Cartan and relativistic spin fluids in a rotating cylinder

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Spin-fluid sources and their metric are obtained for an infinite rotating cylinder. The calculation is performed in general relativity and the Cartan theory. The spin fluids are significantly different in the two theories.

I. INTRODUCTION

Rotating fluids are often used in astrophysical and cosmological model calculations. Rotation can be included in relativistic fluid calculations in several ways. The fluid can be contained within a space-time $g_{\mu\nu}$ ($\mu,\nu = 0\text{–}3$) that has an off-diagonal component g_{0i} ($i = 1\text{–}3$). This can introduce fluid rotation, which is described by the vorticity tensor[1]

$$\omega_{\mu\nu} = U_{[\mu;\nu]} + \dot{U}_{[\mu} U_{\nu]}, \tag{1}$$

where U_μ is the fluid velocity and $\dot{U}_\mu = U_{\mu;\nu} U^\nu$ the acceleration.

In addition to the rotation given by $\omega_{\mu\nu}$, fluids can have an angular velocity that is related to an intrinsic spin density. In these spin fluids, the rotation associated with the spin density can be described by the tensor $\widetilde{\omega}_{\mu\nu}$,[2]

$$\widetilde{\omega}_{\mu\nu} = \dot{a}_\mu^{(i)} a_{\nu(i)}, \tag{2}$$

where $a_\mu^{(i)}$ is an orthonormal tetrad chosen so that $a^\mu{}_{(0)}$ lies along the velocity U^μ. Tetrad indices are in parentheses $(i) = (0\text{–}3)$ and are raised and lowered with the Minkowski metric $\eta_{(ij)} = (-1, +1, +1, +1)$. Tsoubelis,[3] using the spin-fluid stress-energy tensor of Ray and Smalley,[4,5] has shown that a Cartan spin fluid has the same frame dragging properties as a fluid with conventional vorticity. His result indicates that a spin fluid can serve as a source of angular momentum for astrophysical and cosmological models. Spin-fluid interiors can be matched to stationary vacuum exteriors.

In addition to direct astrophysical applications, spin fluids can also be used to model superfluid rotation detectors like those being developed by Cerdonio[6] for use in experimental relativity. The purpose of this paper is to explore the use of rotating spin-fluid sources. The calculation is performed in both general relativity and the Einstein Cartan self-consistent formalism. We impose cylindrical symmetry since the most immediate application of spin fluids is to rotating detectors. For constant rotational velocity, we find metric solutions of the van Stockum[7] type containing an unaccelerated spin fluid. There are some interesting differences between the Cartan and general relativistic fluid sources. The sources are compared and discussed in the last part of the paper. In the next sections we briefly review the spin-fluid stress-energy tensor and write the field equations leading to our solutions.

II. FORMALISM

A. Metric and tetrad

The space-time we consider is stationary and cylindrically symmetric with metric

$$ds^2 = -f\,dt^2 - 2K\,d\phi\,dt + l\,d\phi^2 + e^\mu(dr^2 + dz^2). \tag{3}$$

The orthonormal tetrad that diagonalizes this metric is

$$\begin{aligned} a^\mu{}_{(0)} &= (1/\sqrt{f},0,0,0),\\ a^\mu{}_{(1)} &= (0,e^{-\mu/2},0,0),\\ a^\mu{}_{(2)} &= (-K/D\sqrt{f},0,0,\sqrt{f}/D),\\ a^\mu{}_{(3)} &= (0,0,0,e^{-\mu/2}), \end{aligned} \tag{4}$$

where $D^2 = fl + K^2$. In terms of this tetrad, the comoving velocity is

$$U^\mu = a^\mu{}_{(0)} = (1/\sqrt{f},0,0,0). \tag{5}$$

B. Stress-energy tensor

A spin fluid is a fluid with an angular momentum density $S_{\mu\nu}$ defined throughout its extent. The spin density is constrained by the Frenkel[8] condition

$$U^\mu S_{\mu\nu} = 0. \tag{6}$$

Ray and Smalley[4] have developed a self-consistent Lagrangian formulation of the Einstein–Cartan theory with spin density. The stress-energy tensor derived from the Lagrangian variation can be written[5] as the sum of two parts:

$$T^{\mu\nu} = T^{\mu\nu}(\text{spin}) + T^{\mu\nu}(\text{fluid}), \tag{7}$$

where $T^{\mu\nu}$(fluid) is the perfect fluid stress-energy tensor

$$T^{\mu\nu}(\text{fluid}) = (\varepsilon + p) U^\mu U^\nu + p g^{\mu\nu}, \tag{8}$$

where ε is the energy density and p the pressure.

