Dual Role of Viscosity in the Instability of Revolving Fluids of Variable Density

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The stability of a viscous fluid between rotating cylinders and with a radial temperature gradient against the formation of axisymmetric disturbances (Taylor vortices) is considered, and it has been found that viscosity has a dual role. If the circulation increases radially outward (so that the flow would be stable in the absence of density variation) but the density decreases with the radial distance, the situation can arise that viscosity actually has a destabilizing effect. In the opposite circumstance, thermal diffusivity is always destabilizing. Detailed results for small spacing of the cylinders and sufficient conditions for stability of a revolving fluid of variable density or entropy also are given.

I. INTRODUCTION

FOR a homogeneous fluid flowing between rotating concentric cylinders, Synge[1] has shown that if the circulation increases with radial distance the flow is always stable. If the density of the fluid also has a radial variation, the arguments of von Kármán, as presented in the book by Lin,[2] can be used to show that, in the *absence* of viscosity, the flow in the case of a liquid is always stable if $\rho\Gamma^2$ increases with the radial distance, with ρ denoting the density and Γ the circulation of the fluid in its undisturbed state. In this paper the stability of a liquid between coaxial rotating cylinders and with a radial density gradient is considered. The disturbances are assumed to be axisymmetric, not two dimensional, as assumed by Chandrasekhar.[3] The chief points to be made are that, even if $\rho\Gamma^2$ increases with the radial distance, the flow can actually become less stable or even unstable (1) for increasing kinematic viscosity if Γ increases but ρ decreases outward, and (2) for increasing thermal diffusivity if Γ decreases but ρ increases outward, and that the flow is stable if both ρ (of a liquid) and Γ increase with radial distance.

The destabilizing effect of thermal diffusivity in case (2) is almost exactly the same as in the very interesting case of gravitational instability discovered by Stommel *et al.*[4] In their case a fluid (water) has a temperature increasing and a salinity increasing with height in such a way that their combined effect is to make the density decrease with height, and, *superficially*, to make the fluid stable. However, since the thermal diffusivity of water is much greater than its salinity diffusivity, a displaced particle will harmonize with its new surroundings more readily in temperature than in salinity. Consequently, the original cause for stability can be diminished to such a degree that instability due to adverse salinity gradient actually occurs. In the present case (2) thermal diffusivity plays exactly the same role, although the cause of instability is not gravity but centripetal acceleration.

The destabilizing effect in case (1) lies in the fact that as a material ring is displaced, its density may harmonize with that in the ring's new surroundings less readily than the circulation along it, because of the diffusive effect of viscosity, and consequently the product $\rho\Gamma^2$ of the ring may exceed that in its new surroundings, causing the ring to move further.

Inasmuch as viscosity is a momentum diffusivity, its destabilizing effect in case (1) is similar to that of thermal diffusivity in case (2) or in the case considered by Stommel *et al.*, but not entirely. Aside from being an agent for momentum diffusion, viscosity is always a dissipative agent responsible for the eventual conversion of kinetic energy into heat. Thus, except in the case of Tollmien-Schlichting waves for which viscosity also plays a dual role, the effect of viscosity has always been to stabilize a flow. Certainly in the case of flow of a homogeneous fluid between rotating cylinders, Sir Geoffrey Taylor[5] showed, among other specific results, that viscosity is stabilizing. Therefore, if viscosity is found to be destabilizing in the case of a nonhomogeneous fluid, it must have a dual role which thermal diffusivity cannot have. In the case of Bernard cells, thermal diffusivity is entirely stabilizing. In the case of

[1] J. L. Synge, Proc. Roy. Soc. (London) **A167**, 250–256 (1938).
[2] C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, New York, 1955), pp. 49–50.
[3] S. Chandrasekhar, J. Ratl. Mech. Analysis **3**, 181–207 (1954).
[4] H. Stommel, A. B. Arons, and D. Blanchard, Deep-Sea Research **3**, 152–153 (1956).
[5] G. I. Taylor, Phil. Trans. Roy. Soc. London **A223**, 289–343 (1923).

