calculation also withstand the test (30). Again, this confirms the accuracy of both the equations and numerical results up to the first-order approximation. Keeping (26) in mind, one knows that the stability of the flow can be concluded from the computation results of K. Using the relation (30), values of K for the second mode can be calculated very quickly from that of the first mode in the interchanged system, if care is taken that the arguments m_a and m_b in K^{I} and K^{II} of Eq. (30) are interchanged. Besides, if we set $m_b = 1$, the system reduces to the superposition of two layers of fluid; Yih's[1] results are reproduced analytically and numerically. This provides another test of the accuracy of the algebra and the computation in this investigation.

The results of the calculation will be presented graphically and discussed in the following section.

VI. DISCUSSION OF THE GRAPHS

Since there are two modes of wave motion, two eigenvalues of the phase velocities and hence two amplitude ratios can be found to satisfy the governing equations and the boundary conditions for a fixed value of n, m_a, and m_b. Numerical calculation shows that the two possible amplitude ratios can be

either positive or negative. This would indicate that the two interfaces can oscillate either in phase or 180 deg out of phase whatever the mode of the disturbance. From here on the term amplitude ratio will be used to mean its absolute value. The computed results demonstrate that of the two possible amplitude ratios one is always greater than 1 and the other always less than 1, no matter what the assigned values of parameters n, m_a, and m_b are. From the definition in (28), we know that the mode with an absolute amplitude ratio greater than 1 is primarily an upper-interface mode while the mode with an amplitude ratio less than 1 is primarily a lower-interface mode. In order to avoid confusion, we will call the mode belonging to the upper interface "the first mode" and denote it by the symbol I. Similarly, the other mode will be called the second mode and denoted by the symbol II.

The numerical calculations show that for certain values of n, m_a, and m_b, the eigenvalues might become complex even in the zeroth-order approximations. Since the complex roots of a quadratic equation of real coefficients always appear in a conjugate pair, one of the two modes is unstable. For the special cases $m_a = m_b$ and $m_a = 2m_b$, the curves of the limiting values of n for which the resonance will occur for each assigned value of m_a are shown in Fig. 3. For these two cases, the flow becomes highly unstable if the values of n and m_a fall in the regions below these two curves in Fig. 3. The phenomenon of resonance in the flow field with two or more discontinuity surfaces is not strange. Taylor[2] and Goldstein[7] found the existence of resonance in the inviscid flows of two superposed fluids separated by a layer of intermediate density in which the velocity varies continuously from that of the lower fluid to that of the upper fluid. The unstable range of relative velocities, they found, is a very narrow one in the immediate neighborhood of the relative velocity where the backward-moving free waves at the upper surface of separation move at the same speed as the forward-moving wave at the lower surface of separation. Thus, the instability might be regarded as being due to a free wave at the lower surface forcing a free wave at the upper surface when their velocities in space coincide.

The results of the calculation also show another interesting feature. We will discuss the results of mode I and of mode II individually. For mode I, the growth rate of disturbance is increased with increasing n if the fluid of intermediate layer is "less

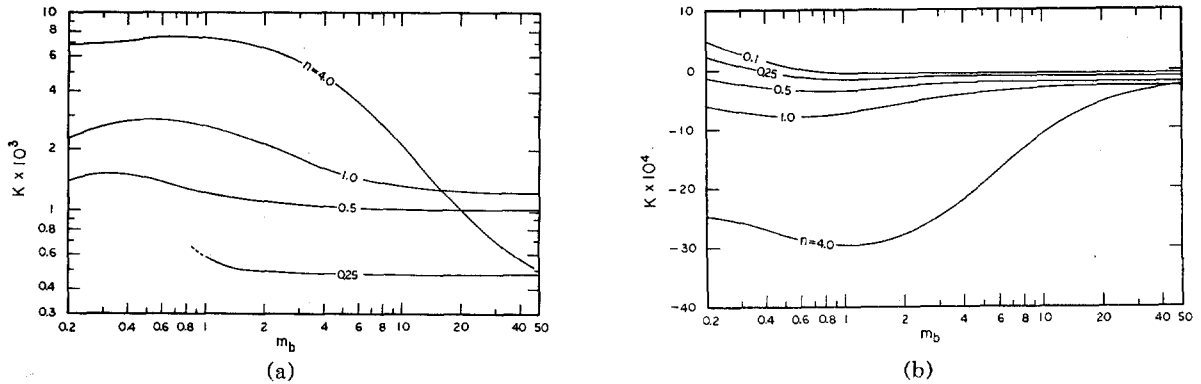[7] S. Goldstein, Proc. Roy. Soc. (London) A132, 524 (1931).