The spin-fluid portion of the stress-energy tensor is

$$T^{\mu\nu}(\text{spin}) = 2U^{(\mu}S^{\nu)\sigma}\dot{U}_\sigma + \nabla^*_\sigma[U^{(\mu}S^{\nu)\sigma}] - \widetilde{\omega}^{\sigma(\mu}S^{\nu)}{}_\sigma. \tag{9}$$

The spin density $S^{\mu\nu}$ is related to the proper torsion $\hat{S}_{\mu\nu}{}^\sigma$ by

$$\hat{S}_{\mu\nu}{}^\sigma = \tfrac{1}{2}(8\pi G) S_{\mu\nu} U^\sigma. \tag{10}$$

The proper torsion is the trace-free part of the torsion $S_{\mu\nu}{}^\sigma$,

$$\hat{S}_{\mu\nu}{}^\sigma = S_{\mu\nu}{}^\sigma + \tfrac{2}{3}\delta^\sigma{}_{[\mu} S_{\nu]\alpha}{}^\alpha. \tag{11}$$

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The quantity $S_{\nu\alpha}{}^{\alpha}$ is the torsion vector and describes that part of the torsion that does not satisfy the Frenkel condition. It is zero in our calculation. The ∇^*_σ derivative is $\nabla^*_\sigma = \nabla_\sigma + 2S_{\sigma\alpha}{}^{\alpha}$. It will be the covariant derivative in both calculations. Although the stress-energy tensor was constructed for self-consistent Cartan fluids, it is also valid in a self-consistent general relativity.[5] The only difference is in the derivative operator ∇^*_σ. It is the ordinary covariant derivative in general relativity. It is also the covariant derivative in the Cartan theory but with an additional spin connection calculated from the torsion:

$$\Gamma^{\sigma}{}_{\mu\nu} = S_{\mu\nu}{}^{\sigma} - S_{\nu}{}^{\sigma}{}_{\mu} + S^{\sigma}{}_{\mu\nu}. \tag{12}$$

Before writing the stress-energy tensor, it is useful to define the parameters describing the fluid.

C. Fluid parameters

The fluid parameters of interest are the acceleration and the rotation function $\omega_{\mu\nu}$. The acceleration is the same in general relativity and the Cartan theory,

$$\dot{U}_r = f_r/2\sqrt{f}. \tag{13}$$

We are interested in unaccelerated fluids so that f is a constant. We will choose

$$f = 1. \tag{14}$$

The fluid angular speed in Cartan theory is

$$\omega_{r\phi} = K_r/2\sqrt{f} - S_{r\phi}/2, \quad \omega_{r0} = f_r/2\sqrt{f}. \tag{15}$$

The term $S_{r\phi}$ in $\omega_{r\phi}$ comes from the spin connection. In general relativity these velocities are

$$\omega_{r\phi} = K_r/2\sqrt{f}, \quad \omega_{r0} = f_r/2\sqrt{f}. \tag{16}$$

Only $\omega_{r\phi}$ will be nonzero in the unaccelerated fluid. Because we have an unaccelerated fluid, the angular velocity associated with the spin, $\tilde{\omega}_{\mu\nu}$, is identical to the fluid vorticity $\omega_{\mu\nu}$.

