A note on energy bounds for boson matter*

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Two new proofs are given of the Dyson and Lenard lower bound for the energy of matter with boson electrons. Another result is a new inequality for the two-point correlation function.

1. INTRODUCTION AND RESULTS

We consider a system of like positively charged particles described by a field ϕ and like negatively charged particles described by a field ψ. The charge is denoted by ϵ, any masses that appear satisfy either $m=\infty$ or $2m=1$, and the total number of particles is $2N$. $\rho'(x, y)$ is defined by

$$\rho'(x, y)=\langle\bar{\phi}\phi(x)\,\bar{\phi}\phi(y)\rangle \tag{1.1}$$

$$=\langle:\bar{\phi}\phi(x)\,\bar{\phi}\phi(y):\rangle+\delta(x-y)\langle\bar{\phi}(x)\,\phi(x)\rangle \tag{1.2}$$

and thus is simply related to the usual two-point correlation function, the first term in (1.2). We note the following two theorems.

Theorem 1: If both sets of particles are bosons and not both masses are ∞, then there is a constant c such that the ground state energy, E_N, satisfies

$$E_N \geqslant -cN^{5/3}. \tag{1.3}$$

Theorem 2: There is a constant c such that if $f(r)$ is a right-continuous monotonically decreasing nonnegative function and

$$\int_{|x-y|<\alpha}\rho'=D, \tag{1.4}$$

then

$$\int_{|x-y|\geqslant\alpha}\rho' f(|x-y|)\leqslant c(D/\alpha^3)\left[\alpha^3 f(\alpha)+\int_\alpha^\infty f(r)\,r^2\,dr\right]. \tag{1.5}$$

Theorem 1 is a slightly strengthened form of a theorem of Dyson and Lenard.[1] (The original theorem requires both masses to be finite.[2] It is believed that the best exponent is 7/5 rather than 5/3.[6] Curiously both proofs we present of Theorem 1, very different, when pursued, are limited by a configuration of linear size $\sim 1/N^{1/3}$ and average spacing $\sim 1/N^{2/3}$.

Theorem 2 follows from the Packing inequality, Fact 2 of Ref. 4, by an easy argument. It is used in our second proof of Theorem 1. The proof of Theorem 2 is given in Sec. 4.

The body of the paper presents two proofs of Theorem 1 (one proof yields the original theorem) in Sec. 2 and Sec. 3. We feel the techniques of this paper are interesting and aesthetic in their own right—but our motivation is to use these techniques to generalize these theorems to a form where they will be useful in developing cluster expansions. Along this line, work is in progress to extend the results of Refs. 7 and 8.

2. FIRST PROOF OF THEOREM 1

Our first proof requires that both masses be finite:

$$H=-\sum\Delta_i+\tfrac{1}{2}\sum_{i\neq j}(\pm)\epsilon^2/|x_i-x_j|. \tag{2.1}$$

We use the electrostatic inequality, an easy inequality from Ref. 1, to obtain

$$H\geqslant-\sum\Delta_i-c\sum(1/R_i), \tag{2.2}$$

where R_i is the distance between the ith particle and its nearest neighbor. (For interest we repeat the remark from Ref. 1 that, to improve on the 5/3 power in Theorem 1, one would have to improve on this estimate.)

To prove Theorem 2 from (2.2) we note it is sufficient to show

$$-\Delta-c/R\geqslant-c_1N^{2/3}, \tag{2.3}$$

where (2.3) describes the motion of one particle in the "field" of N fixed particles. R is the minimum distance to one of the fixed particles. [For notational reasons the $(2N-1)$ fixed particles each particle in (2.2) sees has been changed to N.] This inequality is implied by

$$\left\|\frac{1}{(-\Delta+c_1N^{2/3})^{1/2}}\,\frac{c}{R}\,\frac{1}{(-\Delta+c_1N^{2/3})^{1/2}}\right\|\leqslant 1. \tag{2.4}$$

We note that

$$c/R\leqslant c_1N^{2/3}/2+(c/R)\chi, \tag{2.5}$$

where χ is the characteristic function of the set where

$$c/R\geqslant\tfrac{1}{2}c_1N^{2/3}. \tag{2.6}$$

It is enough now to show

$$\left\|\frac{1}{(-\Delta)^{1/2}}\left(\frac{c}{R}\,\chi\right)\frac{1}{(-\Delta)^{1/2}}\right\|_{\mathrm{H.S.}}\leqslant\tfrac{1}{2}. \tag{2.7}$$

The subscript indicates the Hilbert–Schmidt norm. We use the Sobolev inequality[9]

$$\left|\int d^3x\int d^3y\,g(x)f(y)/|x-y|^2\right|$$
$$\leqslant c_2\|f\|_{3/2}\|g\|_{3/2} \tag{2.8}$$

to convert (2.7) to

$$\|(c/R)\chi\|_{3/2}\leqslant c_3. \tag{2.9}$$

It is easy to see

$$\|(c/R)\chi\|_{3/2}\leqslant\left[4\pi N\int_0^{2c/c_1N^{2/3}}(c/r)^{3/2}r^2\,dr\right]^{2/3} \tag{2.10}$$
$$\leqslant[(c^3/c_1^{3/2})c_4]^{2/3}.$$

so if c_1 is large enough, (2.3) holds and the collapse inequality of Ref. 1 has been proven.

