On the Wigner Supermultiplet Scheme*

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Calculation of Wigner and Racah coefficients for the group $SU(4) \supset [SU(2) \times SU(2)]$ make it possible to perform the spin–isospin sums in the cfp (fractional parentage coefficients) expansion of the matrix elements of one- and two-body operators in the Wigner supermultiplet scheme. The $SU(4)$ coefficients needed to evaluate one- and two-particle cfp's, the matrix elements of one-body operators, and the diagonal matrix elements of two-body operators are calculated in general algebraic form for many-particle states characterized by the $SU(4)$ irreducible representations $[yy0]$, $[y\,y-1\,0]$, $[yy1]$, $[y11]$, $[y\,y-1\,y-1]$, $[y10]$, $[yy\,y-1]$, $[y00]$, and $[yyy]$, whose states are specified completely by the spin and isospin quantum numbers (y = arbitrary integer). Applications are made to the calculation of the matrix elements of the complete space-scalar part of the Coulomb interaction and the space-scalar part of the particle–hole interaction for nucleons in different major oscillator shells.

1. INTRODUCTION

Since the decomposition of a many-nucleon wavefunction into its space times its spin–isospin part forms a good starting basis for shell-model calculations for many nuclei up to and through the first half of the $2s$, $1d$ shell, a detailed study of the Wigner supermultiplet scheme may still be in order, more than 30 years after the classic work[1] of Wigner, in which he first introduced the concept of spin–isospin supermultiplets and classified the many-nucleon spin–isospin wavefunctions according to the irreducible representations of the group $SU(4)$. The recognition that the spin–isospin part of the one- and several-particle fractional parentage coefficients (cfp)[2] can be identified with simple Wigner coefficients for the group $SU(4)$ leads to the possibility of performing spin–isospin cfp sums in a general way. In all nuclear matrix elements, the dependence on the Wigner supermultiplet quantum numbers, total spin, and isospin can thus be expressed directly by factors which are given simply in terms of $SU(4)$ Wigner and Racah coefficients. Because of recent general interest in the unitary groups, much detailed work[3-5] has been carried out on the groups $U(n)$. In particular, Biedenharn and Louck[3] have shown that complete algebraic formulas for the matrix elements of elementary operators (needed for the calculation of many-particle cfp's) can be read off at once from patterns assigned to these operators by means of a so-called pattern calculus. Unfortunately, these results cannot be applied to the Wigner supermultiplet scheme, since they apply only if the symmetry classification is given by the canonical chain of unitary groups, such as $U(4) \supset U(3) \supset U(2) \supset U(1)$. (In the classification scheme based on a canonical chain of subgroups, the states of a given irreducible representation are completely specified by the representation labels of all the subgroups in the chain.) Unfortunately, the group chains of actual physical interest in spectroscopy rarely coincide with these mathematically natural or canonical chains. In the Wigner supermultiplet classification, the physics dictates that the representations of $SU(4)$ be reduced according to the subgroup $SU(2) \times SU(2)$, where the direct product of the two $SU(2)$ groups is generated by the commuting spin and isospin operators. Neither of these is related to the group $U(2)$ in the canonical chain. Since the group $SU(4)$ and the 6-dimensional rotation group $O(6)$ have Lie algebras of identical structure, the Wigner supermultiplet scheme can also be considered from the point of view of the group $O(6)$. The canonical group chain $O(6) \supset O(5) \supset O(4) \supset O(3) \supset O(2)$ is again not the physically relevant one. Although *either* ordinary spin *or* isospin can be put into correspondence with the representations of the group $O(3)$ in this chain, the other is not a good quantum number in this canonical classification scheme for $O(6)$. The calculation of the needed supermultiplet Wigner coefficients must thus proceed from the specific properties of the group chain $SU(4) \supset [SU(2) \times SU(2)]$. Such calculations are complicated by the state-labeling problem. In the most general irreducible representation of $SU(4)$, the states are not completely specified by the spin and isospin quantum numbers alone. Although the additional operators needed for a complete classification of the states of a supermultiplet have been constructed

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[2] H. A. Jahn and H. van Wieringen, Proc. Roy. Soc. (London) **A209**, 502 (1951).

[3] L. C. Biedenharn and J. D. Louck, Commun. Math. Phys. **8**, 89 (1968).

[4] M. Moshinsky, in *Group Theory and the Many Body Problem*, E. Meeron, Ed. (Gordon and Breach, Science Publishers, New York, 1966).

[5] P. Carruthers, *Introduction to Unitary Symmetry* (Interscience Publishers, Inc., New York, 1966).

by Moshinsky and Nagel,[6] the algebraic structure of the eigenvectors associated with such operators is rather complicated and makes it very difficult to derive general algebraic formulas for the Wigner coefficients for the group $SU(4) \supset [SU(2) \times SU(2)]$, valid for all irreducible representations.[7] With relatively few exceptions, however, the $SU(4)$ representations of practical importance for shell-model calculations can be shown to fall into a few simple classes in which states of a given spin–isospin (S, T) occur at most once. By restricting the discussion to those irreducible representations in which all states are specified completely by the spin and isospin quantum numbers, it is possible to give general algebraic formulas for the $SU(4)$ Wigner and Racah coefficients needed to exploit to the fullest the properties of the group $SU(4)$ in the study of nuclei. By restricting the irreducible representations to those of the above type, a further simplification is achieved in connection with the multiplicity problem associated with the fact that Kronecker products of two $SU(4)$ representations are, in general, not simply reducible. For the irreducible representations of the above type and the Kronecker products which occur in nuclear model calculations, these multiplicities are never greater than two.

Our general approach is similar to that of a recent contribution by Kukulin, Smirnov, and Majling[8] on the Racah algebra of $SU(4)$. However, the aim of the present work is quite different insofar as it attempts to give explicit formulas, in general algebraic form, for most of the $SU(4)$ coefficients needed in shell model studies. The aim is not only to facilitate calculations but to derive explicit algebraic formulas which make it possible to study general trends in the dependence of nuclear matrix elements on the Wigner supermultiplet, spin, and isospin quantum numbers.