Stommel's "salt fountain" or in the present case (2), thermal diffusivity is entirely destabilizing. It cannot play *at once* the dual role which viscosity *simultaneously* plays. Since the destabilizing effect of viscosity is so rare that it is now exclusively associated with Tollmien-Schlichting waves, it seems worthwhile to present an essentially distinct instance of the same effect.

The second part of this paper is a discussion of the effect of compressibility on the stability of a revolving gas. A new criterion for stability is derived by neglecting the effects of viscosity and diffusivity. But since either viscosity or diffusivity can be destabilizing, as will be shown, even this criterion is not very useful and must be replaced by a more stringent one in the form of two conditions to be *simultaneously* satisfied.

II. FORMULATION OF THE PROBLEM

The radii of the cylinders will be denoted by r_1 and r_2 (with $r_2 > r_1$). The angular velocities of the cylinders will be denoted by Ω_1 and Ω_2. Thus the velocity of mean flow is, in cylindrical coordinates (r, θ, z),

$$V = A_* r + (B_*/r), \tag{1}$$

in which

$$A_* = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad B_* = -\frac{(\Omega_2 - \Omega_1) r_1^2 r_2^2}{r_2^2 - r_1^2}. \tag{2}$$

If the temperatures at the walls are T_1 and T_2, the temperature distribution in the primary flow is given by

$$T = T_1 + (T_2 - T_1)\frac{\ln r - \ln r_1}{\ln r_2 - \ln r_1}, \tag{3}$$

The density distribution is then given by

$$\rho = \rho_1[1 - \alpha(T - T_1)], \tag{4}$$

in which α is the coefficient of volume expansion.

If axisymmetry of the disturbance is assumed, the linearized equations of motion are, with θ as the temperature perturbation,

$$\rho_1\left[\frac{\partial u}{\partial t} - \frac{2Vv}{r} + \alpha\theta\frac{V^2}{r}\right] = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{5}$$

$$\rho_1\left(\frac{\partial v}{\partial t} + 2A_* u\right) = \mu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{6}$$

$$\rho_1\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \mu\nabla^2 w, \tag{7}$$

in which u, v, and w are the components of the velocity of the disturbance, p the perturbation pressure, t the time, μ the viscosity, which is assumed constant, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{8}$$

The equation of continuity is

$$[\partial(ru)/\partial r] + [\partial(rw)/\partial z] = 0. \tag{9}$$

The diffusion equation is

$$\partial\theta/\partial t + u(dT/dr) = \kappa\nabla^2\theta, \tag{10}$$

in which, as stated before, θ is the temperature perturbation (not the second coordinate which is not needed because of axisymmetry), and κ is the thermal diffusivity.

Assuming, after Taylor,

$$(u, v, \theta) = [u_1(r), v_1(r), \theta_1(r)]\cos\lambda z\, e^{\sigma t},$$
$$w = w_1(r)\sin\lambda z\, e^{\sigma t},$$

and utilizing the equation of continuity, we can write the equations of motion and of diffusion as

$$\nu L\left(L - \lambda^2 - \frac{\sigma}{\nu}\right)u_1 = 2\lambda^2\frac{V}{r}v_1 - \alpha\lambda^2\frac{V^2}{r}\theta_1, \tag{11}$$

$$\nu[L - \lambda^2 - (\sigma/\nu)]v_1 = 2A_* u_1, \tag{12}$$

$$\kappa[L' - \lambda^2 - (\sigma/\kappa)]\theta_1 = (dT/dr)u_1, \tag{13}$$

in which

$$L' = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \qquad L = L' - \frac{1}{r^2}.$$

The conditions that the velocity components vanish at the boundaries can be written, by virtue of (9), in the following form:

$$u_1 = 0, \quad du_1/dr = 0, \quad v_1 = 0 \quad \text{at } r = r_1 \text{ and } r_2. \tag{14a}$$

The boundary conditions for θ_1 are, if the walls are assumed much more conductive than the fluid,

$$\theta_1 = 0 \quad \text{at} \quad r = r_1 \text{ and } r_2. \tag{14b}$$

The differential system consisting of (11)–(13) and the boundary conditions define an eigenvalue problem.