III. FIELD EQUATIONS AND SOLUTIONS—GENERAL RELATIVITY

A. Field equations

Using the fluid parameters discussed in the last section, the stress-energy tensor components are found to be

$$\begin{aligned}
T_{00} &= \varepsilon - W_0,\\
T_{rr} &= pe^{\mu} + (S_{r\phi}/D^2)K_r,\\
T_{zz} &= pe^{\mu},\\
T_{\phi\phi} &= \varepsilon K^2 + D^2 p - KW_\phi + S_{r\phi}e^{-\mu}K_r,\\
T_{0\phi} &= \varepsilon K - \tfrac{1}{2}(W_\phi + KW_0).
\end{aligned} \tag{17}$$

The quantity W_μ is the spin divergence $W_\mu = (S_\mu{}^{\nu})_{;\nu}$. The field equations in general relativity are

$$G_{\mu\nu} = T_{\mu\nu}.$$

We have taken $8\pi G = 1$. It is convenient to use the tetrad indexed components in writing the field equations:

$$\begin{aligned}
G_{(ij)} &= a_{(i)}{}^{\mu} a_{(j)}{}^{\nu} G_{\mu\nu},\\
T_{(ij)} &= a_{(i)}{}^{\mu} a_{(j)}{}^{\nu} T_{\mu\nu},
\end{aligned} \tag{18}$$

$$G_{(ij)} = T_{(ij)}. \tag{19}$$

The field equations are

$$(00)\colon\ \frac{3K_r^2}{4D^2} - \frac{D_{rr}}{D} - \frac{\mu_{rr}}{2} = \varepsilon e^{\mu} + S_{r\phi}\frac{K_r}{D^2}, \tag{20}$$

$$(11)\colon\ \frac{K_r{}^2}{4D^2} + \frac{\mu_r D_r}{2D} = pe^{\mu} + S_{r\phi}\frac{K_r}{D^2}, \tag{21}$$

$$(22)\colon\ \frac{K_r{}^2}{4D^2} + \frac{\mu_{rr}}{2} = pe^{\mu} + S_{r\phi}\frac{K_r}{D^2}, \tag{22}$$

$$(33)\colon\ \frac{-K_r{}^2}{4D^2} - \frac{\mu_r D_r}{2D} + \frac{D_{rr}}{D} = pe^{\mu}, \tag{23}$$

$$(02) + (20)\colon\ \left(\frac{K_r}{D}\right)_r = \left(\frac{S_{r\phi}}{D}\right)_r. \tag{24}$$

B. Solutions

1. Spin and vorticity

One of the most interesting results of the calculation follows from integrating Eq. (24),

$$K_r/2D = S_{r\phi}/2D + c', \tag{25}$$

where c' is an integration constant, or

$$\omega_{r\phi}/D = S_{r\phi}/2D + c'. \tag{26}$$

This is a driving relation between the spin density and the vorticity or spin angular speed. Associated with the spin and rotation tensors are vector functions

$$\begin{aligned}
S^{\mu} &= (\eta^{\mu\nu\alpha\beta}/2\sqrt{-g})\,U_\nu S_{\alpha\beta},\\
\omega^{\mu} &= (\eta^{\mu\nu\alpha\beta}/2\sqrt{-g})\,U_\nu \omega_{\alpha\beta}.
\end{aligned} \tag{27}$$

The driving relation is linear when written in terms of these vectors:

$$\omega_z = S_z/2 + c, \tag{28}$$

where $c = c'\eta^{0r\phi z}$.

2. Metric functions

Comparing Eqs. (21) and (22) the metric potential $\mu(r)$ can be related to D:

$$\mu_r(D_r/D) = \mu_{rr}$$

or

$$\mu_r = AD, \tag{29}$$

where A is a constant of integration.

Using this and eliminating the pressure between Eqs. (20) and (21), a differential equation for D is found:

$$D_{rr}/D - AD_r + 2(S_z{}^2/4 - c^2) = 0. \tag{30}$$

One solution of this differential equation is

$$D = r. \tag{31}$$

This solution corresponds to a constant spin S_z, and therefore a constant vorticity ω_z,

$$S_z = \text{const}, \quad \omega_z = \text{const}. \tag{32}$$

Equation (30) then gives the constant A:

$$A = 2(S_z{}^2/4 - c^2). \tag{33}$$

The other metric functions are

$$K=\omega_z r^2+c_1, \quad u=Ar^2/2+c_2, \quad l=r^2-K^2, \tag{34}$$

where c_1 and c_2 are integration constants. This is a metric of the van Stockum type. The comparison will be made in detail in the last part of the paper.

