Three-Dimensional Motion of a Liquid Film Induced by Surface-Tension Variation or Gravity

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Steady flows of a thin layer of viscous liquid on a horizontal plane induced by the nonuniformity of surface tension at its free surface are treated. If the film is very thin, surface-tension effects dominate gravity effects. Under that circumstance and away from vertical boundaries, a binomial of depth h of the liquid layer is a harmonic function of the Cartesian coordinates x and y in a horizontal plane, and the surface tension is a function of h. Near any vertical boundary there is a velocity boundary layer whose thickness is of the order of h. The velocity distribution in this boundary layer is given explicitly. The diffusion of the surface material affecting the surface tension is considered. Steady flows of a liquid film induced by gravity are also discussed. Simple solutions are possible if the film flows over a horizontal plane.

I. INTRODUCTION

Consider a thin liquid layer on a horizontal plane with a depth h at any point on that plane, very much smaller than any horizontal scale L defined by the spacing or size of vertical boundaries confining the liquid. Figure 1 shows an example of the horizontal geometry, the vertical boundaries being those of the circular cylinder and two plane vertical walls. Figure 2 shows a longitudinal cross section of the flow, which does not cut the cylinder.

If the surface tesnion at the free surface is not uniform, it will cause the fluid to move by surface traction. Only steady flows so induced will be considered, so that the depth h and the surface tension σ are both functions of the horizontal Cartesian coordinates x and y only, and independent of the time. The vertical coordinate is denoted by z.

We shall assume that, for a horizontal bottom and a thin film, the effect of surface tension dominates the effect of gravity. More specifically, this implies that

$$\Delta\sigma \gg \rho g h_0^2, \tag{1}$$

in which $\Delta\sigma$ is a characteristic variation in σ, ρ is the liquid density, g is the gravitational acceleration, and h_0 is a vertical scale, which can be taken to be the maximum of h. The analysis of this case will now be discussed in detail.

Steady flows of a liquid film induced solely by gravity will be treated in the last section of this paper.

II. ANALYSIS FOR FLOWS INDUCED BY SURFACE-TENSION VARIATION

In addition to the limitations stated in Sec. I, we assume the Reynolds number to be so small that the inertial terms can be neglected. The change in h is

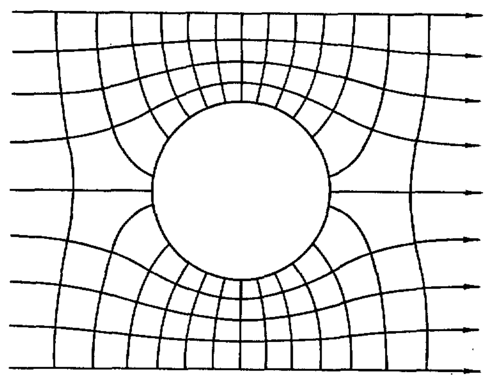

FIG. 1. A plan view of a typical flow pattern. Lines with arrow heads are streamlines. The other lines are lines of constant surface tension or constant depth.

assumed to be gradual, so that the curvature of the free surface is of the order of h/L^2, and the pressure caused by surface curvature is of the order of $\sigma h/L^2$. The shear stress caused by the nonuniformity of surface tension is of the order of $\Delta\sigma/L$. It is therefore clear, that under the assumption that L is much greater than h, the effect of the surface shear concomitant with the nonuniformity of surface tension dominates the effect of pressure induced by surface curvature. Aside from surface curvature and surface tension, the pressure is also affected by gravity; but if (1) holds, the gravity effect can be neglected. Under these assumptions the pressure gradient can be taken to be zero, and the pressure treated as constant throughout the liquid. The first two equations governing steady flows then become simply

$$0 = \mu\nabla^2 u, \qquad 0 = \mu\nabla^2 v, \tag{2}$$

in which ∇^2 is the Laplacian operator, and u and v are the velocity components in the directions of increasing x and y, respectively. The velocity component w, as well as $\partial w/\partial z$, is zero at the bottom, and hence w is of the order of $u(h/L)^2$ throughout.