FIG. 4. (a) Variation of K with the viscosity ratio m_b for various values of n for the upper-interface mode. $m_a = 0.4$, $\gamma_a = \gamma_b = 1$. (b) Variation of K with the viscosity ratio m_b for various values of n for the upper-interface mode. $m_a = 2.5$, $\gamma_a = \gamma_b = 1$.

viscous" than that of the upper layer; but, the growth rate is decreased or the damping rate increased with increasing n if the upper fluid becomes "less viscous" than the intermediate one. This feature is true for any value of m_b as long as n is of order of 1 or less than 1, but having some exceptions when $n > 1$, for instance, $m_b > 10$. A symmetrical feature can be found in mode II. For mode II the growth rate of the disturbance is increased with increasing n if the fluid of the intermediate layer is "less viscous" than that of the lower layer; but, the growth rate is decreased or the damping rate is increased with increasing n as soon as the lower fluid becomes "less viscous" than the intermediate one. This feature is true for any value of m_a as long as n is near or less than 1, but having some exceptions, for instance $m_a > 20$, when $n > 1$. The symmetry of the situation confirms the aforementioned intuitive idea that the two modes are, respectively, dominated by the two interfaces. For the special case $m_a = 0.4$ and 2.5, the variations of K of the first mode with m_b for various values of the parameter n are illustrated in Figs. 4(a) and (b).

If m_a or m_b becomes unity, the flow field reduces to the two-layer case treated by Yih.[1] To see the influence of an additional surface of discontinuity in viscosity on the stability of the flow, the variations in K of mode I with m_b are plotted in Figs. 5(a), (b), and (c) for the special values of $n = 0.5, 4.0,$ and 1.0. In those plots, m_a is used as a parameter. Keeping (26) in mind, we see from Figs. 5(a) and (b) that the additional interface, the lower interface in this case, in the flow field has no appreciable effect on stability or instability if $n > 1$ when $m_b > 1$ or $n < 1$ when $m_b < 1$. When the three layers become equal in depth, the additional discontinuity surface has an

appreciable effect on stability whether $m_b > 1$ or $m_b < 1$. Figure 5(a) shows that, for $n = 0.4$, the flow is unstable if $m_a < 1$ with any values of m_b, but is stable if $m_a > 1$ for $m_b > 1$. The damping or amplification rate becomes constant when the magnitude of m_b is larger than 10. Figure 5(b) shows that, for $n = 4.0$, the flow is stable or unstable accordingly as $m_a > 1$ or $m_a < 1$. The smaller the values of m_b, the more unstable is the flow; the larger the values of m_b, the more stable the flow. When $m_b > 20$, apparently neither the damping nor amplification rate will change with m_b. For $m_a = 10$, the damping rate does not change with m_a.

If we plot K of mode II against m_b with m_a as the parameter for $n = 0.5, 4.0,$ and 1.0, we will have a set of curves similar to those shown in Figs. 5(a), (b), and (c). The tendency to variations of K with m_b and m_a for mode II is exactly the same as that for variations of K with m_a and m_b for mode I. This symmetrical behavior is a consequence of the fact that each mode of waves motion is dominated by one surface of discontinuity.

VII. CONCLUSIONS

As a result of this investigation, the following conclusions can be drawn for plane Couette flows of three layers:

(1) Two modes of disturbance are found. They are dominated, respectively, by the two interfaces.