3. Pressure and energy density

Since the choice $D=r$ requires S_z and ω_z to be constant, the driving relation, Eq. (28), can be reparametrized by defining

$$c=\omega_z(2-n) \tag{35}$$

and

$$S_z=2\omega_z(n-1). \tag{36}$$

The pressure and energy density follow from the field equations

$$pe^{\mu}=-\omega_z S_z=\omega_z{}^2 2(1-n), \tag{37}$$

$$\varepsilon e^{\mu}=2\omega_z{}^2(5-3n). \tag{38}$$

The relation between the pressure and energy density is

$$\varepsilon-3p=e^{-\mu}4\omega_z{}^2, \tag{39}$$

where $n=1$ is a spinless dust solution. Positive pressure and energy require the range of n to be

$$-\infty\leqslant n<1. \tag{40}$$

IV. FIELD EQUATIONS AND SOLUTIONS—CARTAN

A. Field equations

The tetrad indexed stress-energy tensor is

$$\begin{aligned}
T_{(00)}&=\varepsilon-W_0/\sqrt{f},\\
T_{(11)}&=p+e^{-\mu}(S_{r\phi}/D^2)(K_r-S_{r\phi}),\\
T_{(22)}&=p+(S_{r\phi}e^{-\mu}/D^2)(K_r-S_{r\phi}),\\
T_{(33)}&=p,\\
T_{(02)}&=(KW_0-W_\phi)/2D.
\end{aligned} \tag{41}$$

The field equations in the Cartan theory are

$$G_{\mu\nu}=T_{\mu\nu}+\nabla^*_\sigma(T_{\mu\nu}{}^{\sigma}+T^{\sigma}{}_{\mu\nu}-T_{\nu}{}^{\sigma}{}_{\mu}), \tag{42}$$

where $T_{\mu\nu}{}^{\sigma}$ is the modified torsion

$$T_{\mu\nu}{}^{\sigma}=S_{\mu\nu}{}^{\sigma}+2\delta^{\sigma}{}_{[\mu}S_{\nu]\alpha}{}^{\alpha}. \tag{43}$$

Since the torsion vector is zero, $T_{\mu\nu}{}^{\sigma}$ is just the Cartan spin connection and ∇^*_σ is the Cartan covariant derivative. The field equations are

$$(00){:}\ \frac{3K_r{}^2}{4D^2}-\frac{D_{rr}}{D}-\frac{\mu_{rr}}{2}=\varepsilon e^{\mu}+S_{r\phi}\frac{K_r}{D^2}-\frac{S_{r\phi}{}^2}{4D^2}, \tag{44}$$

$$(11){:}\ \frac{K_r{}^2}{4D^2}+\frac{\mu_r D_r}{2D}=pe^{\mu}+S_{r\phi}\frac{K_r}{D^2}-\frac{3}{4}\frac{S_{r\phi}{}^2}{D^2}, \tag{45}$$

$$(22){:}\ \frac{K_r{}^2}{4D^2}+\frac{\mu_{rr}}{2}=pe^{\mu}+S_{r\phi}\frac{K_r}{D^2}-\frac{3S_{r\phi}{}^2}{4D^2}, \tag{46}$$

$$(33){:}\ -\frac{K_r{}^2}{4D^2}-\frac{\mu_r D_r}{2D}+\frac{D_{rr}}{D}=pe^{\mu}-\frac{S_{r\phi}{}^2}{4D^2}, \tag{47}$$

$$(20)+(02){:}\ \left(\frac{K_r}{2D}\right)_r=\left(\frac{S_{r\phi}}{2D}\right)_r. \tag{48}$$

B. Solutions

1. Spin and angular speed

Integrating (48), a result similar to Eq. (25) is found:

$$K_r/2D=S_{r\phi}/2D+c'. \tag{49}$$

The Cartan vorticity is

$$\omega_{r\phi}=K_r/2-S_{r\phi}/2. \tag{50}$$

Equation (50) is not a driving relation between vorticity and spin but a statement that the Cartan vorticity is constant.

