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Collapse of Spherical Cavities in Viscoelastic Fluids

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An analysis is given of the collapse of a spherical cavity in a large body of an incompressible viscoelastic liquid. Proceeding from a linear rheological model for the liquid, one obtains a nonlinear integro-differential equation for the motion of the cavity. Analytical solutions are derived for certain limiting values of the parameters governing collapse, and some numerical solutions are presented for various other values. As one of the more interesting results of this work, it is found that elasticity in the liquid can significantly retard the collapse of a void and produce prolonged, oscillatory motion whenever the relaxation time of the fluid is moderately large in comparison to the Rayleigh collapse time. This is in sharp contrast to the catastrophic collapse which will aways occur for voids in purely viscous liquids. Both numerical and approximate analytical solutions are presented to demonstrate the damping effect of liquid viscosity on the cavity motion.

I. INTRODUCTION

The term cavitation usually refers to the phenomenon of growth and collapse of flow-induced voids or vapor bubbles in liquids. The effects resulting from cavitation are known to produce metal erosion, luminescence, and increases in various chemical reaction rates.

In the previous works on this subject, attention has mainly been restricted to classical liquids. The earliest theoretical treatment is apparently that of Lord Rayleigh,[1] who considered the collapse of a spherical void in an inviscid liquid. In later theoretical works, attempts have been made to account for viscous effects in both the bubble phase and in the surrounding liquid and most of the analyses have dealt with Newtonian[2–4] or purely viscous fluids.[5] An interesting question arises as to the effects that elasticity might have on cavitation in viscoelastic liquids. In other contexts, it has been observed that the presence of elasticity such as that produced by the addition of small amounts of high polymers, can drastically change the flow behavior of liquids. Hence, one might well inquire as to the possible and perhaps beneficial effects of viscoelasticity on bubble collapse, such as suppression or reduction in the intensity of cavitation.

An analysis of bubble growth in viscoelastic fluids has already been given by Street,[6] but because of the applications contemplated in his analysis, inertial effects were neglected. It is precisely these effects, however, that tend to predominate in the collapse phenomena usually associated with cavitation. This provides part of the motivation for the present work, in which we shall focus our attention primarily on the collapse of spherical voids (i.e., regions containing no gas) in an idealized viscoelastic fluid.

We recall that previous studies have shown that collapsing cavities which contain permanent gases will generally always rebound short of actual collapse, such that the cavity radius never actually decreases to zero. On the other hand, a void will generally always collapse to zero radius, at least in purely viscous fluids. It is, therefore, interesting to reconsider this question of rebound versus complete collapse for the case of a void in a viscoelastic fluid.

II. EQUATIONS OF MOTION

Here we wish to treat the motion of a spherical bubble contained in a large body of an incompressible liquid. Initially at time $t = 0$ the system is at rest, with a bubble radius R_0 and a uniform pressure P_0. It has previously been shown[2,3] that the equation for the spherically symmetric motion for a bubble, in which there is no condensation or evaporation of fluid, can be reduced to

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{P_l - P_0}{\rho} - \frac{1}{\rho}\int_R^\infty (\nabla\cdot\boldsymbol{\tau})_r\, dr, \tag{1}$$

where $(\nabla\cdot\boldsymbol{\tau})_r$ denotes the radial component of $\nabla\cdot\boldsymbol{\tau}$, the divergence of the deviatoric or "extra" stress for the liquid phase, and $r = R(t)$ is the radial position of the bubble-liquid interface with P_l and P_0 denoting the pressure in the liquid at $r = R(t)$ and $r = \infty$, respectively. The dots denote derivatives with respect to t, and ρ is the liquid density.