(2) For the case of equal densities, the growth rate or damping rate of disturbance for the lower-interface mode can be calculated from that for the upper-interface mode simply by the relation

$$K^{\mathrm{I}}(n, m_a, m_b) = (m_a/m_b)K^{\mathrm{II}}(n, m_b, m_a),$$
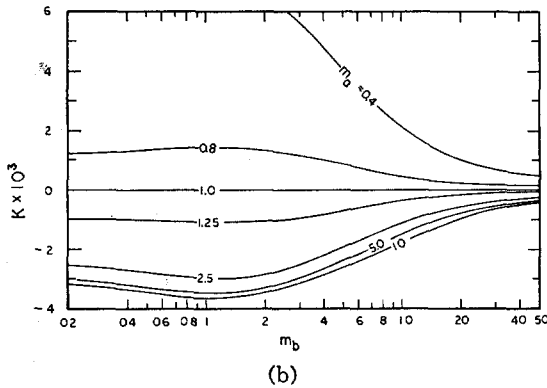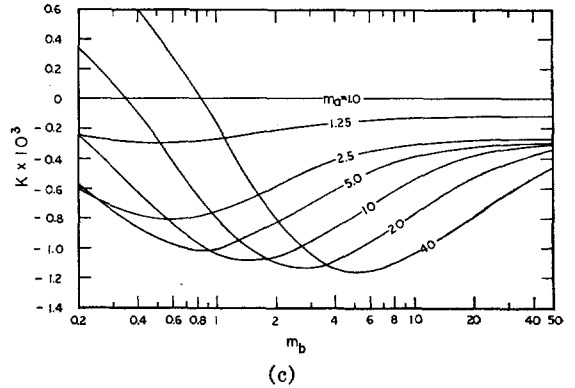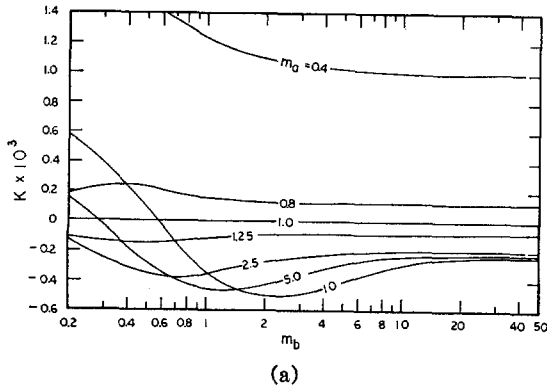
or vice versa.

(a)

(c)

(b)

FIG. 5. (a) Variation of K as a function of the viscosity ratio m_b for various vlaues of m_a for the upper-interface mode. $n = 0.5$, $\gamma_a = \gamma_b = 1$. (b) Variation of K as a function of the viscosity ratio m_b for various values of m_a for the upper-interface mode. $n = 4$, $\gamma_a = \gamma_b = 1$. (c) Variation of K as a function of the viscosity ratio m_b for various values of m_a for the upper-interface mode. $n = 1$, $\gamma_a = \gamma_b = 1$.

(3) The flow was found to be highly unstable for certain values of the depth ratio and the viscosity ratios. This can be regarded as owing to a sort of resonance which prevails when a free wave at the lower interface forces a free wave at the upper interface, as Taylor pointed out in his problem of inviscid fluids.

(4) As the depth ratio n for the upper interfacial mode is increased, the flow becomes more unstable if $m_a < 1$ and more stable or less unstable if $m_a > 1$. A symmetric situation exists for the lower interfacial mode.

(5) The existence of one surface of viscosity discontinuity will cause instability. The presence of an additional surface of discontinuity may or may not

be stabilizing, depending on the values of the depth ratio and the viscosity ratio at the additional interface. The detailed results are given in Sec. VI.

