Unsteady Boundary Layers on Vibrating Spheres in a Uniform Stream

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The influence of radial vibrations on forced convection from a sphere placed in a uniform stream of an incompressible fluid is studied mathematically. Consideration is given to first-order perturbations of the harmonic vibrations of the spherical surface for both high and low frequencies. Theoretical results are obtained by means of a Kármán–Pohlhausen method for velocity and temperature distributions, skin friction, heat-transfer rate and the shift in the separation point of the boundary layer. Numerical calculations are carried out for boundary-layer thickness and velocity profiles in the fluctuating boundary layer.

INTRODUCTION

In recent years considerable attention has been focused on the response of boundary-layer flows to vibrations and flow oscillations[1-7]. This paper is devoted to examining the influence of small, harmonic, radial vibrations of a sphere placed in a free stream of an incompressible fluid. These vibrations induce oscillations of velocity and temperature in the boundary layer. The amplitude and phase angle of these oscillations are obtained for both high and low frequencies.

Also studied are the response of both skin friction and heat-transfer rate and the shift in the separation point of the boundary layer due to the harmonic vibrations of the spherical surface.

ANALYSIS

The physical system analyzed consists of a vibrating sphere placed in a stream $U_0(x)$ of an incompressible fluid. A spherical coordinate system r, θ, ϕ is fixed at the center of the sphere. The sphere is undergoing harmonic vibrations in the radial direction. Considerations of compressibility require the radial velocity of the sphere to be of a lower order than the velocity of sound in the fluid. The harmonic vibration of the sphere is represented by its displacement as $r_0 + \epsilon r_0 e^{i\omega t}$, where r_0 is the mean radius, ϵr_0 is the amplitude of vibration, ω is the frequency of the vibration and t is time.

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The sphere is maintained at a uniform temperature, T_W and is in contact with the fluid at temperature T_∞ and pressure P_∞ flowing in the ϕ direction with velocity U_∞ at infinity. The fluid has constant properties. The governing equations of mass, momentum, and energy may be written in spherical coordinates as

continuity

$$\frac{1}{(r^*)^2}\frac{\partial}{\partial r^*}[(r^*)^2 v^*] + \frac{1}{r^*\sin\theta^*}\frac{\partial}{\partial\theta^*}(v^*\sin\theta^*) = 0, \quad (1)$$

r^*-component momentum

$$\frac{\partial u^*}{\partial t} + v^*\frac{\partial u^*}{\partial r^*} + \frac{u^*}{r^*}\frac{\partial u^*}{\partial\theta^*} + \frac{u^* v^*}{r^*} = -\frac{1}{\rho r}\frac{\partial p^*}{\partial\theta^*} + \nu\left[\nabla^2 u^* + \frac{2}{(r^*)^2}\frac{\partial v^*}{\partial\theta^*} - \frac{u^*}{(r^*)^2\sin^2\theta^*}\right], \quad (2)$$

θ^*-component momentum

$$\frac{\partial v^*}{\partial t} + v^*\frac{\partial v^*}{\partial r^*} + \frac{2u^*}{r^*}\frac{\partial v^*}{\partial\theta^*} - \frac{u^{*2}}{r^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial r^*} + \nu\left[\nabla^2 v^* + \frac{2v^*}{(r^*)^2} - \frac{2}{(r^*)^2}\frac{\partial u^*}{\partial\theta^*} - \frac{2u^*}{(r^*)^2}\cot\theta^*\right], \quad (3)$$

energy

$$\frac{\partial T^*}{\partial t} + v^*\frac{\partial T^*}{\partial r^*} + \frac{u^*}{r^*}\frac{\partial T^*}{\partial\theta^*} = \alpha\left\{\frac{1}{(r^*)^2}\frac{\partial}{\partial r^*}\left[(r^*)^2\frac{\partial T^*}{\partial r^*}\right] + \frac{1}{(r^*)^2}\cdot\frac{1}{\sin\theta^*}\frac{\partial}{\partial\theta^*}\left(\sin\theta^*\frac{\partial T^*}{\partial\theta^*}\right)\right\}, \quad (4)$$

where u^* and v^* are the velocity components in the θ^* and r^* directions, respectively; p^* is the pressure; ν is the kinematic viscosity; ρ is the density; α is the thermal diffusivity; and the superscript * refers to the stationary coordinate system. The boundary conditions are

$$r^* = r_0 + \epsilon r_0 e^{i\omega t} : u^* = 0,\ v^* = \epsilon i\omega r_0 e^{i\omega t},\ T^* = T_w;$$
$$r^* \to \infty \qquad : u^* = U_0(x), \qquad T^* = T_\infty.$$