Irrespective of the fluid rheology, the radial velocity at any radial position r in the liquid is required by continuity, incompressibility, and the assumed symmetry to be

$$u = \frac{\dot{R}R^2}{r^2}. \tag{2}$$

By the usual force balance at the bubble–liquid interface in the cavity, the term P_l in Eq. (1) can be expressed in terms of surface tension and the radial stresses as

$$\tau_{rr,g} + P_g = P_l + \tau_{rr,l} + \frac{2\sigma}{R}, \qquad (3)$$

where g refers to any gas which may be present in the cavity and l refers to the liquid phase. As in Fogler,[4] here we adopt the sign convention of Bird, Stewart, and Lightfoot for the stress tensor: The symbol $\boldsymbol{\tau}$ (or τ_{ij}) denotes the deviatoric stress tensor reckoned as a compressive stress.[7] Since neither surface elasticity nor viscosity are considered in this analysis, the surface tension force is given by the static surface tension σ.

For bubbles containing an ideal gas in a uniform state the gas-phase stress at the bubble surface is equal to the pressure alone; hence $\tau_{rr,g} = 0$,[4] and the liquid-phase interfacial pressure is given by

$$P_l = P_g - \frac{2\sigma}{R} - \tau_{rr,}\Big|_{r=R}. \qquad (4)$$

Furthermore, the term $(\nabla\cdot\boldsymbol{\tau})_r$ which occurs in Eq. (1) can be written in terms of three normal stresses as

$$(\nabla\cdot\boldsymbol{\tau})_r = \frac{\partial\tau_{rr}}{\partial r} + \frac{2\tau_{rr}}{r} - \frac{(\tau_{\phi\phi} + \tau_{\theta\theta})}{r}, \qquad (5)$$

and, since the sum of these deviatoric stresses is by definition zero, one can express the ϕ and θ stresses in terms of the radial stress as

$$\tau_{\theta\theta} + \tau_{\phi\phi} = -\tau_{rr}, \qquad (6)$$

which with Eq. (5) yields

$$(\nabla\cdot\boldsymbol{\tau})_r = \frac{\partial\tau_{rr}}{\partial r} + 3\,\frac{\tau_{rr}}{r}. \qquad (7)$$

Then, upon substituting Eqs. (4) and (7) into Eq. (1), one obtains the equation

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{P_g - P_0}{\rho} - \frac{2\sigma}{\rho R} - \frac{3}{\rho}\int_R^\infty \frac{\tau_{rr}\,dr}{r} \qquad (8)$$

for the bubble radius $R(t)$. In order to complete the description of motion, we must now relate the liquid-phase radial stress τ_{rr} to the bubble motion.

In the case of a general viscoelastic fluid exhibiting long-range memory effects, the stresses will depend on the past history of strain or rate of strain. For the simple, radially symmetric flow field considered in Eq. (2), the strain consists merely of an unsteady simple extension. Hence, we expect that for an isotropic material the instantaneous radial stress $\tau_{rr}(t)$ can be expressed as a functional on the past history of the radial strain rate $e_{rr}(t')$, $0 \leqq t' \leqq t$. Here, as in the following analysis, $t = 0$ corresponds to the beginning of the collapse process, where we assume the liquid to be in a completely "relaxed" state of purely hydrostatic stress.

As with other analyses involving viscoelastic fluids, we must now postulate a relation between the strain and the kinematic history of the motion to be considered, and for this purpose, we adopt the usual material coordinates. Thus, we let r' denote the position at past time t', $0 \leqq t' \leqq t$, of a particle which is at position r at the present time t, so that, with the velocity field given by Eq. (2), we have

$$(r')^3 = r^3 + R^3(t') - R^3(t). \qquad (9)$$

Now, at any position (r, t) the radial deformation rate is given by

$$e_{rr}(r, t) = \frac{\partial u}{\partial r} = -\frac{2\dot{R}R^2}{r^3}, \qquad (10)$$

and, therefore, by Eqs. (9) and (10), the history of the deformation rate is determined by

$$e_{rr}(t', r') = -\frac{2\dot{R}(t')R^2(t')}{r^3 + R^3(t') - R^3(t)}. \qquad (11)$$