III. CASE OF SMALL SPACING

Since the purpose of this paper is to investigate the roles of ν and κ, we shall, for simplicity, assume

$r_2 - r_1 \ll r_1$. In this case it is appropriate to use the dimensionless quantities

$$\xi = (r - r_1)/(r_2 - r_1), \qquad k = \lambda(r_2 - r_1),$$
$$R' = \Omega_1(r_2 - r_1)^2/\nu, \tag{15}$$

in which ξ varies from zero to 1, k is the dimensionless wave number, and R' a Reynolds number. If, furthermore, the substitutions

$$(u_2, v') = (\Omega_1 r_1)^{-1}(u_1, v_1) \quad \text{and} \quad \theta' = \alpha\theta_1 \tag{16}$$

are made, Eqs. (11) to (13) become, if higher powers of $(r_2 - r_1)/r_1$ are neglected in the differential operators on the left-hand sides,

$$(D^2 - k^2)\left(D^2 - k^2 - \frac{\sigma R'}{\Omega_1}\right)u_2$$
$$= 2k^2\omega R'v' - k^2\omega^2 R'\left(1 + \frac{r_2 - r_1}{r_1}\xi\right)\theta', \tag{17}$$

$$[D^2 - k^2 - (\sigma R'/\Omega_1)]v' = 2AR'u_2, \tag{18}$$

$$\left(D^2 - k^2 - \frac{\sigma R'}{\Omega_1}\mathrm{Pr}\right)\theta' = \mathrm{P\acute{e}}\left(\alpha r_1\frac{dT}{dr}\right)u_2. \tag{19}$$

In these equations, Pr is the Prandtl number ν/κ, $D = d/d\xi$,

$$\mathrm{P\acute{e}} = \Omega_1(r_2 - r_1)^2/\kappa \tag{20}$$

is the Péclet number, and

$$\omega = V/r\Omega_1 = A + B[1 + (r_2 - r_1)\xi/r_1]^{-2}, \tag{21}$$

with

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{(r_2^2 - r_1^2)\Omega_1}, \qquad B = \frac{(\Omega_1 - \Omega_2)r_2^2}{(r_2^2 - r_1^2)\Omega_1}. \tag{22}$$

Thus

$$A + B = 1,$$

and

$$\omega = 1 + \alpha'\xi, \qquad \alpha' = (\Omega_2/\Omega_1) - 1, \tag{23}$$

with higher powers in $(r_2 - r_1)/r_1$ neglected. Now, from (3) it follows that

$$\frac{dT}{dr} = \frac{T_2 - T_1}{r_2 - r_1}\left(1 - \frac{r_2 - r_1}{r_1}\xi\right)$$
$$= \beta\left(1 - \frac{r_2 - r_1}{r_1}\xi\right), \tag{24}$$

so that for $r_2 - r_1 \ll r_1$,

$$dT/dr = \beta = (T_2 - T_1)/(r_2 - r_1). \tag{25}$$

It will now be assumed that for neutral stability σ is zero, and not merely equal to a purely imaginary number. Furthermore, the following calculation is based on the assumption that α' is moderately small, so that $\alpha'^2 \ll 1$. Under these assumptions, (17), (18), and (19) become, after the term $\xi(r_2 - r_1)/r_1$ is neglected in (17),

$$(D^2 - k^2)^2 u_2 = 4AR'^2k^2(1 + \alpha'\xi)v_2$$
$$- \mathrm{P\acute{e}}\, R'\alpha\beta r_1 k^2(1 + 2\alpha'\xi)\theta_2, \tag{26}$$

$$(D^2 - k^2)v_2 = u_2, \tag{27}$$

$$(D^2 - k^2)\theta_2 = u_2, \tag{28}$$

with

$$v_2 = v'/2AR' \quad \text{and} \quad \theta_2 = \theta'/\mathrm{P\acute{e}}\,(\alpha\beta r_1). \tag{29}$$