It is convenient to analyze the problem in a frame of reference fixed in the spherical surface, i.e., a coordinate system moving with the surface. This shift in reference systems has an advantage in that the boundary conditions at the surface are simplified. However, this introduces a relative acceleration between the moving system and the stationary system in the radial direction which alters the r^* momentum equation. The substitution of

$$r^* = y + (r_0 + \epsilon r_0 e^{i\omega t}),\ \theta^* = \theta \tag{5}$$

into the governing equations followed by the boundary-layer simplification yields

$$\frac{\partial(uR)}{\partial x} + \frac{\partial(vR)}{\partial y} = 0, \tag{6}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{7}$$

$$\frac{u^2}{r_0} + \epsilon r_0\omega^2 e^{i\omega t} = \frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{8}$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} + u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{9}$$

where $R(x)$ is the radius of the section of the sphere taken at right angles to the direction of the upstream flow. The boundary conditions now become

$$y = 0\ :u = v = 0, \qquad T = T_w;$$

$$y \to \infty\ :u = U_0, \qquad T = T_\infty.$$

The velocity components and temperatures in the moving and stationary coordinates are related as $u^* = u$, $v^* = v - \epsilon r_0 i\omega e^{i\omega t}$, and $T^* = T$.

FLOW BOUNDARY LAYER

Equation (8) is integrated to give the pressure distribution

$$-\frac{1}{\rho}(p - p_\infty) = \int_\delta^y \frac{\partial v}{\partial t}\,\partial y + \int_\delta^y \frac{u^2}{r_0}\,dy + \epsilon r_0\omega^2 e^{i\omega t}(y - \delta). \tag{10}$$

Upon the substitution of $\partial p/\partial x$ obtained from the above expression, Eq. (7) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_\infty}{\partial x} - \frac{\partial}{\partial x}\int_\delta^y \frac{u^2}{r}\,dy + \epsilon r_0\omega^2 e^{i\omega t}\frac{\partial\delta}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{11}$$

where δ is the unperturbed boundary-layer thickness.

The velocity perturbations in the boundary layer caused by the harmonic vibrations are expressed as

$$\begin{aligned} u &= u_0(x, y) + \epsilon u_1(x, y)e^{i\omega t}, \\ v &= v_0(x, y) + \epsilon v_1(x, y)e^{i\omega t}, \end{aligned} \tag{12}$$

where ϵ is assumed so small that the linearization of the governing equations and boundary conditions is possible. u_0 and v_0 are the mean velocity distributions which correspond to the solution of the steady flow over a sphere. u_0 has been found by Tomotika and Imai[8] to be

$$\frac{u_0}{U_0} = (2\eta - 2\eta^3 + \eta^4) + \frac{\lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4), \tag{13}$$

where

$$\eta = \frac{y}{\delta}, \qquad \lambda = \frac{\delta^2}{\nu}\frac{dU_0}{\partial x}.$$

The fluctuating part of the velocity is given by

$$i\omega u_1 + v_1\frac{\partial u_0}{\partial y} + v_0\frac{\partial u_1}{\partial y} + u_1\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_1}{\partial x} = -\int_\delta^y\left(\frac{2u_0}{r_0}\frac{\partial u_1}{\partial x} + \frac{2u_1}{r_0}\frac{\partial u_0}{\partial x}\right)dy + r_0\omega^2\frac{d\delta}{\partial x} + \nu\frac{\partial^2 u_1}{\partial y^2}, \tag{14}$$

$$\frac{\partial(u_1R)}{\partial x} + \frac{\partial(v_1R)}{\partial y} = 0, \tag{15}$$

which are obtained by substituting eq. (12) into eqs. (6) and (11) and retaining the first-order terms of ϵ.