For the present work, we shall employ a rather simple, linear viscoelastic fluid model, in which the normal radial stress is related to the corresponding strain rate by

$$\tau_{rr}(t) = -2\int_0^t N(t - t')e_{rr}(t')\,dt', \qquad (12)$$

where $N(t)$ is a "memory" function or relaxation modulus. On combining Eqs. (11) and (12) we have

$$\tau_{rr} = 4\int_0^t \frac{N(t - t')\dot{R}(t')R^2(t')\,dt'}{r^3 + R^3(t') - R^3(t)}, \qquad (13)$$

and the integral in Eq. (8) becomes

$$\int_R^\infty \frac{\tau_{rr}}{r}\,dr = 4\int_0^t\int_R^\infty \frac{N(t - t')\dot{R}(t')R^2(t')\,dt'\,dr}{r[r^3 + R^3(t') - R^3(t)]}$$
$$= -4\int_0^t \frac{N(t - t')\dot{R}(t')R^2(t')\ln[R(t')/R(t)]\,dt'}{R^3(t') - R^3(t)}. \qquad (14)$$

Under these restrictions the complete equation governing the collapse of a cavity is the nonlinear integrodifferential equation

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{P_g - P_0}{\rho} - \frac{2\sigma}{R\rho} - \frac{12}{\rho}\int_0^t \frac{N(t - t')\dot{R}(t')R^2(t')\ln[R(t')/R(t)]\,dt'}{R^3(t') - R^3(t)}. \qquad (15)$$

For the purpose of the analysis to follow we shall adopt an elementary form of the relaxation modulus N, consisting of a linearly viscous Newtonian contribution and a Maxwellian contribution, as follows:

$$N(t) = \mu\,\delta(t) + G_0 \exp(-t/\lambda), \qquad (16)$$

where δ denotes the delta function, μ a constant viscosity, λ a relaxation time, and G_0 an elastic modulus.

In terms of dimensionless variables, Eq. (15) becomes

$$\psi\ddot{\psi} + \tfrac{3}{2}\dot{\psi}^2 = \frac{P_g - P_0}{P_0} - \frac{2}{N_{\mathrm{We}}\psi} - \frac{4\dot{\psi}}{N_{\mathrm{Re}}\psi} - \frac{12N_{\mathrm{El}}}{N_{\mathrm{Re}}} \cdot \int_0^{t^*} \left[\exp\left(-\frac{(t^*-t_1)}{N_{\mathrm{De}}}\right)\right] \frac{\dot{\psi}_1\psi_1^2 \ln(\psi_1/\psi)}{\psi_1^3 - \psi^3}\, dt_1 \qquad (17)$$

with $\psi(0) = 1$ and $\dot{\psi}(0) = 0$, where

$$N_{\mathrm{De}} = \frac{\lambda}{t_c} \text{ (a Deborah number[8,9])},$$

$$N_{\mathrm{El}} = \frac{G_0 t_c}{\mu} \text{ (an elastic number)},$$

$$N_{\mathrm{Re}} = \frac{\rho R_0^2}{\mu t_c} \text{ (a Reynolds number)},$$

$$N_{\mathrm{We}} = \frac{\rho R_0^3}{t_c^2 \sigma} \text{ (a Weber number)},$$

and $\psi = R/R_0$, $t^* = t/t_c$, $\psi_1 = \psi(t_1)$. Also, $t_c = R_0(\rho/P_0)^{1/2}$ is a characteristic (Rayleigh) collapse time, with P_0 being the initial pressure. In this manner one can readily identify the relevant physical parameters characterizing the collapse process.

In view of the number of parameters, even in this relatively simple model, one is practically forced to consider some special limiting cases where certain effects may be assumed to predominate. Thus, we focus our attention first and foremost on fluids with long relaxation times, corresponding to $N_{\mathrm{De}} \to \infty$. Here, as in the remainder of the analysis, we shall only consider voids, such that $P_g = 0$ in Eq. (17).