3. SECOND PROOF OF THEOREM 1

In this proof, more technical in nature, we lean heavily on the methods of Ref. 4. We make the inessential simplification of considering a system of N positive and N negative particles, the positive particles in fixed classical positions. We also place the system in a unit box (with Neumann or periodic boundary conditions). We need show there is a c with

$$0 \leq cN^{5/3} + H. \tag{3.1}$$

We define

$$A_n = \int_{|x-y|<1/n} \rho' \tag{3.2}$$

and deduce from the Packing inequality[4] that there is a c_2 such that for any $c_1 < 1$ it is possible to find an n satisfying

$$c_1 n^{3/2} \leq A_n \leq c_2 c_1 n^{3/2}. \tag{3.3}$$

This is achieved by starting with $n = N^{2/3}$ and increasing n until (3.3) holds. A and n are now defined as values for which (3.3) holds.

With the notation of Ref. 4 we see

$$H_6 \geq c_3 n(A - N)\, e^{-1} \tag{3.4}$$

and

$$H_4 \geq -2c_3 n N. \tag{3.5}$$

Considering H_5, it is enough to show, to complete the proof, that

$$(-\Delta - V + c_3 n) \geq 0, \tag{3.6}$$

where

$$V = \epsilon^2 \int [\exp(-nr)/r]\, \bar{\phi}\phi. \tag{3.7}$$

Analagous to (2.4) we find, using an H.S. norm, that (3.6) is implied by

$$(1/n^{1/4}) \| V \|_2 \leq c_4 \tag{3.8}$$

or

$$(1/n^{1/4}) \left[\int \rho'(1/n) \exp(-nr)\right]^{1/2} \leq c_5. \tag{3.9}$$

Using Theorem 2, with $f(r) = \exp(-nr)$, (3.9) is implied by

$$(1/n^{3/4}) A^{1/2} \left[n^3 \int r^2 \exp(-nr)\, dr\right]^{1/2} \leq c_6 \tag{3.10}$$

or

$$A/n^{3/2} \leq c_7. \tag{3.11}$$

(3.11) holds if c_1 is small enough. The choices of constants c_i can be made independent of N and the positive particles' configuration.

The devoted reader may observe that if it were desired to prove Theorem 1 for any power larger than 5/3 (instead of exactly 5/3) this second proof could be much simplified, but our intended applications and generalizations require the 5/3 power.

4. PROOF OF THEOREM 2

We start from Fact 2 of Ref. 4 in the form:

Fact 2: There is a constant $c_2 > 0$ such that if $0 \leq R' \leq R$ then

$$\int_{|x-y|<R'} \rho' \geq c_2 (R'/R)^3 \int_{|x-y|<R} \rho'. \tag{4.1}$$

Let $f(r)$ be a right-continuous montonically decreasing nonnegative function. Define $S_\lambda(r)$ by

$$S_\lambda(r) = \begin{cases} 1, & r < \lambda, \\ 0, & r \geq \lambda. \end{cases} \tag{4.2}$$

For $r \geq \alpha$, $f(r)$ may be expressed in the form

$$f(r) = \int_\alpha^\infty d\sigma(\lambda)\, S_\lambda(r) \tag{4.3}$$

with $\sigma(\lambda)$ a positive measure. It is sufficient to prove (1.5) with $f(r) = S_\lambda(r)$, for then this form of (1.5) may be integrated with respect to the measure $\sigma(\lambda)$ to obtain Theorem 2. Substituting, with $\lambda \geq \alpha$, into (1.5), we get

$$\begin{aligned} \int_{\lambda > |x-y| \geq \alpha} \rho' &\leq c(D/\alpha^3)\left(\alpha^3 + \int_\alpha^\lambda r^2\, dr\right) \\ &\leq cD[(\lambda^3 + 2\alpha^3)/3\alpha^3]. \end{aligned} \tag{4.4}$$

This is implied by

$$\int_{|x-y|<\lambda} \rho' \leq c \cdot \int_{|x-y|<\alpha} \rho' \cdot (\lambda^3/3\alpha^3). \tag{4.5}$$

If $c/3 \geq 1/c_2$, then (4.5) holds by Fact 2, yielding (4.4) and the theorem.

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