For low frequency, one writes

$$\begin{aligned} u_1(x, y) &= u_s(x, y) + i\omega u_t(x, y), \\ v_1(x, y) &= v_s(x, y) + i\omega v_t(x, y), \end{aligned} \tag{16}$$

where (u_s, v_s) is the quasi-steady solution as $\omega \to 0$ [i.e., as the vibration of the spherical surface tends to the steady value $(1 + \epsilon)r_0$] and (u_t, v_t) is the acceleration-component of the velocity in the boundary layer. As the frequency approaches zero, the sphere radius and the fluid velocity vary as ϵr_0 and $\epsilon(u_s, v_s)$, respectively. Since (u_0, v_0) are known to be a function of x/r_0 and $y(3U_0/\nu r_0)^{\frac{1}{2}}$ it follows that[9]

$$(u_S, v_S) = \frac{\partial(u_0, v_0)}{\partial r}\,dr = r_0\frac{\partial}{\partial r}(u_0, v_0)$$

or

$$u_S = -\frac{1}{2}y\frac{\partial u_0}{\partial y}, \qquad v_S = \frac{1}{2}\left(v_0 - y\frac{\partial v_0}{\partial y}\right). \tag{17}$$

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Now Eqs. (17) are substituted into Eqs. (14) and (15). Using the fact that (u_s, v_s) is a solution for $(u_t, v_t) \to 0$ one obtains

$$i\omega u_t + v_0 \frac{\partial u_t}{\partial y} + v_t \frac{\partial u_0}{\partial y} + u_0 \frac{\partial u_t}{\partial x} + u_t \frac{\partial u_0}{\partial x} \cdot = -\frac{2}{r_0}\int_\delta^y \left(u_0 \frac{\partial u_t}{\partial x} + u_t \frac{\partial u_0}{\partial x}\right) dy = u_S - r_0\omega i \frac{d\delta}{\partial x} + \nu \frac{\partial^2 u_t}{\partial y^2} \tag{18}$$

and

$$\frac{\partial(u_t R)}{\partial x} + \frac{\partial(v_t R)}{\partial y} = 0. \tag{19}$$

Their boundary conditions are

$$y = 0: u_t = v_t = 0; \quad y \to \infty : u_t = 0.$$

By means of a Kármán–Pohlhausen integral method, Eq. (18) is integrated for sufficiently small values of the frequency. Thus, using equations (17) and (19), one gets

$$\left(i\omega - \frac{dR}{dx} U_0\right) \int_0^\infty u_t \, dy + (1 + R) \frac{d}{dx} \int_0^\infty u_0 u_t \, dy + RU_0 \frac{d}{dx} \int_0^\infty u_t \, dy + 2 \frac{dR}{dx} \int_0^\infty u_t u_0 \, dy + \nu\left(\frac{\partial u_t}{\partial y}\right)_{y=0} + \frac{2}{r_0} \int_0^\infty \left[\int_\delta^y \left(u_0 \frac{\partial u_t}{\partial x} + u_t \frac{\partial u_0}{\partial x}\right) dy\right] dy = r_0\omega^2 \frac{d\delta}{dx}\delta - \int_0^\infty u_S \, dy. \tag{20}$$

Now it is assumed that

$$u_t = [(\delta^2 F(x)/\nu)](1 - \eta)^2[A\eta + (2A - \tfrac{1}{2})\eta^2], \tag{21}$$

which satisfies the boundary conditions $u_t = 0$, $\nu(\partial^2 u_t/\partial y^2)_{y=0} = F(x)$ at $y = 0$ and $u_t = 0$, $\partial u_t/\partial y = 0$, at $y = \infty$, where

$$F(x) = r_0\omega i \frac{d\delta}{dx} - \frac{2}{r_0} \int_0^\delta \left(u_0 \frac{\partial u_t}{\partial x} + u_t \frac{\partial u_0}{\partial x}\right) dy.$$