III. COLLAPSE CRITERIA AT LARGE DEBORAH NUMBERS

A. Large Reynolds Number

To begin with, we treat the case where both the Deborah and Reynolds numbers are large. In this limit, $N_{\mathrm{De}} \to \infty$, $N_{\mathrm{Re}} \to \infty$, the fluid behaves essentially as a purely elastic material, and one obtains in effect a conservative dynamical process characterized by an energy integral. First considering the case where surface tension is negligible, $N_{\mathrm{We}} \to \infty$, and reverting to dimensional variables, one has for the equation of motion,

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = -\frac{P_0}{\rho} - \frac{G}{\rho}, \qquad (18)$$

where

$$G = 12G_0 \int_{R_0}^{R} \frac{R_1^2 \ln(R_1/R)\, dR_1}{R_1^3 - R^3} = \frac{4}{3} G_0 \int_{(R_0/R)^3}^{1} \frac{\ln s}{s-1}\, ds.$$

On multiplying Eq. (18) by $2R^2\rho\, dR$ and integrating, we obtain

$$\rho\dot{R}^2R^3 = \tfrac{2}{3}P_0(R_0^3 - R^3) - 2\int_{R_0}^{R} GR^2\, dR. \qquad (19)$$

One will immediately recognize that this equation is an energy integral, with the left-hand side representing the total kinetic energy of the liquid which is expressed as the difference between the stored elastic energy and the work done by the ambient pressure. Rebound short of collapse is, therefore, possible and will occur at a rebound radius R, which is the root of the equation

$$\tfrac{2}{3}P_0(R_0^3 - R^3) - 2\int_{R_0}^{R} GR^2\, dR = 0, \qquad (20)$$

corresponding to zero kinetic energy in Eq. (19). With the substitution into Eq. (18)

$$y = \frac{1}{s}, \qquad z = \left(\frac{R}{R_0}\right)^3,$$

the integral in Eq. (20) can be written as

$$H = 2\int_{R_0}^{R} GR^2\, dR = -\frac{8R_0^3G_0}{9}\int_1^z \int_1^x \frac{\ln y}{(1-y)y}\, dy\, dx, \qquad (21)$$

which, after changing the order of integration, can be expressed as the infinite series

$$H = \frac{8R_0^3G_0}{9\rho}\left[(1-z)\sum_{n=1}^{\infty}\frac{(1-z)^n}{n^2} - \frac{z}{2}\left(\ln\frac{1}{z}\right)^2\right]. \qquad (22)$$

Thus, Eq. (20) becomes

$$\frac{P_0}{G_0} = \frac{4}{3}\left|\sum_{n=1}^{\infty}\frac{(1-z)^n}{n^2} - z\frac{[\ln(1/z)]^2}{2(1-z)}\right|, \qquad (23)$$

which provides the criterion for rebound, giving the rebound radius $R = R^* \equiv R_0 z^{1/3}$ as a function of P_0/G_0.

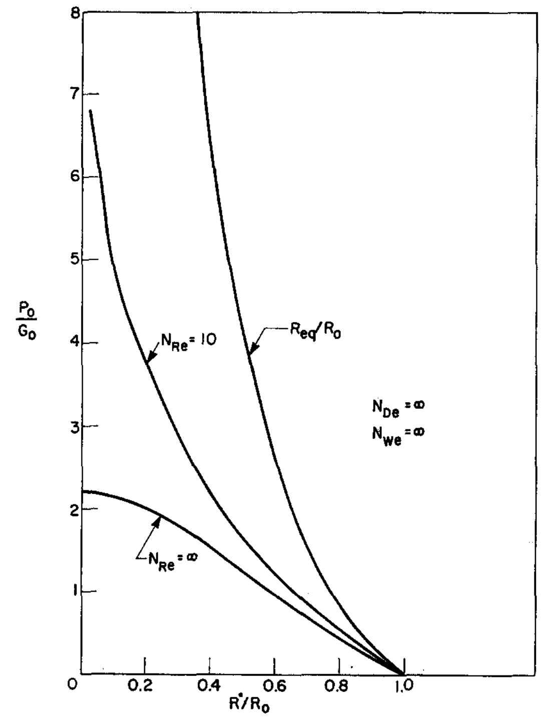

FIG. 1. The initial rebound radius and equilibrium radius as a function of P_0/G_0. The middle curve was computed from Eq. (17).