The equation satisfied by D follows by eliminating the pressure between (44) and (45):

$$-D_{rr}/D+AD_r+2c^2=0. \tag{51}$$

Because of the constraint on the Cartan vorticity, given by Eq. (50), the solution $D=r$ imposes no restrictions on the functional form of the spin density and rotation. The constant A is

$$A=-2c^2, \tag{52}$$

μ is the same as the general relativistic function

$$\mu_r=AD, \quad \mu=Ar^2/2+c_2, \tag{53}$$

and A is, of course, different in general relativity and the Cartan theory.

The functional dependence of the metric potential K depends on the spin. Equation (49) can be written

$$K_r/2D=S_z/2+c, \quad K=\int S_z r\,dr+cr^2+c_1, \tag{54}$$

and

$$l=r^2-k^2. \tag{55}$$

A constant spin, for example, gives

$$K=S_z r^2/2+cr^2+c_1. \tag{56}$$

A constant spin generates a metric of the van Stockum[7] type.

2. Pressure and energy density

The pressure and energy density can be obtained from the field equations

$$pe^{\mu}=-cS_z, \quad \varepsilon e^{\mu}=4c^2+cS_z, \tag{57}$$

with pressure and energy density satisfying

$$\varepsilon+p=4c^2e^{-2}. \tag{58}$$

For the special case of constant spin density, the van Stockum example, we can again define $c=\omega_0(2-n)$, with ω_0 that part of the Cartan vorticity due to $K_r/2D$. This is a useful step to take in order to compare with the general relativity. The results are

$$\begin{aligned}
S_z&=2(n-1)\omega_0,\\
p&=2e^{\mu}\omega_0{}^2(n-2)(n-1),\\
\varepsilon&=2e^{\mu}\omega_0{}^2(n-3)(n-2).
\end{aligned} \tag{59}$$

The range of n for physical solutions is $n\leqslant 1$ and $n>3$, where $n=1$ gives a dust solution without spin.

V. COMPARISON TO THE VAN STOCKUM SOLUTION

The van Stockum interior solution can be written[9] as

$$ds^2 = -dt^2 + 2ar^2\,dt\,d\phi + r^2(1-a^2r^2)d\phi^2 + e^{-a^2r^2}(dr^2+dz^2). \tag{60}$$

The fluid contained in this space-time is dust with energy density

$$\varepsilon = 4a^2e^{a^2r^2}. \tag{61}$$

The general relativistic metric is of this form with $c_1 = c_2 = 0$ in Eq. (34):

$$\begin{aligned} ds^2 &= -dt^2 - 2\omega_z r^2\,dt\,d\phi + r^2(1-\omega_z{}^2r^2)d\phi^2 \\ &\quad + e^{[S_z^2/4 - c^2]r^2}(dr^2+dz^2) \\ &= -dt^2 - 2\omega_z r^2\,dt\,d\phi + r^2(1-\omega_z{}^2r^2)d\phi^2 \\ &\quad + (dr^2+dz^2)e^{\omega_z{}^2(2n-3)r^2}, \end{aligned} \tag{62}$$

where ω_z is the fluid vorticity vector. This is identical to the van Stockum solution for $n=1$ and $|\omega_z| = a$. The fluid is the van Stockum spinless dust. For other n values there is both spin and pressure. If $|n|$ is very large, the fluid equation of state approaches $\varepsilon = 3p$.

The Cartan metric for constant spin density can be written

$$\begin{aligned} ds^2 &= -dt^2 - 2r^2(c+S_z/2)dt\,d\phi \\ &\quad + r^2[1-(c+S_z/2)^2r^2]d\phi^2 \\ &\quad + (dr^2+dz^2)e^{-c^2r^2}. \end{aligned} \tag{63}$$

The constant c is the Cartan vorticity vector, $n=1$, and $c = -a$ generates the spinless dust van Stockum solution. For other n values, there is spin and pressure with an asymptotic equation of state $p = \varepsilon$. These constant spin van Stockum solutions will not match pressures to an exterior vacuum. This is always true for the general relativistic solution with its required constant spin. The Cartan solution allows nonconstant spins, and a vacuum match is possible in this case.