With u_0, u_s, and u_t given as Eqs. (13), (17), and (21), respectively, Eq. (20) now becomes

$$\left(i\omega - \frac{dR}{dx} U_0\right)\left(\frac{9A - 1}{60}\right) \frac{\delta^3 F(x)}{\nu} + RU_0 \frac{d}{dx}\left[\frac{\delta^3 F(x)}{\nu}\left(\frac{9A - 1}{60}\right)\right] + 2\frac{dR}{dx}\frac{\delta^3 F(x) U_0}{\nu}\left(\frac{263A - 32}{2520} + \lambda \frac{10A - 1}{6048}\right)$$
$$+ (1+R)\frac{d}{dx}\left[\frac{\delta^3 F(x) U_0}{\nu}\left(\frac{263A - 32}{2520} + \lambda \frac{10A - 1}{6048}\right)\right] + \frac{2}{r_0}\delta \frac{d}{dx}\left[\frac{\delta^4 F(x) U_0}{\nu} \cdot \left\{-\frac{1327}{1260} A + \frac{463}{1008} - \lambda\left(\frac{2353}{3780} A + \frac{257}{2160}\right)\right\}\right]$$
$$= -U_0 \delta A - r_0\omega^2 i \frac{d\delta}{dx}\delta - \frac{U_0}{2}\delta\left(\frac{3}{10} - \frac{\lambda}{10}\right), \tag{22}$$

where

$$F(x) = r_0\omega i \frac{d\delta}{dx} - \frac{2}{r_0}\frac{d}{dx} \cdot \left[\frac{\delta^3 F(x) U_0}{2}\left(\frac{263A - 32}{2520} + \frac{10A - 1}{6048}\lambda\right)\right]. \tag{23}$$

Because of the small magnitude of δ, the terms having high orders of δ in Eqs. (22) and (23) are neglected. Under the approximation, A and $F(x)$ are found to be

$$A = -i \frac{r_0\omega^2}{U_0}\frac{d\delta}{dx} - \frac{1}{2}\left(\frac{3 - \lambda}{10}\right), \tag{24}$$

$$F(x) = ir_0\omega \, d\delta/dx. \tag{25}$$

Then Eq. (21) may be rewritten as

$$u_t = u_{tr} + iu_{ti}, \tag{26}$$

where

$$u_{tr} = -(r_0\omega)^2 \frac{\delta^2}{\nu U_0}\left(\frac{d\delta}{dx}\right)^2 (1 - \eta)^2(\eta + 2\eta^2),$$

$$u_{ti} = r_0\omega \frac{\delta^2}{\nu}\frac{d\delta}{dx}(1 - \eta)^2 \cdot \left[\frac{1}{2}\left(\frac{3}{10} - \frac{\lambda}{20}\right)(\eta + 2\eta^2) - 2\eta^2\right].$$

The amplitude of the oscillating component u_t is $[(u_s - \omega u_{ti})^2 + (\omega u_{tr})^2]^{\frac{1}{2}}$. Its phase angle, or the time lead of the oscillating component of the velocity ahead of the oscillation of the sphere in the r direction, is $\tan^{-1}[\omega u_{tr}/(u_s - \omega u_{ti})]$. The skin friction τ may be expressed as

$$\tau = -\rho\nu\left(\frac{\partial u_0}{\partial y}\right)_{y=0} + \epsilon e^{i\omega t}\left[\left(\frac{\partial u_S}{\partial y}\right)_{y=0} + i\omega\left(\frac{\partial u_t}{\partial y}\right)_{y=0}\right]$$
$$= \tau_0 + \epsilon e^{i\omega t}\left\{-\frac{1}{2}\tau_0 - i\frac{r_0\omega^2}{\nu}\delta\frac{d\delta}{dx} \cdot \left[\frac{r_0\omega^2}{U_0}\frac{d\delta}{dx} - \frac{1}{2}\left(\frac{3 - \lambda}{10}\right)\right]\right\}, \tag{27}$$

where $\tau_0 = -\rho\nu(\partial u_0/\partial y)_{y=0}$ is the skin friction at steady state. By equating τ to zero in Eq. (27), it is seen that for quasi-steady vibration the separation point remains steady.