To establish the notation, a review of the properties of $SU(4)$ and the supermultiplet scheme is given in Sec. 2. In Sec. 3 the irreducible representations which are completely specified by the spin and isospin quantum numbers are identified. Explicit constructions are given for their state vectors in terms of a sequence of step-down operators acting on the state of highest weight. In Sec. 4 the general properties of the $SU(4)$ Wigner and Racah coefficients are discussed together with the methods used in the actual calculation of the coefficients required for shell-model studies. Tabulations are given in a set of appendices. Some applications are given in Sec. 5. Since the discussion is restricted to the spin–isospin parts of the many-nucleon wavefunctions, the detailed applications necessarily involve the discussion of physical operators which can be approximated by their complete space-scalar parts. The Coulomb interaction, for example, is rather insensitive to the details of the space parts of the wavefunctions. Matrix elements of the space-scalar part of the Coulomb interaction are calculated in Sec. 5 in order to study the dependence of the Coulomb energy on the Wigner supermultiplet, spin, and isospin quantum numbers.[9] Since a space-scalar approximation to the particle–hole interaction may give a good estimate of the full particle–hole interaction energy in $2s, 1d$ shell nuclei,[10] matrix elements for such a particle–hole interaction are derived in general algebraic form through the $SU(4)$ Wigner coefficients which are tabulated in the appendices. Even if an operator cannot be approximated by its complete space-scalar part, the full expression for its matrix elements in terms of cfp expansions can be simplified since the spin–isospin part of the cfp sum can be performed so that the over-all dependence on $[f]$, S, and T is given by the $SU(4)$ Wigner and Racah coefficients of the type tabulated in the appendices. By extending these techniques to the unitary groups needed to classify the space parts of the wavefunctions, the full cfp expansions can in principle be summed in general. The resultant interplay between the supermultiplet and spatial quantum numbers will be discussed in a future publication.

2. PROPERTIES OF $SU(4)$ AND THE SUPERMULTIPLET SCHEME

2.1. Infinitesimal Operators

The supermultiplet scheme is based on the four spin–charge states of a single nucleon $|m_s m_t\rangle$. These are

$$|1\rangle = |+\tfrac{1}{2} +\tfrac{1}{2}\rangle, \quad |2\rangle = |+\tfrac{1}{2} -\tfrac{1}{2}\rangle,$$
$$|3\rangle = |-\tfrac{1}{2} +\tfrac{1}{2}\rangle, \quad |4\rangle = |-\tfrac{1}{2} -\tfrac{1}{2}\rangle. \tag{1}$$

(Note that the first label indicates the spin, the second the isospin quantum number.) These states can also be expressed in terms of the single nucleon-creation operators $a^\dagger_{\alpha i}$, where α stands for the full set of space-quantum numbers, e.g., $\alpha = nlm_l$, and $i = 1, 2, 3$, or 4 stands for the spin–isospin quantum numbers $m_s m_t$ in the sense of Eq. (1), such that $|\alpha, i\rangle = a^\dagger_{\alpha i}|0\rangle$. The infinitesimal operators which generate the unitary transformations in the 4-dimensional space can be built from these fermion creation operators and their

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TABLE I. The infinitesimal operators (Infin. ops.).

Infin. ops.	$SU(4)$ generators	$O(6)$ generators	$SU(4)$ irreducible tensor components (standard phases) $T^{[f]}_{(SM_S)(TM_T)}$
S_0	$\frac{1}{2}(A_{11}+A_{22}-A_{33}-A_{44})$	J_{12}	$T^{[211]}_{(10)(00)}$
T_0	$\frac{1}{2}(A_{11}-A_{22}+A_{33}-A_{44})$	J_{56}	$T^{[211]}_{(00)(10)}$
$E_{00}(Y)$	$\frac{1}{2}(A_{11}-A_{22}-A_{33}+A_{44})$	J_{34}	$-T^{[211]}_{(00)(00)}$
S_+	$(1/\sqrt{2})(A_{13}+A_{24})$	$(1/\sqrt{2})(J_{13}+iJ_{23})$	$-T^{[211]}_{(11)(00)}$
S_-	$(1/\sqrt{2})(A_{31}+A_{42})$	$(1/\sqrt{2})(J_{13}-iJ_{23})$	$T^{[211]}_{(1-1)(00)}$
T_+	$(1/\sqrt{2})(A_{12}+A_{34})$	$(1/\sqrt{2})(J_{45}+iJ_{46})$	$-T^{[211]}_{(00)(11)}$
T_-	$(1/\sqrt{2})(A_{21}+A_{43})$	$(1/\sqrt{2})(J_{45}-iJ_{46})$	$T^{[211]}_{(00)(1-1)}$
E_{10}	$(1/\sqrt{2})(A_{13}-A_{24})$	$(i/\sqrt{2})(J_{14}+iJ_{24})$	$T^{[211]}_{(11)(10)}$
E_{-10}	$(1/\sqrt{2})(A_{31}-A_{42})$	$(-i/\sqrt{2})(J_{14}-iJ_{24})$	$-T^{[211]}_{(1-1)(10)}$
E_{01}	$(1/\sqrt{2})(A_{12}-A_{34})$	$(-i/\sqrt{2})(J_{35}+iJ_{36})$	$T^{[211]}_{(10)(11)}$
E_{0-1}	$(1/\sqrt{2})(A_{21}-A_{43})$	$(i/\sqrt{2})(J_{35}-iJ_{36})$	$-T^{[211]}_{(10)(1-1)}$
E_{11}	A_{14}	$\frac{1}{2}[(J_{15}+iJ_{25})+i(J_{16}+iJ_{26})]$	$-T^{[211]}_{(11)(11)}$
E_{-1-1}	A_{41}	$\frac{1}{2}[(J_{15}-iJ_{25})-i(J_{16}-iJ_{26})]$	$-T^{[211]}_{(1-1)(1-1)}$
E_{1-1}	A_{23}	$\frac{1}{2}[-(J_{15}+iJ_{25})+i(J_{16}+iJ_{26})]$	$T^{[211]}_{(11)(1-1)}$
E_{-11}	A_{32}	$\frac{1}{2}[-(J_{15}-iJ_{25})-i(J_{16}-iJ_{26})]$	$T^{[211]}_{(1-1)(11)}$

conjugate annihilation operators:

$$A_{ij}=\sum_{\alpha} a^{\dagger}_{\alpha i}a_{\alpha j},\quad i,j=1\cdots 4. \tag{2}$$