The boundary conditions are

$$u_2 = 0, \qquad Du_2 = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad 1, \tag{30}$$

$$v_2 = 0, \qquad \theta_2 = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad 1. \tag{31}$$

Since the differential equations and the boundary conditions are identical for v_2 and θ_2, and the left-hand sides of (27) and (28) are not zero, we conclude that

$$\theta_2 = v_2, \tag{32}$$

and can write (26) and (27) as

$$(D^2 - k^2)^2 u_2 = k^2[(C - D) + \alpha'(C - 2D)\xi]v_2, \tag{33}$$

$$(D^2 - k^2)v_2 = u_2. \tag{34}$$

with

$$C = 4AR'^2, \qquad D = \mathrm{P\acute{e}}\, R'\alpha\beta r_1. \tag{35}$$

Comparing this pair of equations with that treated by Chandrasekhar,[6] we see that

$$D - C = T, \tag{36}$$

$$\alpha'(C - 2D)/(C - D) = \alpha_0, \tag{37}$$

in which T is the Taylor number $-4AR'^2$ obtained by Chandrasekhar in the case of no density gradient, and α_0 (denoted by α in his paper) is the value of $(\Omega_2 - \Omega_1)/\Omega_1$ for that case. Chandrasekhar found the critical value (T_c) of T for a variety of values of α_0, all corresponding to Ω_2/Ω_1 less than 1. Later, in considering a case of hydromagnetic instability, Yih[7] encountered a similar differential system and found the relationship between T_c and α_0 (denoted by β in Yih's paper) for three values of α_0 which are positive.[8]

[6] S. Chandrasekhar, Mathematika **1**, 5–13 (1954).
[7] C. -S. Yih, J. Fluid Mech. **5**, 436–44 (1959).
[8] These positive values of α_0 correspond to stability (for whatever T) in the problem studied by Taylor and Chandrasekhar, but not in the problem studied by Yih.

TABLE I. Critical Taylor numbers for various values of Ω_2/Ω_1.

Ω_2/Ω_1	2	1.5	1.25	1	0.50	0.25	
α_0	1	0.5	0.25	0	−0.50	−0.75	
k	3.12[a]	3.12[a]	3.12[a]	3.12	3.12	3.12	
T_c	1138[a]	1366[a]	1518[a]	1708	2275	2725	
Ω_2/Ω_1	0	−0.25	−0.50	−0.60	−0.70	−0.80	−0.90
α_0	−1	−1.25	−1.50	−1.60	−1.70	−1.80	−1.90
k	3.12	3.13	3.20	3.24	3.34	3.49	3.70
T_c	3390	4462	6417	7688	9433	11820	14940
Ω_2/Ω_1	−0.95	−1.00	−1.25	−1.50	−1.75	−2.00	
α_0	−1.95	−2.00	−2.25	−2.50	−2.75	−3.00	
k	3.86	4.00	4.61	5.06	5.60	6.05	
T_c	16760	18680	30460	46190	67590	95630	

[a] Values given by Yih.[7] The rest of the values were given by Chandrasekhar.[6]

The values of T_c against α_0 are given in Table I, with the corresponding wave numbers. The first three lines are reproduced from Yih and the rest from Chandrasekhar.[6] With T_c substituted for T in (36), solution of (36) and (37) yields the parametric relationship between C and D:

$$C = [(\alpha_0/\alpha') - 2]T_c, \tag{38}$$

$$D = [(\alpha_0/\alpha') - 1]T_c. \tag{39}$$

For a given α', values of α_0 are assumed, and the corresponding values of T_c read off from Table I, and C and D are then computed. Since both C and D contain R', in order to separate the effects of viscosity and diffusivity, the value of $D/|C|^{\frac{1}{2}}$ is plotted against C. The curves for various values of α' are given in Fig. 1. For each value of α' (corresponding to a positive value of Ω_2/Ω_1) there is a curve consisting of one or two branches, above which the flow is unstable and below which the flow is stable. The ordinate of the curves is