In the marginal case, that is, rebound at $R = 0$, we have

$$\frac{P_0}{G_0} = \frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{9} = 2.1932\cdots \qquad (24)$$

from Eq. (23), and, therefore, the condition for collapse without rebound is

$$\frac{P_0}{G_0} > \frac{2\pi^2}{9}, \qquad (25)$$

whereas, for rebound short of collapse, $R^* > 0$, we must have

$$\frac{P_0}{G_0} < \frac{2\pi^2}{9}. \qquad (26)$$

In the latter case, Eq. (23) provides us with a plot of rebound radius R^*/R_0 versus the ratio of initial pressure to elastic modulus P_0/G_0 which is displayed as the lower curve in Fig. 1.

If we consider the case of a finite Weber number, where surface tension is included in the equation of motion, the criterion for collapse is no longer independent of the initial bubble radius. By an analysis similar to that above, one can show that the condition now becomes

$$\frac{P_0 + (2\sigma/R_0)}{G_0} > \frac{2\pi^2}{9}, \qquad (27)$$

instead of Eq. (25). From this relation one sees that surface tension effects will tend to be important only in small bubbles.

Next we should like to determine the importance of viscous retardation on the collapse process, corresponding to a finite Reynolds number in Eq. (17). Whenever N_{Re} and N_{De} are finite, the system is no longer conservative and, hence, does not, in general, admit an energy integral like Eq. (19). We are thus forced to treat Eq. (17) with numerical techniques, as will be discussed below. First, however, it is worthwhile to note that for infinite Deborah numbers a cavity is characterized by a certain "equilibrium" radius R_{eq}, as determined by the static balance between pressure, surface tension, and elastic forces. One can easily derive an expression for this radius, and, considering the case of negligible surface tension $N_{We} = \infty$, one finds from Eq. (20) that the condition of static equilibrium is

$$\frac{P_0}{G_0} = \frac{4}{3}\sum_{n=1}^{\infty}\frac{(1-z)^n}{n^2} + \frac{2}{3}(\ln z)^2, \qquad (28)$$

with $R_{eq} = R_0/z^{1/3}$. The upper curve in Fig. 1 gives the corresponding plot of R_{eq}/R_0 versus P_0/G_0. This curve is, of course, independent of the Reynolds number, since it refers to a static situation.

For the purposes of obtaining the numerical solutions, a finite-difference technique was employed to treat Eq. (17). In particular, a modified Milne "four-point predictor" formula was used, and the numerical solutions thus obtained were compared for accuracy with existing numerical solutions for bubble collapse in ordinary liquids.[2,4] In all cases, the solutions were the same.

B. Finite Reynolds Number—Viscous Damping

Figure 2 gives a plot of cavity radius versus time for infinite N_{De} and N_{We}. The cavity is seen to

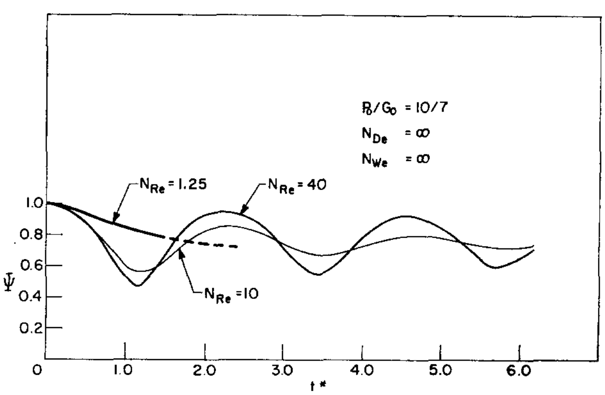

FIG. 2. The influence of the Reynolds number on the damping of stably oscillating cavities.

oscillate about an equilibrium radius subject to viscous damping which increases with N_{Re}^{-1}. Under these circumstances, one would expect to observe critical damping below some threshold value N_{Re_c} say, which we shall refer to here as the critical Reynolds number.