If the values of u and v on the free surface are denoted by U and V, and steady flow is assumed, the equation governing the concentration γ of the surface material is, with κ denoting the diffusivity,

$$\frac{\partial}{\partial x}(U\gamma) + \frac{\partial}{\partial y}(V\gamma) = \kappa\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\gamma, \tag{3}$$

in which diffusion into the body of the liquid is neglected. This neglect is justified, provided the solubility of the surface material in the liquid is small, according to Levich[1] (first equation on page 420). [In a previous paper by this writer,[2] small solubility is implied in Eq. (1) of that paper, which is the one-dimensional counterpart of (3)]. The surface tension σ is dependent on γ. Within any small range the relationship between σ and γ, if not strictly linear, can always be replaced by a linear one

$$\sigma = \sigma_0 - k\gamma,$$

in which σ_0 and k are constants, and k is positive if σ decreases as γ increases. Thus, (3) can be written as

$$(\sigma - \sigma_0)\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) + U\frac{\partial \sigma}{\partial x} + V\frac{\partial \sigma}{\partial y} = \kappa\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma. \tag{4}$$

It should be noted that whereas the convective terms in the equations of motion have been neglected because the Reynolds number U_0h_0/ν (U_0 is a representative velocity) is small, the convective terms in (3) and (4) are retained because the Péclet number U_0L/κ is not assumed small. For clarity, the analysis will now be divided into two parts.

III. THE CORE

Since h is very much smaller than L, it is clear that anywhere except very near the vertical boundaries delineating the flow region, the terms $\partial^2u/\partial z^2$ and $\partial^2v/\partial z^2$ dominate the other terms in the two equations contained in (2), so that these equations can simply be written

$$\frac{\partial^2 u}{\partial z^2} = 0, \qquad \frac{\partial^2 v}{\partial z^2} = 0. \tag{5}$$

Thus, the velocity field is described by

$$u = Uz/h, \qquad v = Vz/h. \tag{6}$$

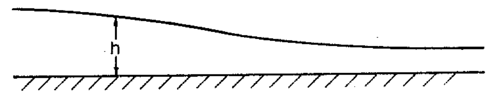

FIG. 2. A cross-sectional view of the flow.

The equation of continuity is

$$\frac{\partial}{\partial x}\int_0^h u\,dz + \frac{\partial}{\partial y}\int_0^h v\,dz = 0, \tag{7}$$

or, by virtue of (6),

$$\frac{\partial}{\partial x}(hU) + \frac{\partial}{\partial y}(hV) = 0. \tag{8}$$

The boundary conditions at the bottom, where $z = 0$, are satisfied by (6). The boundary conditions at the free surface are

$$\mu\frac{\partial u}{\partial z} = \frac{\partial \sigma}{\partial x}, \qquad \mu\frac{\partial v}{\partial z} = \frac{\partial \sigma}{\partial y},$$

or

$$\frac{\mu U}{h} = \frac{\partial \sigma}{\partial x}, \qquad \frac{\mu V}{h} = \frac{\partial \sigma}{\partial y}. \tag{9}$$

Note that these equations would demand that $\sigma_{xz} + \sigma_{yz}$ vanish if w were strictly zero, which it is not. Only w_x and w_y are assumed small in comparison with u_z and v_z. It can also be shown that surface viscosities (both bulk and shear viscosities) contribute terms of negligible magnitude, so long as $h/L \ll 1$. The unknowns σ, U, V, and h are functions of x and y, and are governed by (4), (8), and (9), all of which are nonlinear. At first sight the situation seems rather hopeless. The solution, however, turns out to be very simple.

Substituting (9) in (8) and expanding, we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma = -\frac{2\mu}{h^2}\left(U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right). \tag{10}$$

Now, by virtue of (8), the left-hand side of (4) can be written as

$$U\frac{\partial \sigma}{\partial x} + V\frac{\partial \sigma}{\partial y} - \frac{\sigma - \sigma_0}{h}\left(U\frac{\partial h}{\partial x} + V\frac{\partial h}{\partial y}\right) = h\left(U\frac{\partial}{\partial x}\frac{\sigma - \sigma_0}{h} + V\frac{\partial}{\partial y}\frac{\sigma - \sigma_0}{h}\right),$$

so that (4) can be written as

$$\kappa\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma = h\left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\frac{\sigma - \sigma_0}{h}. \tag{4'}$$

Comparison of (10) with (4′) yields

$$\left(U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)\left(\frac{\sigma - \sigma_0}{h} - \frac{\mu\kappa}{h^2}\right) = 0. \tag{11}$$

[1] V. G. Levich, *Physicochemical Hydrodynamics* (Prentice-Hall. Inc., Englewood Cliffs, New Jersey, 1962).
[2] C.-S. Yih, Phys. Fluids **11**, 477 (1968).