ACKNOWLEDGMENTS

APPENDIX

$$Q_a = \left(m_a + n^2 + 2n - 3m_b - \frac{4m_b}{n} \right)$$

$$\cdot \left[m_b(3n + 4)[2n(1 - m_b)A + a_3 - a_2] - 2n(nA - a_2)\left(n^2 + n + 5m_b + 3nm_b + \frac{2m_b}{n} \right) \right],$$

$$Q_b = \frac{1}{m_a} \left(m_a + n^2 + 2n - 3m_b - \frac{4m_b}{n} \right)$$

$$\cdot [(nA - a_2)(m_a - m_b - n^2 - 2m_b n) + 2m_b n(1 - m_b)A + m_b(a_3 - a_2)]$$

$$+ 2\left(n^2 + n + 5m_b + 3nm_b + \frac{2m_b}{n}\right)\left[\left(1 + 2n + \frac{n^2}{m_a} - \frac{m_b}{m_a}\right)(nA - a_2) - a_2 + a_1\right]$$

$$- \frac{m_b}{nm_a}(3n + 4)(2nm_a + n^2 + m_a - m_b)[2n(1 - m_b)A + a_3 - a_2],$$

$$Q_c = \frac{1}{nm_a^2}\{(m_a - m_b - n^2 - 2nm_b)[m_a(a_1 - a_2) - (nA - a_2)(2m_a n + n^2 - m_a - m_b)]$$

$$- m_b[2n(1 - m_b)A + a_3 - a_2](2nm_a + n^2 + m_a - m_b)\},$$

$$G_1 = \frac{m_a\chi_0(-n)[2n(1 - m_b)A + a_3 - a_2]}{(c_0' + na_2)^2(a_2 - a_1)}c_0'^2 - n(3n + 4) - m_b(6n + 7) - n^2(2 + n)Q_2,$$

$$G_2 = -m_a\left(1 + nQ_2 + \frac{\chi_0(-n)[2n(1 - m_b)A + a_3 - a_1]}{(c_0' + na_2)^2(a_2 - a_1)}c_0'^2\right),$$

$$G_3 = (3n^2 + n^3 Q_2 + 3m_b + 6nm_b)\left[-h(1) + \frac{1}{6}\left(\frac{1}{c_0'F_a^2} - (\gamma_a - 1)(c_0'B_1 + a_1)\right)\right]$$

$$- \{m_a Q_3 - m_b[\gamma_b h_3'(-1 - n) + \gamma_b h_3''(-n) - \gamma_a h_2''(-n)] - 3m_a Q_1\}$$

$$+ [n(3n + 2) + m_b(5 + 6n) + n^2 Q_2(1 + n)][3h_1(1) - h_1'(1)],$$

$$H_1 = -n^3 - 2n^2 + 4m_b + 3nm_b,$$

$$H_2 = [n^3 - m_b(2 + 3n)]\left[-h_1(1) + \frac{1}{6}\left(\frac{1}{c_0'F_a^2} - (\gamma_a - 1)(c_0'B_1 + a_1)\right)\right]$$

$$- \{\gamma_a h_2(-n) - \gamma_b m_b[h_3(-n) - h_3(-1 - n)] + \gamma_b m_b h_3'(-1 - n) - m_b\gamma_b h_3''(-n) + m_b\gamma_a h_2''(-n) - 3m_a Q_1\}$$

$$+ (n^2 + n^3 - 3m_b - 3nm_b)[3h_1(1) - h_1'(1)],$$

in which the primes on h_1, h_2, and h_3 denote differentiation with respect to y before evaluating at the right position, and

$$Q_1 = \frac{m_b}{m_a}\left(\frac{\chi_0(-n)}{c_0''F_b^2} - (\gamma_b - \gamma_a)[c_0''\chi_0'(-n) - 2nA\chi_0(-n) + a_2\chi_0(-n)] - \gamma_b h_3'''(-n) + \gamma_a h_2'''(-n)\right),$$

$$Q_2 = \frac{1}{c_0' + na_2}[2n(1 - m_b)A + a_3 - a_2],$$

$$Q_3 = -\frac{1}{m_a}[\gamma_a h_2'(-n) - m_b\gamma_b h_3'(-n) - \gamma_a Q_2 h_2(-n)].$$