On the Equivalence of Dressing Transformations*

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(Received 6 December 1971; Revised Manuscript Received 22 February 1972)

The equivalence of the representations of the Weyl algebra for ϕ^4_{2+1} in a box induced by the dressing transformation of Glimm and the unitary dressing transformation is studied. Equivalence is shown for the simplified model in which only the most singular portion of the interaction is kept.

I. INTRODUCTION

A method for constructing dressing transformations for certain field theory models has been developed by Glimm.[1] This method employs a sequence of operators that are not unitary. In Refs. 2 and 3 an alternate construction is developed employing unitary operators. In Ref. 4 results are obtained on the equivalence of different dressing transformations constructed for ϕ^4_{2+1} by the method of Glimm. In this paper we obtain some results on the equivalence of the representations of the Weyl algebra induced by the transformations of Glimm and by the unitary transformations for a simplified ϕ^4_{2+1} model—keeping only the most singular terms—in a box.

II. BASIC MODEL AND DEFINITIONS

Corresponding to a cut-off momentum Λ the Hamiltonian is written as

$$H_\Lambda = H_{0\Lambda} + V_\Lambda + \Delta_\Lambda \tag{1}$$

with

$$H_{0\Lambda} = \sum_{|k|\le\Lambda} \omega_k a_k^* a_k, \tag{2}$$

$$\omega_k^2 = k^2 + M^2, \quad V_\Lambda = g\int_0^1 :\phi_\Lambda^4: dx\,|_{0,4+4,0}, \tag{3}$$

$$\Delta_\Lambda = \Delta_\Lambda^{(2)} + \tfrac{1}{2}(\delta M_\Lambda^{(2)})\int :\phi_\Lambda^2: dx\,|_{1,1}. \tag{4}$$

Here ϕ_Λ contains only operators a_k and a_k^* with $|k| \le \Lambda$: $\Delta_\Lambda^{(2)}$ and $\delta M_\Lambda^{(2)}$ are the renormalization constants appropriate to this modified ϕ^4_{2+1} model, and the numbers following the bars indicate that in the Wick expansion of $:\phi_\Lambda^4:$ only the terms with four creation or four annihilation operators are kept, and in the Wick expansion of $:\phi_\Lambda^2:$ only the terms in one creation and one annihilation operator are kept.

We define V_n as the sum of terms in the expansion of V for which the maximum absolute value of a momentum $k_{\max}$ satisfies

$$(n-1)^\alpha \le k_{\max} < n^\alpha: \tag{5}$$

α is a large number chosen to ensure certain properties of the dressing transformations to be constructed. Note that

$$V = \sum_1^\infty V_n. \tag{6}$$

We use the same definition of Γ as used in Refs. 2 and 3, and let $P(s)$ be the projection onto states with particle number spectra $\le s$.

Define

$$A_n = P(n^2)(\Gamma V_n)P(n^2), \tag{7}$$

$$U_n = e^{-A_n}\cdot e^{-A_{n-1}}\cdots e^{-A_1}. \tag{8}$$

The U_n are unitary operators defining the unitary dressing transformation.

Now let $V_n^+(V_n^-)$ be the terms in V_n containing only creation operators (annihilation operators)

$$V_n = V_n^+ + V_n^-. \tag{9}$$

Define

$$B_n = \Gamma V_n^+, \tag{10}$$

$$W_n = \sum_0^{[n/4]} \frac{(-B_n)^s}{s!}, \tag{11}$$

$$T_n = W_n\cdot W_{n-1}\cdots W_1. \tag{12}$$

Equation (11) defines a particular truncation (our construction satisfies condition C_δ of the second paper in Ref. 4). The T_n define Glimm's dressing transformation.

We define $\lambda(n)$:

$$\lambda(n) = \sum_{s=0}^{n} \langle 0|(-\Gamma V_s^-)(\Gamma V_s^+)|0\rangle. \tag{13}$$

Finally we let $\mathfrak{D}$ be the subspace of vectors in Fock space with finite particle number and momentum components in a bounded region of momentum space.

III. GLIMM'S DRESSING TRANSFORMATION

For vectors φ, ψ in $\mathfrak{D}$ we define an inner product

$$\langle\varphi|\psi\rangle_{\mathcal{G}} = \lim_{n\to\infty}\langle T_n\varphi|T_n\psi\rangle\, e^{-\lambda(n)}. \tag{14}$$

Inner products without subscripts indicate inner products in Fock space. The completion of $\mathfrak{D}$ in this product defines a Hilbert space $\mathcal{H}_{\mathcal{G}}$. If G is an appropriate operator—say an element of the Weyl algebra—then the matrix element of the operator G as it appears in this representation is given by

$$\langle\varphi|G|\psi\rangle_{\mathcal{G}} = \lim_{n\to\infty}\langle T_n\varphi|G|T_n\psi\rangle\, e^{-\lambda(n)}. \tag{15}$$

We do not know whether an application of the Gel'fand construction will lead to a Hilbert space larger than $\mathcal{H}_{\mathcal{G}}$.