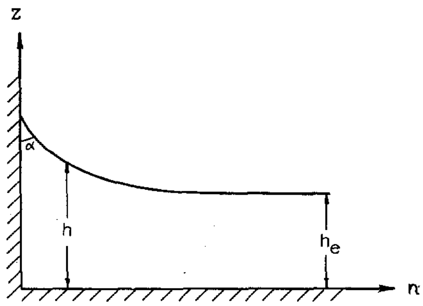

FIG. 3. The boundary-layer region.

Since (8) allows us to define a stream function ψ for the surface flow, in terms of which

$$U = \frac{1}{h}\frac{\partial \psi}{\partial y}, \qquad V = -\frac{1}{h}\frac{\partial \psi}{\partial x}.$$

From (9) it is evident that the constant-σ lines are orthogonal to the streamlines at the surface. From (11) it then follows that

$$\frac{\sigma - \sigma_0}{h} - \frac{\mu\kappa}{h^2} = f(\psi).$$

If along any one constant-depth line σ is also constant (for example, when the upstream flow is parallel),

$$\frac{\sigma - \sigma_0}{h} - \frac{\mu\kappa}{h^2} = C. \tag{12}$$

Equation (12) allows us to write (9) as

$$hU = \frac{\partial}{\partial x}\left(-\kappa h + \frac{C}{3\mu}h^3\right), \quad hV = \frac{\partial}{\partial y}\left(-\kappa h + \frac{C}{3\mu}h^3\right). \tag{13}$$

Substitution of (13) into (8) produces

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(-\kappa h + \frac{C}{3\mu}h^3\right) = 0. \tag{14}$$

Thus, the binomial

$$F(h) = -\kappa h + \frac{C}{3\mu}h^3$$

is a harmonic function of x and y, and for any specified boundary of the flow, (14) can be solved by any of the known methods for solving two-dimensional Laplace equations. Then, U and V are found by (13), and σ by (12), provided σ is known at some constant-h line. The core flow, therefore, can easily be determined. Figure 1 shows a well-known flow pattern merely for the sake of demonstration. The directed lines are streamlines issuing from an upstream reservoir into a downstream reservoir. For steady flow to be possible, the σ and h in the downstream reservoir must be maintained to satisfy (12), with the C determined by conditions in the upstream reservoir.

IV. BOUNDARY LAYERS

Equations (13) show that

$$-\kappa \ln h + \frac{C}{2\mu}h^2$$

is a potential for the surface flow. At a vertical boundary the normal velocity component is zero, so that the normal component of the gradient of h must be zero. Since $F(h)$ is a harmonic function, the tangential component of the gradient of h at such a boundary cannot be zero without the trivial consequence of a constant h all over the field, according to a well-known result in potential theory. Thus, the tangential component of the velocity at any vertical boundary, as found by solving (14) and using (13) and (6), is not zero. However, the physical condition at such a boundary demands that the velocity on it be zero; hence, a boundary layer must exist. In that layer, the normal derivatives of the velocity are of the same order of magnitude as its vertical derivatives.

For clarity, we shall denote the horizontal distance along a vertical boundary by s, the velocity component in the direction of increasing s by q, and the distance normal to the wall by n. Then, in the boundary layer

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial n^2}\right)q = 0. \tag{15}$$

Outside of the boundary layer, q is equal to q_0, which is different from zero, and is given by (13), with h satisfying (14). We shall write

$$q_0 = \frac{Qz}{h_e}, \tag{16}$$

in which Q is the surface speed at the free surface and h_e the depth, both just outside of the boundary layer.

Due to surface tension, the free surface makes an angle α (Fig. 3) with the wall, which is less than $\pi/2$ if the liquid wets the wall, and more than $\pi/2$ if it does not. The value of α depends on the nature of the liquid and of the wall. Since the vertical acceleration is negligible under the assumptions made, the pressure distribution in the liquid is hydrostatic. This means that, for small cot α,

$$\sigma \frac{\partial^2 h}{\partial n^2} = \rho g h. \tag{17}$$

The solution of (17) is

$$h = A \exp\left[-\left(\frac{\rho g}{\sigma}\right)^{1/2} n\right] + h_e, \tag{18}$$

in which

$$A\left(\frac{\rho g}{\sigma}\right)^{1/2} = \cot \alpha. \tag{19}$$

Since within the boundary layer the component of velocity in the direction of n is of the order of h_0/L, as can be deduced from the equation of continuity in differential form in the usual way, σ must be constant at that part of the free surface which is within the boundary layer. Hence, the σ in (18) and (19) is that just outside the boundary layer. The region occupied by the wall is shown in Fig. 3.