To obtain an estimate of this number, we shall make use of the techniques of linear stability theory. Thus, letting ψ_e represent the dimensionless equilibrium radius and ψ' a small perturbation about this radius, we have

$$\psi = \psi_e + \psi', \psi' \ll \psi_e. \quad (29)$$

Then, substituting Eq. (29) into the equation of motion (17) and neglecting terms of the second order in ψ', we obtain the corresponding linearized equation for a collapsing void, which in the case of infinite Weber numbers becomes

$$\psi_e \ddot{\psi}' - 1 + \frac{4\dot{\psi}'}{N_{Re}\psi_e} + 12\left[G(\psi_e) + \left.\frac{\partial G}{\partial \psi}\right|_e \psi'\right] = 0. \quad (30)$$

Since $12\, G(\psi_e) = 1$ by Eq. (18), the preceding equation becomes

$$\ddot{\psi}' + \frac{4\dot{\psi}'}{N_{Re}(\psi_e)^2} - \frac{4G_0}{P_0}\left[\frac{\ln(\psi_e)^3}{(\psi_e)^2(1-\psi_e^3)}\right]\psi' = 0, \quad (31)$$

or, simply,

$$\ddot{\psi}' + b\dot{\psi}' + c\psi' = 0, \quad (32)$$

where b and c are constants. In the usual way, it can be seen that the oscillation of the cavity will be critically damped whenever $b^2 = 4c$. With the appropriate values of these constants from Eq. (31), this criterion becomes

$$N_{Re}^2 = N_{Re_c}^2 = \frac{P_0}{G_0}\frac{1-\psi_e^3}{\psi_e^2 \ln(1/\psi_e^3)}, \quad (33)$$

which on rearrangement and making use of the definition in Eq. (17) becomes

$$N_{Re_c}N_{El} = \frac{1-\psi_e^3}{\psi_e^2 \ln(1/\psi_e^3)}. \quad (34)$$

Since the equilibrium radius corresponding to ψ_e is determined by P_0/G_0, we may express the critical Reynolds number as given by Eq. (34) in terms of P_0/G_0 or, alternatively, in terms of ψ_e. In the latter case one obtains a plot of the critical Reynolds number as a function of the equilibrium radius as shown in Fig. 3.

A physical interpretation can be given to the shape of the curve in the following way. Near $\psi_e = 1$, where the elastic force is, relatively speaking,

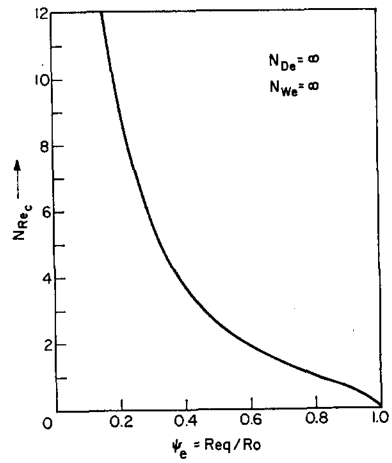

FIG. 3. The "critical" Reynolds number as a function of the equilibrium radius.

not very large, a greater viscous force is required to damp oscillations as the cavity approaches its equilibrium radius. However, when ψ_e is only slightly less than unity (e.g., $\psi_e = 0.7$ as in Fig. 2), the elastic force, which increases rapidly in a nonlinear way, exerts a greater degree of retardation on the motion, and consequently, a smaller viscous force is necessary for critical damping.

Owing to the method of derivation, the present expression for the critical Reynolds number, at which cavities move from their initial radius to their equilibrium radius on a critically damped path, can be regarded as strictly valid only for cavities in which ψ_e is close to unity. For cavities with equilibrium radii close to zero, the departure from equilibrium ψ' at the initial state $\psi = 1$ is effectively much greater than the equilibrium ratio ψ_e, and hence the above linearization technique cannot provide an adequate description of the cavity motion from $\psi = 1$ and $\psi = \psi_e$. One notes, however, that for an equilibrium radius ratio of 0.74, the critical Reynolds number obtained from Fig. 3 is 1.25, and from Fig. 2 it is observed that for this value of the Reynolds number the cavity does indeed approach equilibrium in a critically damped way. Thus, the linearization is evidently valid in this range.