They contain the three components of the spin operator $\mathbf{S}$ and the isospin operator $\mathbf{T}$ together with the nine components of the operator

$$\mathbf{E}=\sum_{\alpha ij}\langle i|\,\boldsymbol{\sigma\tau}\,|j\rangle\, a^{\dagger}_{\alpha i}a_{\alpha j}, \tag{3}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ are the single-particle Pauli spin and isospin operators. As a specific example, let us take

$$\begin{aligned} E_{1-1} &= \sum_{\alpha m_s'\cdots m_t} \langle m_s'm_t'|\,\sigma_+\tau_-\,|m_sm_t\rangle\, a^{\dagger}_{\alpha m_s'm_t'}a_{\alpha m_sm_t} \\ &= \sum_{\alpha} a^{\dagger}_{\alpha\frac{1}{2}-\frac{1}{2}}a_{\alpha-\frac{1}{2}\frac{1}{2}} \\ &= A_{23}. \end{aligned} \tag{4}$$

The 15 operators $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{E}$ generate the group $SU(4)$. Together with the number operator $N_{\rm op}=\sum_i A_{ii}$, they generate the full group $U(4)$. The relation between the A_{ij} and the full set of Wigner supermultiplet operators $\mathbf{S}$, $\mathbf{T}$, $\mathbf{E}$ is shown in Table I. The components of $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{E}$ are all normalized such that the structure constants are ± 1 or 0. The commutation relations for the operators follow from the anticommutation relations of $a^{\dagger}$ and a. They are given by

$$[A_{ij},A_{kl}]=\delta_{jk}A_{il}-\delta_{il}A_{kj}. \tag{5}$$

Since the groups $SU(4)$ and $O(6)$ have Lie algebras of identical structure, the 15 infinitesimal operators can also be expressed in terms of angular-momentum operators J_{ij} $(i,j=1,\cdots,6)$ which generate rotations in a 6-dimensional real space. The specific relationship is shown in Table I. The spin and isospin spaces have been chosen as the (1, 2, 3) and (4, 5, 6) 3-dimensional subspaces of the full 6-dimensional space.

2.2. Irreducible Representations

The irreducible representations of the group $U(4)$ can be specified by the permutation symmetries of the n-nucleon spin–isospin functions. These symmetries are characterized by Young tableaux of 4 rows or partition numbers $[f_1f_2f_3f_4]$ on n objects, where f_i are integers such that $f_1+f_2+f_3+f_4=n$ and $f_1\ge f_2\ge f_3\ge f_4\ge 0$. The partition number f_i specifies the length of the ith row of the Young tableau. Since a totally antisymmetric 4-particle spin–isospin function is invariant under *unimodular* unitary transformations in the 4-dimensional space of the single-particle states, columns of four can be removed from the Young tableaux in restricting the irreducible representations to those of the group $SU(4)$. The irreducible representations of $SU(4)$ are thus specified by 3-rowed Young tableaux or the partition numbers $[f_1-f_4, f_2-f_4, f_3-f_4]$. The irreducible representation labels are often abbreviated by $[f]$. (For economy in writing representations, $[y00]$, $[yy0]$, and $[yyy]$ will sometimes also be denoted by $[y]$, $[y^2]$, and $[y^3]$, respectively.) The irreducible representations of $SU(4)$ can also be specified by the highest weights of the three commuting operators S_0, T_0, E_{00} of the rank 3 group. In an n-particle spin–isospin function of the above symmetry, there can be at most f_1 particles in state $|1\rangle$; of the remaining particles there can be at most f_2 particles in state $|2\rangle$; etc. The highest weights associated with the operators S_0, T_0, E_{00} are thus given by

$$\begin{aligned} P &= \tfrac{1}{2}(f_1 + f_2 - f_3 - f_4), \\ P' &= \tfrac{1}{2}(f_1 - f_2 + f_3 - f_4), \\ P'' &= \tfrac{1}{2}(f_1 - f_2 - f_3 + f_4), \end{aligned} \tag{6}$$

where P = maximum value of S_0 (and therefore S) contained in the representation; P' = maximum value of T_0 for a state having $S_0 = P$; and P'' = maximum value of E_{00} for a state with $S_0 = P$ and $T_0 = P'$. The three supermultiplet quantum numbers (P, P', P'') also specify the $O(6)$ irreducible representations according to the standard Weyl–Gel'fand labeling scheme.[11] (To avoid confusion, $O(6)$ quantum numbers (P, P', P'') will always be enclosed in round parentheses, $SU(4)$ quantum numbers $[f]$ by square brackets.)

2.3. State-Labeling Problem

Since the group $SU(4)$ [or alternately $O(6)$] is a 15-parameter group of rank 3, the states of a given irreducible representation are, in general, specified completely[12] by $\frac{1}{2}(15 - 3)$ or 6 quantum numbers in addition to the irreducible representation labels. The 6 additional quantum numbers could in principle be furnished by the irreducible representation labels of all the subgroups in one of the canonical subgroup chains (Fig. 1). In the chain based on $U(4)$, however, neither the spin nor the isospin operators can be put into correspondence with the subgroup $U(2)$. (Neither can the operators $\mathbf{S}$ nor $\mathbf{T}$ be identified with A_{ij}, $i, j = 1, 2$, or some other pair.) In the chain based on $O(6)$ it is possible to identify m_{31} and m_{21} with *either* the quantum numbers SM_S *or* TM_T. According to the specific choice of Table I, $m_{31} = S$, $m_{21} = M_S$, but in this scheme the operators $\mathbf{T}^2$ and T_0 are not diagonal. A complete classification of the state vectors for $SU(4) \supset [SU(2) \times SU(2)]$ must thus be given by a set of 6 commuting operators which must include, besides $\mathbf{S}^2$, S_0, $\mathbf{T}^2$, T_0, two additional operators. Unfortunately, the eigenvalues of the two additional operators cannot be simply related to irreducible representation labels of a subgroup of $SU(4)$. In the most general irreducible representation of $SU(4)$, the algebraic structure of the eigenvectors associated with these operators is therefore complicated. Moshinsky

$$\begin{array}{ccccccc} f_1 & & f_2 & & f_3 & & f_4 \\ & m_{31} & & m_{32} & & m_{33} & \\ & & m_{21} & & m_{22} & & \\ & & & m_{11} & & & \end{array} \qquad\qquad \begin{array}{ccccc} P & & P' & & P'' \\ & m_{51} & & m_{52} & \\ & m_{41} & & m_{42} & \\ & & m_{31} & & \\ & & & m_{21} & \end{array}$$