If $\cot \alpha$ is not small, the free surface near the wall has to be obtained from the differential equation

$$\frac{\sigma h''}{(1 + h'^2)^{3/2}} = \rho g(h - h_e),$$

in which the accents indicate differentiations with respect to n. After multiplication by h', a first integration of this equation is

$$-2\sigma(1 + h'^2)^{-1/2} = \rho g(h - h_e)^2 - 2\sigma,$$

in which the constant of integration has been determined by the condition that $h = h_e$ when $h' = 0$. At the wall the value of h, denoted by h_w, is determined from

$$-2\sigma(1 + \cot^2\alpha)^{-1/2} = \rho g(h_w - h_e)^2 - 2\sigma.$$

A second integration gives

$$n = -\int_{h_w}^{h} \frac{1 - B(h - h_e)^2}{\{1 - [1 - B(h - h_e)^2]^2\}^{1/2}}\, dh,$$

in which $B = g\rho/2\sigma$. For any B and h_e, the integral can be carried out numerically if necessary. The fluid region near the wall can, therefore, be unambiguously determined for any α.

Equation (15) is to be solved for the region just described with the boundary conditions

(i) $q = 0$ at $n = 0$,

(ii) $q = 0$ at $z = 0$,

(iii) $q \to q_0$ at $n \to \infty$,

(iv) $\dfrac{\partial q}{\partial z} = \dfrac{Q}{h_e}$ at $z = h$.

Boundary condition (iii) is, of course, the usual simplified statement for $q \to q_0$ as n approaches the outer edge of the boundary layer, the "infinity" being used for convenience only. It is also evident that the effect of the boundary layer on the core is neglected.

The problem thus posed for the boundary-layer region is defined and solvable for any value of σ, ρg, and h by numerical methods, such as the method of relaxation. We shall not attempt to give an example of the numerical solution. Instead, we shall give an analytical solution for the special case $\alpha = \pi/2$. Since, in this case

$$\frac{\partial h}{\partial n} = 0, \tag{20}$$

h is equal to h_e throughout, and for simplicity we shall drop the subscript e and consider h as constant throughout the boundary layer. The values of U, V, and hence Q, are also constant in the boundary layer. Although Q is not zero even at the intersection of the free surface with the vertical wall, this fact is not disturbing. The same situation is encountered at the intersection of a stationary and a moving boundary in contact with a viscous fluid.

We can now simply write the solution of (15)

$$q = \frac{Qz}{h} + \sum_{m=1}^{\infty} A_m \sin \frac{(2m-1)\pi}{2h} z \cdot \exp\left(-\frac{(2m-1)\pi}{2h} n\right). \tag{21}$$

The boundary conditions at the horizontal bottom, at the free surface, and just outside of the boundary layer are all exactly satisfied. It remains to determine A_m so that $q = 0$ at $n = 0$. This is accomplished by taking

$$A_m = \frac{2Q}{h}\int_0^h z \sin\frac{(2m-1)\pi}{2h} z\, dz = 8hQ(-1)^{m-1}[(2m-1)\pi]^{-2}, \tag{22}$$

in which Q and h are functions of s only. It should be noted that

(a) the boundary effect dies out exponentially, and the boundary-layer thickness is of the order of h,

(b) the velocity distribution in the boundary layer depends only on the local values of h and Q, and

(c) within the boundary layer the shear forces in horizontal planes are balanced by shear forces in vertical surfaces parallel to the wall.

The foregoing analysis can easily be extended for application to liquid films attached to curved surfaces.

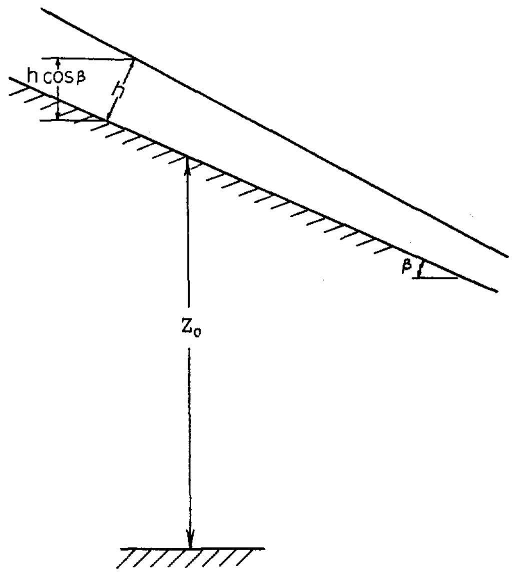

FIG. 4. Cross section in a vertical plane containing the line of steepest descent of the bottom.