For large frequency, another method may be used, which is based on the theory dealing with differential equations containing a large parameter. According to the theory, a first approximation to the solution may be obtained by retaining only the terms of the greatest magnitude and of the highest-order derivative. In the present case, Eq. (14) becomes

$$i\omega u_1 = r_0\omega^2 \frac{d\delta}{dx} + \nu \frac{\partial^2 u_1}{\partial y^2}, \tag{28}$$

subject to the conditions $u_1 = 0$ at $y = 0$ and $u_1 =$ finite. Consequently the resulting solution is

$$u_1 = -ir_0\omega\,(d\delta/dx)\{1 - \exp[-y(i\omega/\nu)^{\frac{1}{2}}]\}. \tag{29}$$

v_1 may be obtained from the equation of continuity as

$$v_1 = ir_0\omega \frac{1}{R}\frac{d}{dx}\left[\frac{d\delta}{dx}R\right]\left(y - \frac{1 - \exp[-y(i\omega/\nu)^{\frac{1}{2}}]}{(i\omega/\nu)^{\frac{1}{2}}}\right). \tag{30}$$

The skin friction is

$$\tau = \tau_0 - \epsilon i\omega\rho\nu r_0\left(\frac{\nu\omega}{\nu}\right)^{\frac{1}{2}} \frac{d\delta}{dx} e^{i\omega t}. \tag{31}$$

From Eq. (31), it is seen that the amplitude of the fluctuating skin friction increases with the $\frac{3}{2}$-power of the frequency but its phase angle lags that of the radial vibrations of the spherical surface by 45 degrees. Equation (31) with $\tau = 0$ shows that the shift in the mean separation point proportional to the $\frac{3}{2}$ power of the frequency.

THERMAL BOUNDARY LAYER

The linearization of the temperature equation (9) and its boundary conditions by

$$T = T_0(x, y) + \epsilon T_1(x, y)e^{i\omega t}$$

results in the expressions

$$u_0 \frac{\partial T_0}{\partial x} + U_0 \frac{\partial T_0}{\partial y} = \alpha \frac{\partial^2 T_0}{\partial y^2} \tag{32}$$

for the mean temperature T_0, with the boundary conditions $T_0 = T_W$ at $y = 0$, and $T_0 \to 0$ as $y \to \infty$;

$$i\omega T_1 + u_0 \frac{\partial T_1}{\partial x} + u_1 \frac{\partial T_0}{\partial x} + v_0 \frac{\partial T_1}{\partial y} + v_1 \frac{\partial T_0}{\partial y} = \alpha \frac{\partial^2 T_1}{\partial y^2} \tag{33}$$

for the fluctuating part of the temperature T_1 with the boundary conditions $T_1 = 0$ at $y = 0$ and ∞. The temperature of the stream may be conveniently taken as zero.

Using the equation of continuity and Eq. (32), one obtains

$$\left(\frac{d}{dx} + \frac{1}{R}\frac{dR}{dx}\right)\int_0^\delta u_0 T_0\, dy = -\alpha\left(\frac{\partial T_0}{\partial y}\right)_{y=0}. \tag{34}$$

Let the mean temperature profile be

$$T_0/T_W = (1 + a\eta)(1 - a\eta)^3, \tag{35}$$

where a is the ratio of the thickness of the velocity boundary layer to that of the temperature boundary layer. Then Eq. (34) has to satisfy the boundary conditions $T_0 = T_W$, $\partial^2 T_0/\partial y^2 = 0$ at $y = 0$ and $T_0 \to 0$, $\partial T_0/\partial y \to 0$, $\partial^2 T_0/\partial y^2 \to 0$ as $y \to \infty$. The combination of Eqs. (13), (34), and (35) results in the equation for a as

$$\left(\frac{d}{dx} + \frac{1}{R}\frac{dR}{dx}\right)U_0\delta\left[\frac{7}{10} - \frac{13}{15}a + \frac{67}{140}a^3 - \frac{7}{36}a^4 + \frac{1}{6}\lambda\left(\frac{9}{20} - \frac{7}{10}a + \frac{1}{140}a^3 - \frac{1}{504}a^4\right)\right] = \frac{2a\alpha}{\delta}. \tag{36}$$

For low frequency, as ω tends to zero, the quasi-steady solution is seen to be

$$T_S = -\tfrac{1}{2}y\,\partial T_0/\partial y. \tag{37}$$

One then writes for a general ω

$$T_1(x, y) = T_S(x, y) + i\omega T_t(x, y) \tag{38}$$

so that for vibrations of a relatively low frequency $-T_t/T_s$ will represent the time lag in the temperature distribution behind its quasi-steady value.