(a) (b)

FIG. 1. Weyl–Gel'fand canonical state labeling schemes based on the group chains (a) $U(4) \supset U(3) \supset U(2) \supset U(1)$ and (b) $O(6) \supset O(5) \supset O(4) \supset O(3) \supset O(2)$. The quantum numbers m_{ni} label the irreducible representations of $U(n)$ in (a) and $O(n)$ in (b). Note that $f_i \equiv m_{4i}$ and $(P, P', P'') \equiv (m_{61}, m_{62}, m_{63})$. The m_{ni} satisfy the branching rule $m_{ni} \leq m_{n-1,i} \leq m_{n,i+1}$. For $U(n)$ the m_{ni} are positive integers. For $O(n)$ they are positive integers or half-integers with the exception of m_{21}, m_{42}, and m_{63}, which may be negative; for these the branching rule becomes $|m_{2k,k}| \leq m_{2k+1,k}$.

and Nagel[6] have suggested that the additional operators be chosen as

$$\Omega = S_i E_{ij} T_j,$$
$$\Phi = S_i S_j E_{ik} E_{jk} + E_{ki} E_{kj} T_i T_j - \epsilon_{ijk} \epsilon_{lmn} S_i E_{jm} E_{kn} T_l, \tag{7}$$

where $i, j, \cdots$ stand for Cartesian components (rather than the spherical ones of Table I) and summation convention is implied by repeated indices. Because of the algebraic difficulties involved in the eigenvalue problem associated with operators such as Ω and Φ, it has not been possible to derive expressions for the Wigner coefficients of the supermultiplet scheme in a completely general way. In actual practice, however, most of the Wigner supermultiplets of importance for shell-model studies fall into a few special classes for which the spin and isospin quantum numbers are sufficient for a complete classification. The present work will be restricted to the study of such irreducible representations. For these the needed $SU(4)$ Wigner and Racah coefficients can be calculated.

2.4. Construction of State Vectors; Step-Up and Step-Down Operators

In the most general irreducible representation of $SU(4)$, the state vectors (or many-particle spin–isospin wavefunctions), can be denoted by

$$|[f]\omega\varphi, SM_S TM_T\rangle, \tag{8}$$

where ω and φ are the eigenvalues of the operators Ω and Φ. In the following sections the discussion will be restricted to those irreducible representations for which all state vectors are uniquely determined by the quantum numbers $SM_S TM_T$. For these the quantum numbers ω and φ are redundant, and the state vectors can be denoted by

$$|[f]SM_S TM_T\rangle. \tag{9}$$

For these representations the full set of state vectors can be constructed by a successive application of step-down operators acting on the state of highest

[11] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Application* (The Macmillan Company, New York, 1963), p. 353.
[12] G. Racah, "Group Theory and Spectroscopy," lecture notes, Princeton (1951); CERN reprint 61-8 (1961).

TABLE II. The step operators $O_{\alpha\beta}$.

$$O_{11} = E_{11}$$
$$O_{01} = E_{01} + S_-E_{11}\frac{1}{S_0+1}$$
$$O_{10} = E_{10} + T_-E_{11}\frac{1}{T_0+1}$$
$$O_{00} = E_{00} + S_-E_{10}\frac{1}{S_0+1} + T_-E_{01}\frac{1}{T_0+1} + S_-T_-E_{11}\frac{1}{(S_0+1)(T_0+1)}$$
$$O_{1-1} = E_{1-1} - T_-E_{10}\frac{1}{T_0} - T_-^2E_{11}\frac{1}{T_0(2T_0+1)}$$
$$O_{-11} = E_{-11} - S_-E_{01}\frac{1}{S_0} - S_-^2E_{11}\frac{1}{S_0(2S_0+1)}$$
$$O_{0-1} = E_{0-1} + S_-E_{1-1}\frac{1}{(S_0+1)} - T_-E_{00}\frac{1}{T_0} - T_-^2E_{01}\frac{1}{T_0(2T_0+1)} - S_-T_-E_{10}\frac{1}{T_0(S_0+1)} - T_-^2S_-E_{11}\frac{1}{(S_0+1)T_0(2T_0+1)}$$
$$O_{-10} = E_{-10} + T_-E_{-11}\frac{1}{(T_0+1)} - S_-E_{00}\frac{1}{S_0} - S_-^2E_{10}\frac{1}{S_0(2S_0+1)} - S_-T_-E_{01}\frac{1}{S_0(T_0+1)} - T_-S_-^2E_{11}\frac{1}{(T_0+1)S_0(2S_0+1)}$$
$$O_{-1-1} = E_{-1-1} - T_-E_{-10}\frac{1}{T_0} - S_-E_{0-1}\frac{1}{S_0} - T_-^2E_{-11}\frac{1}{T_0(2T_0+1)} - S_-^2E_{1-1}\frac{1}{S_0(2S_0+1)} + S_-T_-E_{00}\frac{1}{S_0T_0} + S_-^2T_-E_{10} \times \frac{1}{S_0(2S_0+1)T_0} + S_-T_-^2E_{01}\frac{1}{S_0T_0(2T_0+1)} + S_-^2T_-^2E_{11} \times \frac{1}{S_0T_0(2S_0+1)(2T_0+1)}$$

weight. Since the properties of angular-momentum eigenvectors are well known, it will be sufficient to construct the state vectors with $M_S = S$, $M_T = T$. It will sometimes be convenient to use the short-hand notation (employing curly brackets):