V. FILM FLOW INDUCED BY GRAVITY

Consider a plane boundary described by

$$Z_0 = ax + by + c, \tag{23}$$

in which Z_0 is the elevation of the plane, measured in the direction of the vertical, and x and y are Cartesian coordinates in the plane. The third Cartesian coordinate z is measured in the direction normal to the plane boundary, which we shall call the bottom. The angle of inclination of the bottom to the horizontal will be denoted by β. Boundaries normal to the bottom will be called walls. A liquid film on the bottom will flow under the action of gravity, provided there is a free surface. The depth of the film, measured in the direction of increasing z, is denoted by h. Equations governing the flow of the film under the combined action of gravity and surface-tension variation can be derived. However, they are so complicated that their analytical solutions are unlikely to be obtainable. For this reason we shall consider the effect of surface-tension variation and the effect of gravity separately. The former has been discussed in the foregoing sections. We now consider the effect of gravity alone.

The gravity potential is (Fig. 4)

$$\Omega = gZ = g(Z_0 + h\cos\beta) = g(ax + by + c + h\cos\beta). \tag{24}$$

As before, the velocity component w is negligible. The equations of motion are

$$0 = -\rho\frac{\partial\Omega}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2}, \quad 0 = -\rho\frac{\partial\Omega}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2}, \tag{25}$$

and these are satisfied *in the core* by

$$u = \frac{1}{2\nu}\frac{\partial\Omega}{\partial x}z(2h - z), \quad v = \frac{1}{2\nu}\frac{\partial\Omega}{\partial y}z(2h - z), \tag{26}$$

which also satisfy the zero-shear condition at the free surface. Thus,

$$\int_0^h u\,dz = \frac{1}{3\nu}\frac{\partial\Omega}{\partial x}h^3, \qquad \int_0^h v\,dz = \frac{1}{3\nu}\frac{\partial\Omega}{\partial y}h^3, \tag{27}$$

and the equation of continuity in integral form is

$$\frac{\partial}{\partial x}\left[\left(a + \cos\beta\frac{\partial h}{\partial x}\right)h^3\right] + \frac{\partial}{\partial y}\left[\left(b + \cos\beta\frac{\partial h}{\partial y}\right)h^3\right] = 0. \tag{28}$$

This partial differential equation is to be solved with the boundary condition

$$\frac{\partial\Omega}{\partial n} = 0 \quad \text{or} \quad \frac{\partial Z_0}{\partial n} + \cos\beta\frac{\partial h}{\partial n} = 0 \tag{29}$$

at the walls, where n is measured in the direction normal to them. The differential system for the core is nonlinear. If a solution is obtained, the flow in the boundary layer is again governed by (15). The boundary-layer region is as shown in Fig. 3, except that the bottom has a slope.

We shall deal with the simpler case of the horizontal bottom, for which

$$a = b = \beta = 0.$$

The differential system governing the core is, from (28),

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)h^4 = 0, \tag{30}$$

and

$$\frac{\partial h}{\partial n} = 0 \tag{31}$$

at the boundaries. Since the boundary conditions can be written as

$$\frac{\partial}{\partial n}h^4 = 0 \tag{32}$$

at the boundaries, the differential system is linear in h^4, and all the available methods for solving potential-flow problems can be brought to bear.

After the flow in the core is determined, the flow in the boundary-layer region shown in Fig. 3 can be determined by solving (15). If $\alpha = \pi/2$, the solution is simple, since h is then constant throughout the boundary layer. Again using s to denote the distance measured along the boundary and q the velocity component in the direction of increasing s, we have, outside of the boundary layer,

$$q_0 = \frac{1}{2\nu}\frac{\partial\Omega}{\partial s} z(2h - z), \tag{33}$$

in which $\partial\Omega/\partial s$ is taken just outside of the boundary layer. The solution for q is

$$q = q_0 + \sum_{m=1}^{\infty} A_m \sin\frac{(2m-1)\pi z}{2h} \cdot \exp\left(-\frac{(2m-1)\pi n}{2h}\right), \tag{34}$$

in which

$$A_m = -2\int_0^h q_0 \sin\frac{(2m-1)\pi z}{2h}\,dz = -\frac{2}{\nu}\frac{\partial\Omega}{\partial s}\left(\frac{2h}{(2m-1)\pi}\right)^3. \tag{35}$$

All the comments in Sec. III regarding the nature of the boundary layer are still valid here.

ACKNOWLEDGMENT

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