$$D/|C|^{\frac{1}{2}} \quad \text{or} \quad \text{Pé}\, \alpha\beta r_1/2\, |A|^{\frac{1}{2}}, \tag{40}$$

which is independent of the viscosity. The graphs show that for a given value of this parameter, there is a region in which the flow is destabilized as the viscosity is increased, i.e., the region to the right of the point of relative minimum of the ordinate of the curve for the particular value of α' considered. Outside of this region viscosity is always stabilizing. Furthermore, for negative values of C the curves eventually dip below the horizontal axis. For each such curve the region below the horizontal axis is a region in which thermal diffusivity has a destabilizing effect. Outside of this region thermal diffusivity is always stabilizing.

TABLE II. Classification of cases.

Case 1	$A < 0$,	$\beta > 0$,
Case 2	$A < 0$,	$\beta < 0$,
Case 3	$A = 0$,	$\beta > 0$,
Case 4	$A = 0$,	$\beta < 0$,
Case 5	$A > 0$,	$\beta > 0$,
Case 6	$A > 0$,	$\beta < 0$.

A. Special Case of Nearly Rigid Rotation

For the special case $\Omega_2 \simeq \Omega_1$, $\alpha' \simeq 0$, and by virtue of (37) α_0 also vanishes approximately. This makes $T_c = 1708$. With

$$S = -4AR'^2 + \text{Pé}\, R'\alpha\beta r_1, \tag{41}$$

the dual roles of viscosity and diffusivity can be brought forth very clearly by means of graphs. There are six cases, which are given in Table II. The first four cases are realistic because α' is only nearly, not exactly, zero. The cases in which $\beta = 0$ will not be discussed here, because they have been discussed thoroughly in the existing literature. It may be mentioned here that for α' equal to zero the method of Pellew and Southwell[9] can be used

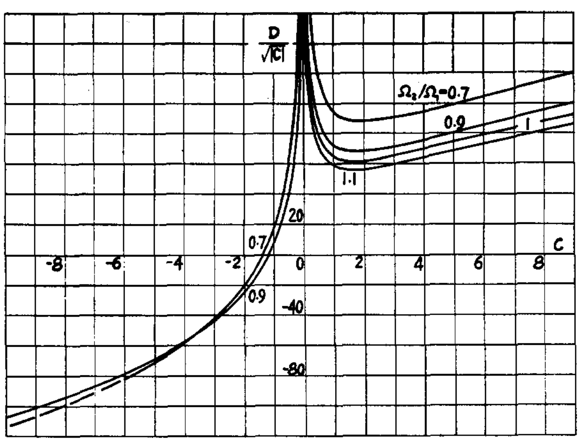

FIG. 1. Stability chart for the case of small spacing. $C = 4AR'^2$, $D = \text{Pé}\, R'\alpha\beta r_1$, so that $D/|C|^{\frac{1}{2}} = (1/|4A|^{\frac{1}{2}})\, \text{Pé}\, \alpha\beta r_1$. For each value of Ω_2/Ω_1, the region above the curve or curves corresponds to instability, and the region below corresponds to stability.

[9] A. Pellew and R. V. Southwell, Proc. Roy. Soc. (London) **A176**, 312–343 (1940).

