A note on energy bounds for boson matter

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Two new proofs are given of the Dyson and Lenard lower bound for the energy of matter with boson electrons. Another result is a new inequality for the two-point correlation function.

1. INTRODUCTION AND RESULTS

We consider a system of like positively charged particles described by a field \( \phi \) and like negatively charged particles described by a field \( \bar{\phi} \). The charge is denoted by \( e \), any masses that appear satisfy either \( m = \infty \) or \( 2m = 1 \), and the total number of particles is \( 2N \).

\[ \rho(x, y) = \langle \bar{\phi}(x) \phi(y) \rangle \]

and thus is simply related to the usual two-point correlation function, the first term in (1.2). We note the following two theorems.

**Theorem 1:** If both sets of particles are bosons and not both masses are \( 0 \), then there is a constant \( c \) such that the ground state energy, \( E_N \), satisfies

\[ E_N \geq -\frac{c}{R} \]

where \( R \) is the distance between the \( i \)th particle and its nearest neighbor. (For interest we repeat the remark from Ref. 1 that, to improve on the \( 5/3 \) power in Theorem 1, one would have to improve on this estimate.)

To prove Theorem 2 from (2.2) we note it is sufficient to show

\[ \frac{1}{c} \left( \frac{1}{R} \right)^{1/2} \leq 1 \]

(1.4)

**Theorem 2:** There is a constant \( c \) such that if \( f(r) \) is a right-continuous monotonically decreasing nonnegative function and

\[ \int_{|x-y| < a} \rho^2 = D \]

then

\[ \int_{|x-y| > a} \rho f(|x-y|) \leq c \left( \frac{D}{\alpha'^2} \right)^{1/2} c^{1/2} \int_0^\infty f(r) r \, dr \]

(1.5)

Theorem 1 is a slightly strengthened form of a theorem of Dyson and Lenard. The original theorem requires both masses to be finite. It is believed that the best exponent is \( 7/5 \) rather than \( 5/3 \). Curiously both proofs we present of Theorem 1, very different, when pursued, are limited by a configuration of linear size \( 1/N^{1/3} \) and average spacing \( 1/N^{2/3} \).

Theorem 2 follows from the Packing inequality, Fact 2 of Ref. 4, by an easy argument. It is used in our second proof of Theorem 1. The proof of Theorem 2 is given in Sec. 4.

The body of the paper presents two proofs of Theorem 1 (one proof yields the original theorem) in Sec. 2 and Sec. 3. We feel the techniques of this paper are interesting and aesthetic in their own right—but our motivation is to use these techniques to generalize these theorems to a form where they will be useful in developing cluster expansions. Along this line, work is in progress to extend the results of Refs. 7 and 8.
3. SECOND PROOF OF THEOREM 1

In this proof, more technical in nature, we lean heavily on the methods of Ref. 4. We make the inessential simplification of considering a system of \( N \) positive and \( N \) negative particles, the positive particles in fixed classical positions. We also place the system in a unit box (with Neumann or periodic boundary conditions). We need show there is a \( c \) with
\[
0 \leq cN^{5/3} + H. \tag{3.1}
\]
We define
\[
A_n = \int_{|x-y|<n}\rho', \tag{3.2}
\]
deduce from the Packing inequality\(^4\) that there is a \( C_2 \) such that for any \( c_1 < 1 \) it is possible to find an \( n \) satisfying
\[
c_n n^{5/2} \leq A_n \leq c_2 C_1 n^{5/2}. \tag{3.3}
\]
This is achieved by starting with \( n = N^{2/3} \) and increasing \( n \) until (3.3) holds. \( A \) and \( n \) are now defined as values for which (3.3) holds.

With the notation of Ref. 4 we see
\[
H_5 \geq c_2 A - N c_1 e^{-1}. \tag{3.4}
\]
and
\[
H_4 \geq -2c_2 n N. \tag{3.5}
\]
Considering \( H_5 \), it is enough to show, to complete the proof, that
\[
(-\Delta - V + c_1 n) > 0, \tag{3.6}
\]
where
\[
V = c_2 \int [\exp(-nr)/r] \partial \phi. \tag{3.7}
\]
Analogous to (2.4) we find, using an H.S. norm, that (3.6) is implied by
\[
(1/n^{1/4}) ||V||_2 \leq c_6 \tag{3.8}
\]
or
\[
(1/n^{1/4}) \left[ \int \rho (1/n) \exp(-nr) \right]^{1/2} \leq c_5. \tag{3.9}
\]
Using Theorem 2, with \( f(r) = \exp(-nr) \), (3.9) is implied by
\[
(1/n^{5/4}) A^{1/2} \left[ n^3 \int \rho^2 \exp(-nr) dr \right]^{1/2} \leq c_6 \tag{3.10}
\]
or
\[
A/n^{3/2} \leq c_7. \tag{3.11}
\]
(3.11) holds if \( c_1 \) is small enough. The choices of constants \( c_i \) can be made independent of \( N \) and the positive particles' configuration. The devoted reader may observe that if it were desired to prove Theorem 1 for any power larger than \( 5/3 \) (instead of exactly \( 5/3 \)) this second proof could be much simplified, but our intended applications and generalizations require the \( 5/3 \) power.

4. PROOF OF THEOREM 2

We start from Fact 2 of Ref. 4 in the form:

**Fact 2:** There is a constant \( c_2 > 0 \) such that if \( 0 \leq R' \leq R \) then
\[
\int_{|x-y|<R'} \rho' \geq c_2 (R'/R)^{3} \int_{|x-y|<R} \rho'. \tag{4.1}
\]
Let \( f(r) \) be a right-continuous monotonically decreasing nonnegative function. Define \( S_1(r) \) by
\[
S_1(r) = \begin{cases} 1, & r < \lambda, \\ 0, & r \geq \lambda. \end{cases} \tag{4.2}
\]
For \( r \geq \alpha \), \( f(r) \) may be expressed in the form
\[
f(r) = \int_{0}^{\infty} d\sigma(\lambda) S_1(\lambda), \tag{4.3}
\]
with \( \sigma(\lambda) \) a positive measure. It is sufficient to prove (1.5) with \( f(r) = S_1(\lambda) \), for then this form of (1.5) may be integrated with respect to the measure \( \sigma(\lambda) \) to obtain Theorem 2. Substituting, with \( \lambda \geq \alpha \), into (1.5), we get
\[
\int_{|x-y|<R} \rho' \leq c_2 (R'/R)^{3} \int_{|x-y|<R} \rho'. \tag{4.4}
\]
This is implied by
\[
\int_{|x-y|<R} \rho' \leq c_2 (R'/R)^{3} (R'^{3} + \int_{0}^{\infty} \rho' \sigma(\lambda) d\lambda). \tag{4.5}
\]
If \( c_3 \geq 1/c_2 \), then (4.5) holds by Fact 2, yielding (4.4) and the theorem.

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\(^1\)F. J. Dyson and A. Lenard, J. Math. Phys., 8, 423 (1967).\(^2\)
\(^2\)Dyson and Lenard in Ref. 3 obtained a proof of Theorem 1 in conjunction with their proof of stability of matter.