$$|[f]\{ST\}\rangle \equiv |[f]SM_S = S,\ TM_T = T\rangle. \quad (10)$$

A step operator $O_{\alpha\beta}$ is defined by

$$O_{\alpha\beta}\,|[f]\{ST\}\rangle = N_{\alpha\beta}^{[f]}(S,T)\,|[f]\{S+\alpha, T+\beta\}\rangle, \quad (11)$$

where

$$N_{\alpha\beta}^{[f]}(S,T) = [\langle[f]\{ST\}|\,O_{\alpha\beta}^{\dagger}O_{\alpha\beta}\,|[f]\{ST\}\rangle]^{\frac{1}{2}} = [\langle[f]\{ST\}|\,O_{-\alpha-\beta}O_{\alpha\beta}\,|[f]\{ST\}\rangle]^{\frac{1}{2}} \quad (12)$$

and

$$N_{\alpha\beta}^{[f]}(S,T) = N_{-\alpha-\beta}^{[f]}(S+\alpha, T+\beta). \quad (13)$$

The choice of the positive sign for the normalization factors, Eq. (12), specifies the phase convention used in this investigation. The step operators $O_{\alpha\beta}$ must satisfy the commutation relations

$$[S_+, O_{\alpha\beta}]\,|[f]\{ST\}\rangle = 0,$$
$$[T_+, O_{\alpha\beta}]\,|[f]\{ST\}\rangle = 0. \quad (14)$$

These equations are sufficient to construct the operators. There are 9 basic step operators corresponding to the $9E$ generators. The operators are shown in Table II. [The operators $O_{\alpha\beta}$ are not unique, but the lack of uniqueness can involve only trivial additional factors or combinations. For example, if $O_{\alpha\beta}$ satisfies Eqs. (14), so does $O_{\alpha\beta} + O'T_+$, where O' is any arbitrary operator. The lack of uniqueness in O_{10} can be illustrated by noting that $O_{1-1}O_{01} = O_{10}' \neq O_{10}$.] It will be convenient to express the nine E generators in terms of the O operators. The results are shown in Table III.

2.5. Irreducible Tensor Operators; Tensor Character of the Generators

The 15 infinitesimal operators $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{E}_{ab}$ transform according to the regular representation [211] and have spin–isospin spherical tensor character $ST = 10, 01$, and 11, respectively.

The components of an $SU(4)$ irreducible tensor operator $T_{(SM_S)(TM_T)}^{[f]}$ can be defined by the commutator equations

$$[E_{ab}, T_{(SM_S)(TM_T)}^{[f]}] = \sum_{S'T'} \langle[f](S'M_S+a)(T'M_T+b)| \times E_{ab}\,|[f](SM_S)(TM_T)\rangle\, T_{(S'M_S+a)(T'M_T+b)}^{[f]} \quad (15)$$

TABLE III. Expressions for the E operators in terms of the O operators.

$$
\begin{aligned}
E_{11} &= O_{11}\\
E_{01} &= O_{01} - S_-O_{11}\frac{1}{(S_0+1)}\\
E_{10} &= O_{10} - T_-O_{11}\frac{1}{(T_0+1)}\\
E_{00} &= O_{00} - S_-O_{10}\frac{1}{(S_0+1)} - T_-O_{01}\frac{1}{(T_0+1)} + S_-T_-O_{11}\frac{1}{(S_0+1)(T_0+1)}\\
E_{1-1} &= O_{1-1} + T_-O_{10}\frac{1}{T_0} - T_-^2O_{11}\frac{1}{(T_0+1)(2T_0+1)}\\
E_{-11} &= O_{-11} + S_-O_{01}\frac{1}{S_0} - S_-^2O_{11}\frac{1}{(S_0+1)(2S_0+1)}\\
E_{0-1} &= O_{0-1} + T_-O_{00}\frac{1}{T_0} - S_-O_{1-1}\frac{1}{(S_0+1)} - T_-S_-O_{10}\frac{1}{T_0(S_0+1)}\\
&\quad - T_-^2O_{01}\frac{1}{(2T_0+1)(T_0+1)} + S_-T_-^2O_{11}\frac{1}{(S_0+1)(T_0+1)(2T_0+1)}\\
E_{-10} &= O_{-10} + S_-O_{00}\frac{1}{S_0} - T_-O_{-11}\frac{1}{(T_0+1)} - T_-S_-O_{01}\frac{1}{S_0(T_0+1)}\\
&\quad - S_-^2O_{10}\frac{1}{(2S_0+1)(S_0+1)} + T_-S_-^2O_{11}\frac{1}{(T_0+1)(S_0+1)(2S_0+1)}\\
E_{-1-1} &= O_{-1-1} + S_-O_{0-1}\frac{1}{S_0} + T_-O_{-10}\frac{1}{T_0} + T_-S_-O_{00}\frac{1}{S_0T_0}\\
&\quad - S_-^2O_{1-1}\frac{1}{(S_0+1)(2S_0+1)} + T_-^2O_{-11}\frac{1}{(T_0+1)(2T_0+1)}\\
&\quad - T_-^2S_-O_{01}\frac{1}{S_0(2T_0+1)(T_0+1)} - T_-S_-^2O_{10}\frac{1}{T_0(2S_0+1)(S_0+1)}\\
&\quad + T_-^2S_-^2O_{11}\frac{1}{(T_0+1)(S_0+1)(2S_0+1)(2T_0+1)}
\end{aligned}
$$

with the analogous well-known commutator equations involving $\mathbf{S}$ and $\mathbf{T}$; e.g.,

$$[S_a, T^{[f]}_{(SM_S)(TM_T)}] = \langle SM_S + a|\, S_a\, |SM_S\rangle\, T^{[f]}_{(SM_S+a)(TM_T)}. \quad (16)$$

The operators $\mathbf{S}$, $\mathbf{T}$, and E_{ab} themselves have irreducible tensor character $T^{[211]}$. It is important to determine the phases implied by the standard defining equations (15). The full set of states of the representation [211] can be constructed by the step-down operators O_{0-1} and O_{-10} acting on the highest-weight state with $\{ST\} = \{11\}$. The normalization coefficients (13) are, in this case,

$$N^{[211]}_{0-1}(1,1) = N^{[211]}_{-10}(1,1) = +1. \quad (17)$$

With these and the relations of Table III, the matrix elements of E_{ab} can be evaluated. The relation between the components of $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{E}$ and the standard components of the irreducible tensor operator $T^{[211]}$ then follow from Eqs. (15) and (16). The results are shown in the last column of Table I. The over-all phase is fixed so that the components of $\mathbf{S}$ and $\mathbf{T}$ have phases according to the standard conventions for spherical tensors. [Note the minus signs for the $SU(4)$ tensor components with $(SM_S)(TM_T) = (11)(00)$ and $(00)(11)$ in Table I.]