For large but finite Deborah numbers, Fig. 4 shows the numerically computed motion of the cavity. One observes complete collapse for $P_0/G_0 = 100$, with a collapse time very nearly equal to the Rayleigh collapse time for an inviscid, nonelastic liquid. Furthermore, it is evident that for the case $N_{De} = 1000$ shown there, the motion on the first few cycles is effectively the same as for $N_{De} = \infty$. Also, it can be observed that the Reynolds number has a significant effect on the initial motion only

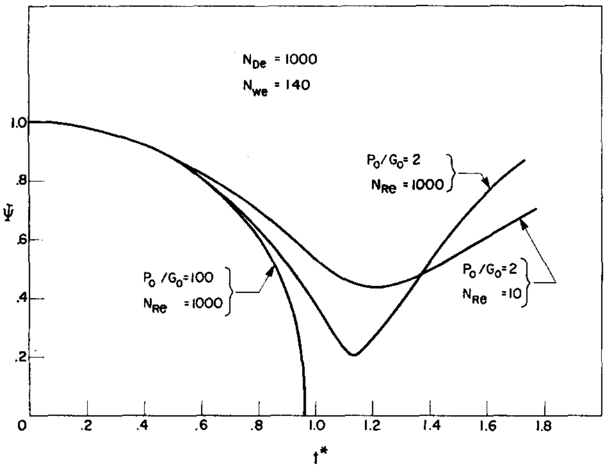

FIG. 4. The effect of the Reynolds number and the P_0/G_0 ratio on the collapse of a cavity.

when it is numerically on the order of magnitude of ten or less. Because of the greater energy dissipation at the lower Reynolds numbers, it appears that the rebound radius decreases with the increasing Reynolds numbers.

IV. COLLAPSE AT SMALL DEBORAH NUMBERS

While it is evident that for any finite Deborah number a void must eventually collapse to zero radius, it is nonetheless of interest to investigate how collapse is delayed by the elasticity of the fluid. In particular, we may consider the first cycle of motion, as in Fig. 5. For a given P_0/G_0, the rebound radius on the first cycle decreases with decreasing Deborah number as shown there. If the fluid is "inviscid" ($N_{Re} = \infty$) the critical Deborah number at which the cavity collapses completely on the first cycle is 0.51 for a P_0/G_0 ratio of 1.43, whereas for

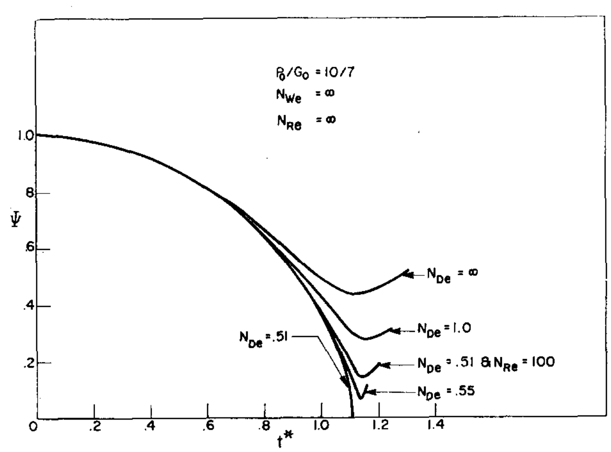

FIG. 5. The effect of the Deborah number on the initial motion of a cavity.