IV. UNITARY DRESSING TRANSFORMATION

For vectors φ, ψ in $\mathfrak{D}$ we define an inner product

$$\langle\varphi|\psi\rangle_U = \langle\varphi|\psi\rangle. \tag{16}$$

Thus the completion of $\mathfrak{D}$ in this product, $\mathcal{H}_U$, is formally identical to Fock space. If G is an appropriate operator then we shall define a transformed operator G' and define

$$\langle\varphi|G'|\psi\rangle_U = \langle\varphi|G'|\psi\rangle, \tag{17}$$

so that all the information on the unitary dressing transformation is contained in the transformation from G to G'. Formally,

$$G' = \lim_{n\to\infty} U_n^{-1} G U_n. \tag{18}$$

Initially the limit in (18) is defined only for vectors in $\mathfrak{D}$, but since $U_n^{-1}GU_n$ are uniformly bounded by the norm of G, the limit holds on all of $\mathcal{H}_U$. By the methods of Ref. 3 the limit in (18) holds strongly for vectors in $\mathfrak{D}$ and G in the Weyl algebra. The transformation G to G' can clearly be extended to a uniformly closed set of operators.

V. RELATION BETWEEN THE TWO TRANSFORMATIONS

We define the operators R_n:

$$R_n = e^{-(1/2)\lambda(n)} U_n^{-1} T_n, \tag{19}$$

defined initially on $\mathfrak{D}$. We now define

$$R = \lim_{n\to\infty} R_n. \tag{20}$$

We claim R is an isometric operator relating the two dressing transformations as follows:

$$T_{m,n} = W_m \cdot W_{m-1} \cdots W_{n+1}, \tag{21}$$

$$\begin{aligned}\langle\varphi|\psi\rangle_{\mathcal{G}} &= \lim_{m\to\infty} \langle T_m\varphi| T_m\psi\rangle e^{-\lambda(m)}\\ &= \lim_{m\to\infty} \langle U_m^{-1}T_{m,n}U_nR_n\varphi| U_m^{-1}T_{m,n}U_nR_n\psi\rangle e^{-\lambda(m)+\lambda(n)}\\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle U_m^{-1}T_{m,n}U_nR_n\varphi| U_m^{-1}T_{m,n}U_nR_n\psi\rangle\\ &\quad\times e^{-\lambda(m)+\lambda(n)}\\ &= \lim_{n\to\infty} \langle R_n\varphi|R_n\psi\rangle = \langle R\varphi|R\psi\rangle = \langle R\varphi|R\psi\rangle_U,\end{aligned} \tag{22}$$

$$\begin{aligned}\langle\varphi|G|\psi\rangle &= \lim_{m\to\infty} \langle T_m\varphi|G|T_m\psi\rangle e^{-\lambda(m)}\\ &= \lim_{m\to\infty} \langle U_m^{-1}T_{m,n}U_nR_n\varphi| U_m^{-1}GU_m|U^{-1}T_{m,n}U_nR_n\psi\rangle\\ &\quad\times e^{-\lambda(m)+\lambda(n)}\\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle U_m^{-1}T_{m,n}U_nR_n\varphi| U_m^{-1}GU_m|\\ &\quad\times U_m^{-1}T_{m,n}U_nR_n\psi\rangle e^{-\lambda(m)+\lambda(n)}\\ &= \langle R\varphi|G'|R\psi\rangle_U.\end{aligned} \tag{23}$$

The statements of (18), (22), and (23) detail the relation between the two dressing procedures:

$$\langle\varphi|\psi\rangle = \langle R\varphi|R\psi\rangle_U, \tag{22'}$$

$$\langle\varphi|G|\psi\rangle_{\mathcal{G}} = \langle R\varphi|G'|R\psi\rangle_U. \tag{23'}$$

$$G' = \lim_{n\to\infty} U_n^{-1}GU_n, \tag{18'}$$

We do not know whether the range of R is all of $\mathcal{H}_U$. The statements of (22) and (23) follow from the following three theorems.

Theorem 1: Let G be an element of the Weyl algebra, $\psi\in\mathfrak{D}$; then

$$\lim_{n\to\infty} U_n^{-1}GU_n|\psi\rangle$$

exists as a strong limit.