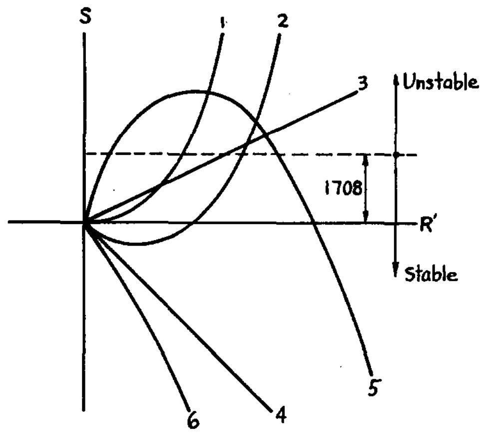

FIG. 2. A schematic drawing showing the stabilizing or destabilizing effect of viscosity for the special case $\Omega_2 \simeq \Omega_1$. Here $S = -4AR'^2 + \text{Pé}\, R'\alpha\beta r_1$, and the numbering of the curves corresponds to the six cases (or subcases) considered.

to show that for neutral stability $\sigma = 0$ exactly.

If S is plotted against R' (Fig. 2), the curves are parabolas (concave upward) in cases 1 and 2, straight lines in cases 3 and 4, and parabolas (concave downward) in cases 5 and 6. In cases 1 and 3 the effect of viscosity is stabilizing. The reverse is true in cases 4 and 6, although the destabilizing effect is never great enough to make the flow actually unstable. In case 2 the effect of viscosity is mainly to stabilize, although there is a region in which viscosity can render the flow less stable. This destabilizing effect is again never great enough to make the flow actually unstable. Case 5 is the most interesting. To the left of the crown of the parabola, viscosity has a stabilizing effect. To the right, it has a destabilizing effect. The destabilizing effect can actually bring about instability if the value of S at the crown of the parabola is greater than 1708 (as shown in Fig. 2). In general, the effect of viscosity is to stabilize if $dS/dR' > 0$, and to destabilize if $dS/dR' < 0$.

S can also be plotted against Pé (Fig. 3). The effect of diffusivity is clear-cut. In cases 1, 3, and 5, diffusivity has a stabilizing effect. In cases 2, 4, and 6, the opposite is true. Except in case 2, the destabilizing effect is never great enough to make the flow actually unstable.

From Figs. 2 and 3 it can be seen that in cases 4 and 6, i.e., when neither the circulation nor the density decreases outward, the flow is stable. This is not limited either to the case of small spacing or to the special case of $\Omega_2 \simeq \Omega_1$, so long as Ω_2/Ω_1 is positive. Indeed, there are only two causes for instability: outwardly decreasing circulation and outwardly decreasing density. When both causes are present, viscosity and thermal diffusivity can only be stabilizing. When only the first cause is present, thermal diffusivity has the effect of removing the stabilizing effect of the positive radial density gradient and of bringing out the destabilizing effect of circulation variation, possibly to such an extent for actual instability to occur. When only the second cause is present, viscosity has a dual role, as mentioned before. On the one hand it is, as usual, a dissipative agent. On the other, it tends to equalize the circulation of a displaced fluid ring with that of its new surrounding, and thus to remove the stabilizing effect of the outward increase of circulation and to bring the destabilizing effect of negative density gradient in the radial direction into prominence. When neither cause is present, the flow is definitely stable. Consequently, the criterion that the flow is stable

$$d(\rho V^2 r^2)/dr \geq 0,$$

derived by Rayleigh and von Kármán (see Lin[2]) from different viewpoints is not only not necessary, but also not sufficient for stability, if diffusive effects are taken into account. The criterion is much more stringent when ρ also varies, and now reads that the flow is stable if

$$d\rho/dr \geq 0 \quad \text{and} \quad d(Vr)^2/dr \geq 0, \tag{42}$$

or, in the case under consideration, if

$$\beta \leq 0 \quad \text{and} \quad A \geq 0. \tag{43}$$

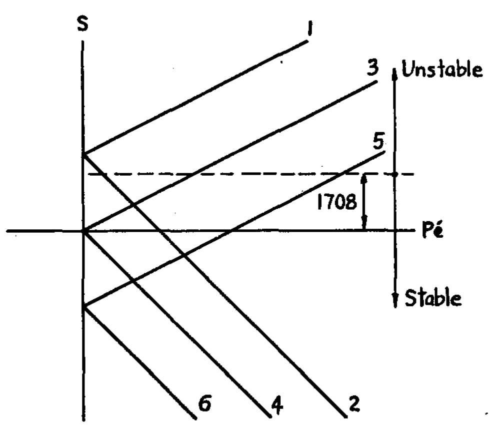

FIG. 3. A schematic drawing showing the stabilizing or destabilizing effect of thermal diffusivity. The symbol S and the numbering of the curves have the same meanings as in Fig. 2.