VI. THE VACUUM MATCH FOR THE CARTAN SOLUTIONS

The general solution for the Cartan calculations is

$$\begin{aligned} &ds^2 = -dt^2 - 2K\,d\phi\,dt + l\,d\phi^2 + e^{\mu}(dr^2+dz^2), \\ &\mu = -c^2r^2, \\ &K_r/2r = S_{r\phi}/2r + c, \\ &pe^{\mu} = -S_{r\phi}c/r, \\ &l = r^2 - K^2, \\ &\varepsilon e^{\mu} = cS_{r\phi}/r + 4c^2, \end{aligned} \tag{64}$$

where $S_{r\phi}$ and K_r are not yet specified.

This metric can be matched to a van Stockum exterior solution found by Bonner[10]:

$$\begin{aligned} &e^{\bar\mu} = e^{-1/4}(R/r)^{1/2}, \\ &\bar l = (rR/4)[3+\log(r/R)], \\ &\bar K = -\tfrac{1}{2}r[1+\log(r/R)], \\ &\bar f = (r/R)[1-\log(r/R)]. \end{aligned} \tag{65}$$

This metric is one of three exterior van Stockum metrics found by Bonner; it is Petrov II. Here R is the boundary between interior and exterior.

The matching conditions[11] identify the first and second fundamental forms of the bounding surface, $r=R$. Matching the metrics, one finds two conditions

$$c^2R^2 = \tfrac{1}{4}, \tag{66}$$

$$K(R) = -R/2. \tag{67}$$

Identifying the second fundamental forms, there is one new relation

$$K_r(R) - S_{r\phi}(R) = \bar K_r(R)$$

or

$$K_r(R) - S_{r\phi}(R) = -1. \tag{68}$$

Using the relation requiring the Cartan vorticity to be constant,

$$K_r - S_{r\phi} = 2rc,$$

the sign of c in Eq. (66) is determined as

$$c = -1/2R. \tag{69}$$

The functional form of $K(r)$ and $S_{r\phi}(r)$ is constrained by these matching conditions:

$$K = \int S_{r\phi}\,dr + r^2c + C_0,$$

where C_0 is an integration constant. At $r=R$, this becomes

$$K(R) = \int_{(r=R)} S_{r\phi}\,dr - \frac{R}{2} + C_0. \tag{70}$$

Therefore at the boundary we have the spin constraint

$$\int_{(r=R)} S_{r\phi}\,dr = \text{const.} \tag{71}$$

In addition, if the pressure is to be zero at the vacuum boundary then the spin density is constrained to be zero at the boundary:

$$S_{r\phi}(R) = 0 \tag{72}$$

Any function satisfying (71) and (72) is acceptable. Choosing $S_{r\phi}(r) = rf(r)$, $f(r)$ regular at $r=0$, will produce finite pressure and energy at the origin.

VII. DISCUSSION AND CONCLUSION

General relativity and the Cartan theory both admit physical spin-fluid solutions for a rotating cylinder. The solutions in general relativity are only for constant spin and angular speed. The Cartan theory also has a constant spin solution but it is only one of many possibilities. Both constant spin sources lead to van Stockum-type metrics. The general relativistic source approaches the equation of state $\varepsilon = 3p$, and the Cartan source approaches $\varepsilon = p$, in the high spin limit.

There is another important difference between the Cartan and relativistic solutions in the case of constant spin. In both theories, the spin and $\omega_0 = K_r/2D$ are related through

$$S_z = 2\omega_0(n-1).$$

Both theories allow $n \leqslant 1$, giving S_z the opposite sense to

ω_0. The Cartan theory also allows $n > 3$; S_z and ω_0 can be parallel.

There is a very interesting driving relation between spin and cylinder speed in general relativity:

$$S_z/2 = \omega_z - c.$$

Vortex formation in rotating superfluids is one of the possible applications of the spin-fluid formalism. This driving relation, although in the continuum limit, could be interpreted in this context. Here c would be the critical speed for vortex formation to begin.

In conclusion, there are some similarities and also some fundamental differences between the two spin-fluid sources used for this calculation. With spin fluids being used as models in astrophysics and condensed matter physics, it is hopeful that a test of the two theories might occur in the near future.

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