Theorem 2: Let $\psi\in\mathfrak{D}$; then

$$\lim_{n\to\infty} R_n|\psi\rangle$$

exists as a strong limit.

Theorem 3: Let $\psi\in\mathfrak{D}$; then

$$\lim_{n\to\infty} \frac{|(U_{m(n)}^{-1}T_{m(n),n}U_n - 1)R_n|\psi\rangle|}{|R_n|\psi\rangle|} = 0,$$

where $m(n)$ is an arbitrary function of n satisfying $m(n) > n$. Theorem 1 may be proved by the methods of Ref. 3, as noted before, much more simply than the result of Ref. 3. We shall prove Theorem 2 and merely comment that Theorem 3 is proved the same way.

VI. PROOF OF THEOREM 2

It is clearly sufficient to prove

$$\sum_{n=m}^{\infty} |(R_n - R_{n-1})|\psi\rangle| < \infty. \tag{24}$$

Now

$$R_n - R_{n-1} = e^{A_1}\cdot e^{A_2}\cdots e^{A_{n-1}}(\Delta_n e^{\Gamma V_n^-} - 1)W_{n-1} \times W_{n-2}\cdots W_1 e^{-(1/2)\lambda(n-1)} \tag{25}$$

with

$$\Delta = e^{P\Gamma VP}\sum_0^{[n/4]} \frac{(-\Gamma V^+)^s}{s!} e^{-(1/2)\lambda} e^{-\Gamma V^-}, \tag{26}$$

where to simplify notation we have abbreviated

$$\begin{aligned}\Delta &= \Delta_n,\\ P\Gamma VP &= P(n^2)\Gamma V_n P(n^2),\\ \lambda &= \lambda(n) - \lambda(n-1),\\ V^- &= V_n^-.\end{aligned} \tag{27}$$

For a given $\psi\in\mathfrak{D}$ and n large enough, the $e^{\Gamma V_n^-}$ in (25) can be dropped so that

$$|(R_n - R_{n-1})|\psi\rangle| = |(\Delta_n - 1)T_{n-1}|\psi\rangle| e^{-(1/2)\lambda(n-1)}. \tag{28}$$

It is known by the work of Glimm (and can also easily be proven using estimate 5 of Ref. 3) that $|T_s|\psi\rangle| e^{-(1/2)\lambda(s)}$ is uniformly bounded in s so that it is sufficient to consider

$$\frac{|(\Delta_n - 1)T_{n-1}|\psi\rangle| e^{-(1/2)\lambda(n-1)}}{|T_{n-1}|\psi\rangle| e^{-(1/2)\lambda(n-1)}} = \delta_n \tag{29}$$

and to show $\sum\delta_n < \infty$. We proceed to study $\Delta - 1$ (using the fact that for n large P may sometimes be replaced by 1):

$$\begin{aligned}\Delta - 1 &= \int_0^1 dt\, \frac{d}{dt}\left(e^{tP\Gamma VP}\sum_0^{[n/4]} \frac{(-t\Gamma V^+)^s}{s!} e^{-(1/2)t^2\lambda} e^{-t\Gamma V^-}\right)\\ &= \int_0^1 dt\, e^{tP\Gamma VP}[C + E_1 + E_2] e^{-t\Gamma V^-} e^{-(1/2)t^2\lambda},\end{aligned} \tag{30}$$

$$E_1 = \Gamma V^+ \cdot \frac{(-t\Gamma V^+)^{[n/4]}}{[n/4]!}, \tag{31}$$

$$E_2 = (-t\lambda)\frac{(-t\Gamma V^+)^{[n/4]}}{[n/4]!}, \tag{32}$$

$$\begin{aligned}C &= \sum_0^{[n/4]-1} \frac{(-t\Gamma V^+)^s}{s!}(-t\lambda + [\Gamma V^-, -t\Gamma V^+])\\ &+ \sum_{r=2}^{4}\sum_{s=0}^{[n/4]-r} \frac{(-t\Gamma V^+)^s}{s!}\frac{1}{r!}[\Gamma V^-, \underbrace{-t\Gamma V^+, \ldots, -t\Gamma V^+}_{r\text{ terms}}],\end{aligned} \tag{33}$$

where the abbreviations

$$[A,B,C] = [[A,B],C],$$

etc., are used.