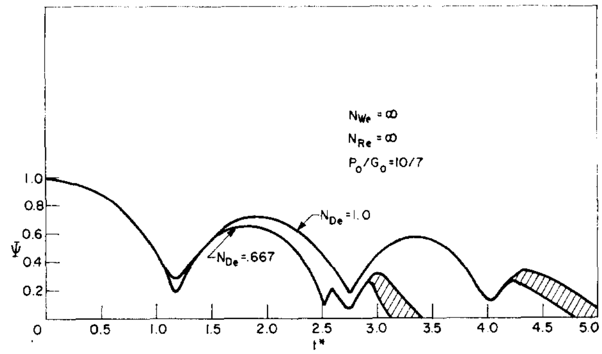

FIG. 6. The effect of the Deborah number on the cavity motion.

a finite Reynolds number the cavity no longer collapses on the first cycle at $N_{De} = 0.51$, but instead rebounds as shown in the figure.

For various cases, the numerical solutions were carried out for several cycles of the motion, and some of the results are shown in Fig. 6. In this figure one observes that for a Deborah number of $\frac{2}{3}$, the cavity collapses in approximately three major cycles. One also notes that the maximum radius reached after each rebound decreases in an almost linear fashion for the first few oscillations when $N_{De} = 1$. The modulation within the later cycles and the exact radius values in final stages of collapse are uncertain at this time, since numerical integration difficulties were encountered at long times. (The longest time shown represents some 30–40 min of IBM 360 computation time for a single run.)

V. CONCLUSIONS

The results of the preceding analysis indicate that elastic effects may well have a strong influence on cavitation in viscoelastic liquids. We should certainly expect such effects to occur at high Deborah numbers λ/t_c, where the relaxation time λ of the fluid is long compared with the classical Rayleigh collapse time t_c.

In particular, for the Maxwellian liquid considered here, the present analysis shows that in the limit of large Deborah numbers $\lambda/t_c \to \infty$, a spherical void may either collapse or undergo oscillations about an equilibrium radius, depending on whether the ratio of ambient pressure to the elastic modulus of the fluid exceeds a definite, critical value. The presence of viscosity in the fluid tends to damp the oscillations, and a critical-damping phenomenon occurs for Reynolds numbers below a certain value.

Even for finite and moderate Deborah numbers, $\lambda/t_c = O(1)$, the ultimate collapse of a void is delayed for several cycles of expansion and contraction.

Although we have not considered the possible effects of gases or vapors in the collapsing cavity in detail, we should not expect such effects to greatly alter the role of liquid elasticity in the collapse process. In fact, one might reasonably anticipate that the combined effects of volume elasticity in the gas and shape elasticity in the liquid would reinforce one another in such a way as to retard or completely suppress the collapse of bubbles. From the results of previous studies of gas-filled bubbles in Newtonian fluids, we might also expect that, in many instances, the effects of liquid elasticity would be important at a much earlier stage in the collapse process. In such cases, the buildup of the liquid-phase momentum, which gives rise to catastrophic collapse, would be greatly suppressed.

In addition to any experimental work which may be suggested by the present study, it would also be of some interest to theoretically investigate the hydrodynamic stability of the spherically symmetric motion of cavities collapsing in viscoelastic liquids. While one might be tempted to employ a somewhat more refined rheological model for the liquid, this would probably lead to rather difficult analytical and computational problems, without necessarily providing much additional insight on the physics of the collapse phenomenon.

ACKNOWLEDGMENT

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[1] Lord Rayleigh, Phil. Mag. **34**, 94 (1917).
[2] H. G. Flynn, in *Physical Acoustics*, W. P. Manson, Ed. (Academic Press Inc., New York, 1964), Pt. B, p. 57.
[3] M. S. Plesset, in *Cavitation in Real Liquids*, R. Davies, Ed. (Elsevier Publishing Company, Amsterdam, 1964), p. 1.
[4] H. S. Fogler, Chem. Engr. Sci. **24**, 1043 (1969).
[5] W. J. Yang and H. C. Yeh, A.I.Ch.E. J. **12**, 927 (1966).
[6] J. R. Street, Trans. Soc. Rheol. **12**, 103 (1968).
[7] R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena* (John Wiley & Sons, Inc., New York, 1960), Chap. 3.
[8] M. Reiner, Phys. Today, January **17**, 62 (1964).
[9] A. B. Metzner, J. L. White, and M. M. Denn, A.I.Ch.E. J. **12**, 863 (1966).