2.6. Conjugate Representations

If a many-nucleon spin–isospin wavefunction transforms according to the $SU(4)$ irreducible representation $[f]$, the conjugate of such a function transforms according to the conjugate representation, to be denoted by $[f^*]$, where for

$$
\begin{aligned}
[f] &= [f_1 - f_4, f_2 - f_4, f_3 - f_4],\\
[f^*] &= [f_1 - f_4, f_1 - f_3, f_1 - f_2], \quad (18a)
\end{aligned}
$$

or, in terms of the supermultiplet quantum numbers P, P', P'',

$$(P, P', P'')^* \equiv (P, P', -P''). \quad (18b)$$

The conjugate representations $[f^*]$ can be pictured in terms of spin–isospin functions for $n = 3f_1 - f_2 - f_3 - f_4$ nucleons which have been lifted out of a configuration whose Young tableaux are given by $(f_1 - f_4)$ columns of 4—the well-known particle–hole relationship. Note also that the single-nucleon creation operators $a^\dagger$ have tensor character [1], while the annihilation operators a have tensor character [111]. The irreducible representations of $SU(2)$ are self-conjugate. For spherical tensors, conjugation implies only $M_S \to -M_S$, $M_T \to -M_T$. The conjugate of a state vector is thus given by

$$|[f]SM_S, TM_T\rangle^* = (-1)^\eta (-1)^{S-M_S+T-M_T} |[f^*]S - M_S, T - M_T\rangle. \quad (19)$$

The phase factor has been split into two pieces: one gives the dependence on the spin–isospin quantum numbers standard for spherical tensor operators; the second, denoted by η, is independent of M_S, M_T, but is a function of S, T, and $[f]$: $\eta = \eta([f], S, T)$. The phase factors for the representations of interest in this investigation are evaluated in Sec. 3. The dependence on $[f]$ is a matter of arbitrary phase conventions. The choice of phase conventions adopted in this work (Sec. 3) is such that the irreducible tensor character of the single-nucleon creation and annihilation operators is given by

$$a^\dagger_{\alpha m_s m_t} \to T^{[1]}_{(\frac{1}{2}m_s)(\frac{1}{2}m_t)},$$
$$a_{\alpha m_s m_t} \to (-1)^{\frac{1}{2}-m_s+\frac{1}{2}-m_t} T^{[111]}_{(\frac{1}{2}-m_s)(\frac{1}{2}-m_t)}. \quad (20)$$

In addition, the operators **S** and **T** are to have the conjugation properties standard for ordinary spherical tensor operators. This implies

$$(-1)^{\eta([211]1,0)} = (-1)^{\eta([211]0,1)} = +1. \quad (21)$$

2.7. Casimir Invariant

The quadratic Casimir operator is of particular importance. It can be expressed as

$$\mathbf{C} = \sum_{i<j=1}^{6} J_{ij}^2 = \sum_{ab} E_{ab}E_{ba} + \mathbf{S}^2 + \mathbf{T}^2. \quad (22a)$$

This can be brought into the form

$$\mathbf{C} = 2(E_{-1-1}E_{11} + E_{-11}E_{1-1} + E_{-10}E_{10} + E_{0-1}E_{01} + S_-S_+ + T_-T_+) + S_0^2 + 4S_0 + T_0^2 + 2T_0 + E_{00}^2, \quad (22b)$$

from which the eigenvalue of the Casimir operator can be read off by acting on the state of highest weight with $S_0 = P$, $T_0 = P'$, $E_{00} = P''$. This gives the Casimir invariant

$$C(SU_4) = P(P+4) + P'(P'+2) + P''^2. \quad (23)$$

2.8. Kronecker Products

The Kronecker product of two irreducible representations of $U(4)$ is given by the Littlewood rules for outer multiplication of $[f]$-symmetric states.[13]

The one-and two-particle coefficients of fractional parentage are simply related to the matrix elements of $a^\dagger$ (or a) and $a^\dagger a^\dagger$ (or aa), respectively. These coefficients are therefore related to the coupling coefficients for the products $[f_1] \times [f_2]$, where $[f_1]$ is the representation for an arbitrary n-nucleon spin–isospin symmetry, and $[f_2]$ is a one-particle (or hole), or two-particle (or two-hole) representation; that is, $[f_2] = [1]$ (or $[1^3]$); or the antisymmetrically coupled two-particle representation $[1^2]$ (which is self-conjugate), or the symmetrically coupled two-particle representation $[2]$ (or its conjugate $[2^3]$). All such products are simply reducible. In addition to these, the Kronecker products of particular interest for nuclear physics applications are those needed for the evaluation of matrix elements of the one-body operators $(a^\dagger a)$ and the two-body operators $(a^\dagger a^\dagger aa)$. From the reduction of the Kronecker product

$$[1] \times [1^3] = [0] + [211], \quad (24)$$

it can be seen that the one-body operators are either $SU(4)$ scalars or $[211]$ tensors. Similarly, from the reduction of the products

$$[11] \times [11] = [0] + [211] + [22],$$
$$[2] \times [2^3] = [0] + [211] + [422], \quad (25)$$

it can be seen that the two-body operators transform according to the representations $[0]$, $[211]$, $[22]$, and $[422]$. [Products involving symmetrically coupled pair-creation operators with antisymmetrically coupled pair-annihilation operators (or vice versa) have not been included, since they would arise only in the case of the relatively uncommon two-body operators which are antisymmetric in *both* the space *and* spin–isospin variables.] The Kronecker products $[f_1] \times [f_2]$ with $[f_2] = [211]$, $[22]$, or $[422]$ are not simply reducible. If $[f_1]$ is the most general irreducible representation of $SU(4)$, these products will contain the irreducible representation $[f_1]$ itself with multiplicities as high as 3, 2, or 6, respectively. For the special representations to be considered in this investigation, however, these multiplicities are never greater than 2; and in this case the nature of the operators themselves furnishes a canonical method of resolving the multiplicity problem.