On the vectors it acts upon for n large we get

$$|\Delta - 1| \le \int_0^1 dt\, |C + E_1 + E_2| e^{-(1/2)t^2\lambda}. \tag{34}$$

Consider the first term in C

$$\sum_0^{[n/4]-1} \frac{(-t\Gamma V^+)^s}{s!} (-t\lambda + [\Gamma V^-, -t\Gamma V^+]). \tag{35}$$

$|-t\lambda + [\Gamma V^-, -t\Gamma V^+]|$ on the vector upon which it acts may by an N_τ estimate be seen to satisfy

$$|-t\lambda + [\Gamma V^-, -t\Gamma V^+]| < e/N^\beta, \tag{36}$$

where β can be made arbitrarily large by choosing α large. The contribution of this term to $|\Delta - 1|$ contributes a term that for β large ensures $\sum \delta_N < \infty$, provided we can bound $\sum_0^{[n/4]-1} [(-t\Gamma V^+)^s/s!] e^{-(1/2)t^2\lambda}$. The conclusion of the proof is contained in the result that the norm of this term is uniformly bounded in n upon the vectors it acts on. One piece of this is the estimate

$$\sum_0^\infty \frac{x^s}{\sqrt{s!}} \le e' e^{+(1/2)x^2}, \quad x \ge 0, \tag{37}$$

The remaining problem is to estimate $|(\Gamma V_n^+)^s T_{n-1}|\psi\rangle| / |T_{n-1}|\psi\rangle|$.

Assume ψ has particle number restricted by r and all momenta less than or equal to L. If $n > N^{1/\alpha} + 1$, then

$$(\Gamma V_n^-) T_{n-1} |\psi\rangle = 0 \tag{38}$$

and $(\Gamma V^+)^d T_{n-1}|\psi\rangle$ has particle number less than $r + 4d + \sum_{s=0}^{n-1} \frac{1}{4} s \cdot 4$, which is less than $r + 4d + \frac{1}{2}n^2$. For $d \le [\frac{1}{4}n]$ as is restricted in the definition of W_n there is a constant c such that the particle number is $\le cn^2$ for all n. Using the method of the Estimate 5 of Ref. 3, we get for n satisfying the condition preceding (38)

$$\frac{|(\Gamma V^+)^d T_{n-1}|\psi\rangle|}{|T_{n-1}|\psi\rangle|} \le \left[d + \left(\frac{f}{cn^2}\right)^\gamma\right]^{1/2} \times \left[(d-1) + \left(\frac{f}{cn^2}\right)^\gamma\right]^{1/2} \cdots \left[1 + \left(\frac{f}{cn^2}\right)^\gamma\right]^{1/2} \times \lambda^{d/2} \tag{39}$$

for some f and γ, with γ arbitrarily large as α gets large. [In this application of the method of Estimate 5 the grading is not by total number of particles, but by the power of (ΓV^-) that can be applied before the zero vector is obtained.] Estimates (36), (37), and (39)—(36) modified slightly for other terms in $\Delta - 1$—combined yield the theorem.

VII. CONCLUSION

The following is a small selection of questions that remain to be answered. Is the range of R all of $\mathcal{K}_U$? Does the use of the Gel'fand construction lead to an expansion of H_S? Is there any way of describing the representation induced by (18)? How unique are the dressing transformations?

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* This work was supported in part by NSF Grant GP 17523.

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A Tachyon Dust Metric in General Relativity

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(Received 28 July 1971)

Special relativity allows the possibility of a class of particles, called tachyons, which travel with speeds greater than the speed of light in vacuum. These particles have spacelike 4-velocities. Since tachyons have energy and momentum, they will contribute to the gravitational field through the energy–momentum tensor. One question then is what types of solutions to the Einstein field equations will tachyons yield. We consider a metric which admits a four-parameter isometry group. When this metric is used in the field equations using a dust energy–momentum tensor, solutions exist only for spacelike 4-velocity of the dust. We interpret these as solutions for a tachyon dust. Exact solutions to the field equations are obtained.

1. INTRODUCTION

We are interested in solving Einstein's field equations with a dust energy–momentum tensor[1]

$$G^{ij} = R^{ij} - \tfrac{1}{2} g^{ij} R = +\rho U^i U^j + \Lambda g^{ij}, \tag{1.1}$$

where g^{ij} is the metric tensor, R^{ij} the Ricci tensor, R the scalar curvature, Λ the cosmological constant, U^i the 4-velocity of the dust normalized to unity and $\rho > 0$ the energy density. In order to solve the field equations, we assume the space–time possesses symmetries. We shall consider space–times which admit the four-parameter isometry group defined by the commutators[2]

$$[X_1, X_2] = 0, \quad [X_2, X_3] = 0, \quad [X_1, X_3] = X_1, \tag{1.2a}$$

$$[X_1, X_4] = 0, \quad [X_2, X_4] = X_2, \quad [X_3, X_4] = 0. \tag{1.2b}$$

The X_a are the infinitesimal operators of the group and are related to the killing vectors ξ_a^i which generate the group by