Using Eqs. (16) and (38) plus the fact that T_s is a solution of Eq. (33) when $\omega = 0$, one deduces the equation for T_t in the form

$$i\omega T_t + u_0 \frac{\partial T_t}{\partial x} + v_0 \frac{\partial T_t}{\partial y} - \alpha \frac{\partial^2 T_t}{\partial y^2} = -T_S - u_t \frac{\partial T_0}{\partial x} - v_t \frac{\partial T_0}{\partial y} \tag{39}$$

subject to the boundary conditions $T_t = 0$ at $y = 0, \infty$. The term $-T_s$ represents the thermal inertia of the fluid which resists the quasi-steady fluctuations of temperature, and tends to produce a phase lag in the heat transfer. The remaining terms represent the additional heat transfer due to convection caused by the acceleration-dependent component of the velocity fluctuations. Integrating Eq. (39) over

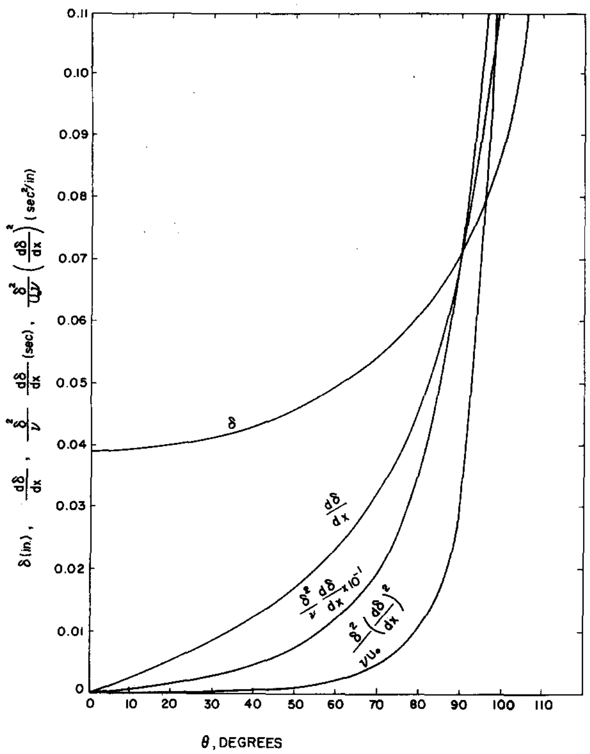

FIG. 1. Distributions of δ, $d\delta/dx$, $(\delta^2/\nu)(d\delta/dx)$, and $(\delta^2/U_0\nu)(d\delta/dx)^2$ around a sphere.

the whole boundary-layer, we get

$$i\omega\int_0^\delta T_t\,dy + \left(\frac{1}{R}\frac{dR}{dx}+\frac{d}{dx}\right)\int_0^\delta T_t u_0\,dy + \alpha\left(\frac{\partial T_t}{\partial y}\right)_{y=0} = -\frac{1}{2}\int_0^\delta T_0\,dy - \left(\frac{1}{R}\frac{dR}{dx}+\frac{d}{dx}\right)\int_0^\delta T_0 u_t\,dy - \epsilon i\omega r_0 T_w e^{i\omega t}. \qquad (40)$$

A profile for T_t is assumed

$$\frac{T_t}{T_w} = \frac{D}{2a^4}(a\eta - 3a^3\eta^3 + 2a^4\eta^4), \qquad (41)$$

which satisfies the boundary conditions

$$T_t = \partial^2 T_t/\partial y^2 = 0 \quad \text{at} \quad \eta = 0$$

and

$$T_t = \partial T_t/\partial y = 0 \quad \text{at} \quad \eta = 1/a.$$

Now Eqs. (13), (21), (35), and (41) are substituted into Eq. (40). The expression for D is then obtained as