IV. EFFECT OF COMPRESSIBILITY

Since the stability of the flow of a gas between rotating cylinders kept at different temperatures bears upon that of the flow of a gas around a convex surface, which has been discussed by several authors, it is appropriate to discuss here very briefly the effect of compressibility.

With V denoting the velocity parallel to a solid surface and r the normal distance measured from the center of curvature, the criterion that for

$$d(\rho V^2 r^2)/dr \geq 0 \tag{44}$$

the flow of a fluid along a convex surface must be stable is correct only if the fluid is a liquid and if the effects of viscosity and diffusivity are neglected. For a gas, a material ring displaced from $r = r_1$ to $r = r_2$ will not maintain its density, and it is not true that the strongest destabilizing effect is obtained if we take $\tilde{\rho}_1 = \rho_1$, in which $\tilde{\rho}_1$ is the density of the fluid ring when it has reached its new position $r = r_2$. In fact, the radial pressure gradient is $\rho V^2/r$, so that the pressure increases outward, and the fluid in a material ring will be compressed as it moves out, and have a density $\tilde{\rho}_1$ greater than its original density ρ_1. Thus, assuming isentropic process, we obtain

$$\tilde{\rho}_1 = \rho_1 (p_2/p_1)^{1/\gamma}, \tag{45}$$

and (44) should be replaced by

$$d(\rho V^2 r^2 / p^{1/\gamma})/dr \geq 0, \tag{46}$$

or

$$d[V^2 r^2 \exp(-S/c_p)]/dr \geq 0, \tag{47}$$

in which S is now the entropy and c_p the specific heat of the gas at constant pressure. The criterion (44) obtained by Lees[10] and Lessen[11] is therefore approximately correct if the change of pressure is small. The pressure gradient, being equal to $\rho V^2/r$, is usually not large enough to cause any significant pressure variation with r, provided the pressure at $r = r_1$ is not exceedingly low. If only the stability of the boundary layer (which is usually very thin) is considered, the approximate criterion used by these authors is practically identical with (46), and, within the implied limitations, usually quite accurate. However, if $\rho V^2 r^2$ does not change with r, the destabilizing effect of pressure variation is quite critical, and, but for mitigating circumstances brought about by diffusive and dissipative agents, would actually cause instability. This is true both in gas flow between cylinders and in gas flows around a convex surface.

Actually, the density $\tilde{\rho}_1$ defined by (45) is a potential density (at a reference pressure p_2, say), the concept of which is very familiar to meteorologists. Thus if we identify the ρ in (44) as the potential density, (44) and (46) become identical.

Neglecting pressure variation with r, Lees[10] has reached the conclusion that cooling at a convex surface can never cause instability. This is again usually true, provided the effects of diffusivities can be neglected. But even under this provision it may not be true when pressure variation plays an important role, such as when the pressure at the surface is low (so that pressure variation contributes heavily to density variation), or when there exists a region where $d(Vr)/dr$ is nearly zero (presumably outside of the boundary layer).

As has been demonstrated for the case of incompressible fluids, the effects of viscosity and thermal diffusivity can be destabilizing, so that even (47) is insufficient for the stability of real fluids. The sufficient conditions for stability of real fluids are more stringent, and are

$$dS/dr \leq 0 \quad \textit{and} \quad d(Vr)^2/dr \geq 0.$$

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[10] L. Lees, J. Aeronaut. Sci. **25**, 407–8 (1958).

[11] M. Lessen, "Hydrodynamic stability of curved laminar compressible flows," IAS Preprint No. 812 (cited by Lees) (1958).