3. THE SPECIAL $SU(4)$ REPRESENTATIONS; CONSTRUCTION OF STATE VECTORS

Most of the Wigner supermultiplets of actual importance in shell-model studies fall into a few special classes for which the spin and isospin quantum numbers are sufficient for a complete classification of the states of a given irreducible representation. The reduction of the irreducible representations of $SU(4)$ into the representations of $SU(2) \times SU(2)$ has been

[13] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley Publ. Co., Reading, Mass., 1962), Sec. 7-12.

TABLE IV. Branching formula for $[yy0] \rightarrow (S, T)$.

(S, T)						
$(y, 0)$						
$(y-1, 1)$						
$(y-2, 2)$	$(y-2, 0)$					
$(y-3, 3)$	$(y-3, 1)$					
$(y-4, 4)$	$(y-4, 2)$	$(y-4, 0)$				
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$(y-2i, 0)$		
$\cdot$	$\cdot$	$\cdot$				
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$		
$\cdot$	$\cdot$	$\cdot$		$\cdot$		
$(3, y-3)$	$(3, y-5)$	$(3, y-7)$	$\cdots$	$\cdot$	$\cdots$	
$(2, y-2)$	$(2, y-4)$	$(2, y-6)$	$\cdots$	$(2, y-2i-2)$	$\cdots$	
$(1, y-1)$	$(1, y-3)$	$(1, y-5)$	$\cdots$	$(1, y-2i-1)$	$\cdots$	
$(0, y)$	$(0, y-2)$	$(0, y-4)$	$\cdots$	$(0, y-2i)$	$\cdots$	$(0, 0)$[a]

[a] The last column has the entry (0, 0) for y = even integer; or (1, 0) and (0, 1) for y = odd integer.

TABLE V. Branching formula for $[y\,y-1\,0]$ or $[yy1] \rightarrow (S, T)$.

(S, T)						
$(y-\frac{1}{2}, \frac{1}{2})$						
$(y-\frac{3}{2}, \frac{3}{2})$	$(y-\frac{3}{2}, \frac{1}{2})$					
$(y-\frac{5}{2}, \frac{5}{2})$	$(y-\frac{5}{2}, \frac{3}{2})$	$(y-\frac{5}{2}, \frac{1}{2})$				
$(y-\frac{7}{2}, \frac{7}{2})$	$(y-\frac{7}{2}, \frac{5}{2})$	$(y-\frac{7}{2}, \frac{3}{2})$	$(y-\frac{7}{2}, \frac{1}{2})$			
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot\cdot$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$		
$(\frac{3}{2}, y-\frac{3}{2})$	$(\frac{3}{2}, y-\frac{5}{2})$	$(\frac{3}{2}, y-\frac{7}{2})$	$(\frac{3}{2}, y-\frac{9}{2})$	$\cdots$	$(\frac{3}{2}, \frac{1}{2})$	
$(\frac{1}{2}, y-\frac{1}{2})$	$(\frac{1}{2}, y-\frac{3}{2})$	$(\frac{1}{2}, y-\frac{5}{2})$	$(\frac{1}{2}, y-\frac{7}{2})$	$\cdots$	$(\frac{1}{2}, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$

discussed in general algebraic form by Racah.[14] Racah's technique leads to the branching law giving the set of possible ST values in a given irreducible representation $[f]$, together with their multiplicities. In particular, it can be seen that these multiplicities are never greater than one in the following classes of $SU(4)$ representations:

$$\begin{array}{lllll} [yy0] & [y\,y-1\,0] & [y00] & [y10] & [y11] \\ & [yy1] & [yyy] & [yy\,y-1] & [y\,y-1\,y-1], \end{array} \tag{26}$$

where y = arbitrary integer (including zero, when possible), and where conjugate pairs of representations have been arranged in the same columns. Many-nucleon wavefunctions made up entirely of pairs coupled to an orbital angular momentum of zero (seniority zero functions) have spin–isospin wavefunctions which transform according to the self-conjugate representations $[yy0]$. States with seniority 1 have spin–isospin wavefunctions which transform according to the representations $[y\,y-1\,0]$ or $[yy1]$. These representations are therefore of special interest in the study of spin-charge independent pairing interactions.[15] The irreducible representations $[y00]$ imply totally symmetric spin–isospin wavefunctions, hence totally antisymmetric spatial wavefunctions. Such functions are therefore of interest in nuclear problems only for small values of the integer y. However, they are relatively simple and are therefore included in the present investigation.

The set of possible ST values for the irreducible representations $[yy0]$ is listed in Table IV. They are arranged in columns for which $y - S - T = 0, 2, 4, \cdots$, even integer only. The set of possible ST values for $[y\,y-1\,0]$ or $[yy1]$ is listed in Table V, where they are arranged in columns for which

$$y - S - T = 0, 1, 2, 3, \cdots;$$

that is, $y - S - T$ can be alternately even or odd. In the irreducible representation $[y00]$ or its conjugate $[yyy]$, the possible ST values are restricted to those with $T = S$, where $S = \frac{1}{2}y,\ \frac{1}{2}y - 1,\ \frac{1}{2}y - 2, \cdots$, ending in $S = 0$ (or $\frac{1}{2}$) for y = even (or odd) integer. In the representation $[y10]$ or its conjugate $[yy\,y-1]$, the possible ST values are restricted to the sets with $T = S$ or $T = S \pm 1$, starting with

$$\{ST\} = \{\tfrac{1}{2}(y+1)\,\tfrac{1}{2}(y-1)\}, \{\tfrac{1}{2}(y-1)\,\tfrac{1}{2}(y+1)\}, \{\tfrac{1}{2}(y-1)\,\tfrac{1}{2}(y-1)\}, \cdots,$$

and ending with $\{11\}, \{10\}, \{01\}$, or $\{\frac{3}{2}\,\frac{1}{2}\}, \{\frac{1}{2}\,\frac{3}{2}\}, \{\frac{1}{2}\,\frac{1}{2}\}$, for y = odd or even integer, respectively. In the representation $[y11]$ or its conjugate $[y\,y-1\,y-1]$,

[14] G. Racah, Rev. Mod. Phys. **21**, 494 (1949).