$$\frac{DT_w\delta}{2a^4}\left[i\omega\left(\frac{a}{2}-\frac{3}{4}a^3+\frac{2}{5}a^4\right)+\frac{\alpha a}{\delta^2}\right] + \left(\frac{1}{R}\frac{dR}{dx}+\frac{d}{dx}\right)\left[T_w U_0\frac{D\delta}{2a^4}\left\{\frac{1}{10}a+\frac{9}{280}a^3 - \frac{1}{18}a^4 + \frac{\lambda}{16}\left(\frac{1}{60}a - \frac{3}{280}a^3 + \frac{1}{252}a^4\right)\right\}\right]$$
$$= -\frac{1}{2}T_w\delta\left(1 - a + \frac{1}{2}a^3 - \frac{1}{5}a^4\right) - \left(\frac{1}{R}\frac{dR}{dx}+\frac{d}{dx}\right)\left[\frac{\delta^3 F(x) T_w}{\nu}\left\{\frac{19}{10}A - \frac{31}{60} + \frac{13}{15}aA - \frac{29}{60}a + \frac{3}{70}Aa^3 - \frac{a^3}{168} - \frac{Aa^4}{72} + \frac{1}{504}a^4\right\}\right]. \qquad (42)$$

The heat transfer rate per unit area q fluctuates according to

$$q = -k\left\{\left(\frac{\partial T_0}{\partial y}\right)_{y=0} + \epsilon e^{i\omega t}\left[\left(\frac{\partial T_S}{\partial y}\right)_{y=0} + i\omega\left(\frac{\partial T_t}{\partial y}\right)_{y=0}\right]\right\} = q_0 - \epsilon e^{i\omega t}\left(\frac{1}{2}q_0 + i\omega k\frac{T_w D}{2a^3\delta}\right), \qquad (43)$$

where k is the thermal conductivity of the fluid and $q_0 = -k(\partial T_0/\partial y)_{y=0}$ is the heat transfer rate per unit area at steady state.

Another approach is used when ω is large. The dominant terms of Eq. (39) are (as discussed in the previous section for the flow boundary-layer)

$$\alpha\frac{\partial^2 T_1}{\partial y^2} - i\omega T_1 = u_1\frac{\partial T_0}{\partial x} + v_1\frac{\partial T_0}{\partial y}. \qquad (44)$$

With the substitution of Eqs. (29), (30), and (35), the terms on the right-hand side of Eq. (44) may be expressed as functions of y. The resulting solution, which may be readily obtained, is not presented here because of its great length. Instead, two approximate solutions of the equation are given; one for the outer region and the other for the inner region of the thermal boundary layer. In the outer region away from the surface, both u_1 and v_1 do not change very rapidly and the conduction effect is small. By neglecting the term $\alpha\,\partial^2 T_1/\partial y^2$, Eq. (44) gives

$$T_1 = \frac{i}{\omega}\left(u_1\frac{\partial T_0}{\partial x} + v_1\frac{\partial T_0}{\partial y}\right) \qquad (45)$$

for the temperature profile. In the inner region of the thermal boundary layer where the changes in u_1 and v_1 are quite rapid for this approximation, a linear profile is assumed for the fluid temperature near the surface, i.e., $T_0 = T_W + y(\partial T_0/\partial y)_{y=0}$. Then Eq. (40) may be solved as

$$T_1 = \left(\frac{\partial^2 T_0}{\partial y\,\partial x}\right)_{y=0} r_0\frac{d\delta}{dx}\left(y + \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\,y\exp\left[-y\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right] + \frac{2\,\mathrm{Pr}}{(i\omega/\nu)^{\frac{1}{2}}(1-\mathrm{Pr})^2}\left\{\exp\left[-y\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}}\right]\right.\right.$$

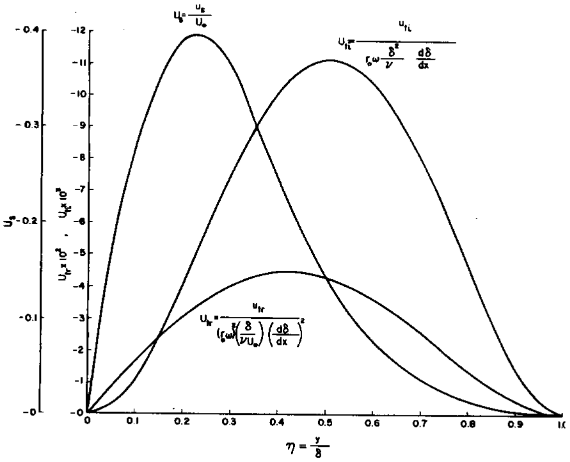

FIG. 2. Profiles of velocity components u_s, u_{tr} and u_{ti} for water flowing around a vibrating sphere with Re = 4070 (low-frequency approximation) at $x/r_0 = 1$.

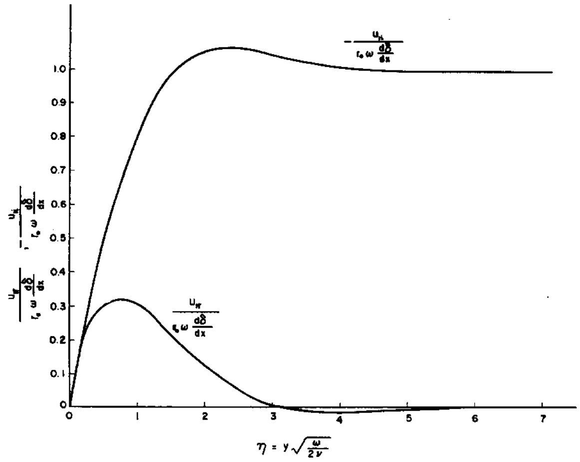

FIG. 3. Profiles of real and imaginary components u_{1r} and u_{1i} for water flowing around a vibrating sphere with Re = 4070 (high-frequency approximation).

$$- \exp\left[-y\left(\frac{i\omega}{k}\right)^{\frac{1}{2}}\right]\Bigg\}\Bigg) - \left(\frac{\partial T_0}{\partial y}\right)_{y=0} \frac{r_0}{R}\frac{d}{dx}\left[\frac{d\delta}{dx}R\right] \cdot \left[y - \frac{1-\Pr+\Pr\exp[-y(i\omega/\nu)^{\frac{1}{2}}] - \exp[-y(i\omega/k)^{\frac{1}{2}}]}{(i\omega/\nu)^{\frac{1}{2}}(1-\Pr)}\right]. \quad (46)$$

NUMERICAL RESULTS AND DISCUSSION

A numerical example is given for a system consisting of water flowing around a vibrating sphere with a steady mean radius $r_0 = 1$ in. The whole system is at a uniform temperature of 200 °F. For a water velocity of 1 in./sec, the flow corresponds to Re = 4070. Using the results obtained in Ref. 8 the boundary layer thickness δ and its x derivative are numerically evaluated for the ideal velocity distribution in potential irrotational flow past a sphere of radius r_0 as given by $U_0(x) = \frac{3}{2}U_\infty \sin(x/r_0)$. The results are graphically presented in Fig. 1.

The profiles of the velocity components u_s and u_t at low frequency, as expressed by Eqs. (17) and (26), are illustrated in Fig. 2. u_{tr} and u_{ti} represent the real and imaginary parts of u_t respectively. It is disclosed that the amplitude of u_s is independent of the frequency, while those of u_{tr} and u_{ti} are proportional to $-(r_0\omega)^2(\delta^2/\nu U_0)(d\delta/dx)^2$ and $(r_0\omega)(\delta^2/\nu)(d\delta/dx)$, respectively. The dependency of the parameters $(\delta^2/\nu U_0)(d\delta/dx)^2$ and $(\delta^2/\nu)(d\delta/dx)$ on x is shown in Fig. 1. It is observed that the changes in the magnitude of these two parameters are very small near the forward stagnation point and become very rapid near the separation point. Similar changes in the amplitudes of both u_{tr} and u_{ti} are expected.

The velocity profile of the oscillating component u_1 as described by Eq. (29) for high frequency is presented in Fig. 3. It shows that u_{1r} and u_{1i}, the real and imaginary parts of u_1, respectively, are directly proportional to $\omega\, d\delta/dx$. Since the change in $d\delta/dx$ is gradual along the surface, as shown in Fig. 1, the variation of the u_1 profile along the surface would also be gradual. Near the separation point, these variations become quite rapid.