[15] S. C. Pang, Nucl. Phys. **A128**, 497 (1969).

TABLE VI. Construction of state vectors. The overall normalization coefficients, denoted by $\mathcal{N}$, are given by the appropriate products of the normalization coefficients $N^{[f]}_{\alpha\beta}(S, T)$ given in Table VII.

$$
\begin{aligned}
&|[yy0]\{ST\}\rangle = \mathcal{N} O^{q}_{-1-1}\, O^{p}_{-11}\, |[yy0]\{y0\}\rangle; && p = \tfrac{1}{2}(y - S + T),\ q = \tfrac{1}{2}(y - S - T) \\
&|[y\,y-1\,0]\{ST\}\rangle = \mathcal{N} O^{q}_{-11}\, O^{p}_{-10}\, |[y\,y-1\,0]\{y-\tfrac{1}{2}, \tfrac{1}{2}\}\rangle; && p = (y - S - T),\ q = T - \tfrac{1}{2} \\
&|[y00]\{ST\}\rangle = \mathcal{N} O^{p}_{-1-1}\, |[y00]\{\tfrac{1}{2}y, \tfrac{1}{2}y\}\rangle; && p = \tfrac{1}{2}y - S \\
&|[y10]\{ST\}\rangle = \mathcal{N} O^{r}_{-10}\, O^{q}_{-11}\, O^{p}_{-1-1}\, |[y10]\{\tfrac{1}{2}(y+1), \tfrac{1}{2}(y-1)\}\rangle \\
&\qquad \text{with } p = \tfrac{1}{2}(y+1) - S,\ q = 0,\ r = 0, \quad \text{for } ST = S, S-1, \\
&\qquad\qquad p = \tfrac{1}{2}(y+1) - S,\ q = 0,\ r = 1, \quad \text{for } ST = S-1, S-1, \\
&\qquad\qquad p = \tfrac{1}{2}(y+1) - S,\ q = 1,\ r = 0, \quad \text{for } ST = S-1, S \\
&|[y11]\{ST\}\rangle = \mathcal{N} O^{r}_{0-1}\, O^{q}_{-10}\, O^{p}_{-1-1}\, |[y11]\{\tfrac{1}{2}y, \tfrac{1}{2}y\}\rangle \\
&\qquad \text{with } p = \tfrac{1}{2}y - S,\ q = 0,\ r = 0, \quad \text{for } ST = SS, \\
&\qquad\qquad p = \tfrac{1}{2}y - S,\ q = 0,\ r = 1, \quad \text{for } ST = S, S-1, \\
&\qquad\qquad p = \tfrac{1}{2}y - S,\ q = 1,\ r = 0, \quad \text{for } ST = S-1, S
\end{aligned}
$$

the possible ST values are also restricted by the conditions $T = S$ or $T = S \pm 1$. Now the $\{ST\}$ values start with $\{\tfrac{1}{2}y\,\tfrac{1}{2}y\}$, $\{\tfrac{1}{2}y\,\tfrac{1}{2}y - 1\}$, $\{\tfrac{1}{2}y - 1\,\tfrac{1}{2}y\}$, $\cdots$, but end with $\{11\}$, $\{10\}$, $\{01\}$, or $\{\tfrac{3}{2}\,\tfrac{1}{2}\}$, $\{\tfrac{1}{2}\,\tfrac{3}{2}\}$, $\{\tfrac{1}{2}\}$ for $y =$ even or odd integer, respectively.

For the irreducible representations of the above types, the full set of state vectors can be constructed by a successive application of step-down operators $O_{\alpha\beta}$, defined by Eqs. (11)–(13), beginning with the operator acting on the highest-weight state: $\{ST\} = \{PP'\}$. The details of the construction for the five special classes of $SU(4)$ representations (26) are shown in Table VI. The basic numbers in these constructions are the normalization coefficients $N^{[f]}_{\alpha\beta}(S, T)$. Once these are determined, the matrix elements of the generators E_{ab} can be calculated with the aid of the relations of Tables II and III. The matrix elements of E_{ab} in turn can be taken to form the starting point for the calculation of the $SU(4)$ Wigner coefficients. The normalization coefficients needed for the construction of the state vectors for the representations (26) are given in Table VII. The details of the technique used in their calculation are illustrated by two examples in Appendix A.

State vectors $|\{ST\}\rangle$ for the representations conjugate to those included in Tables VI and VII must be constructed by exactly the same sequence of step operators. The phase factors needed to relate a state vector to its conjugate are thus determined by the integers p, q, r defined in Table VI. It is convenient to define the conjugation operator K:

$$|[f]SM_S, TM_T\rangle^* \equiv K\,|[f]SM_S, TM_T\rangle, \tag{27}$$

where $KcK^{-1} = c^*$ ($c =$ complex number). When applied to the infinitesimal operators, the conjugation operator has the transformation properties

$$KJ_{ij}K^{-1} = -J_{ij}, \quad KE_{ab}K^{-1} = -E_{-a-b},$$
$$KS_aK^{-1} = -S_{-a}, \quad KT_aK^{-1} = -T_{-a}.$$

From these properties and the relations of Tables II and III, it follows that

$$
\begin{aligned}
KO_{\alpha\beta}\,&|[f]\{ST\}\rangle \\
&= (-1)^{1+\eta([f],S,T)} \frac{2^{S+\alpha+T+\beta}}{(2(S+\alpha))!\,(2(T+\beta))!} \\
&\quad \times S_-^{2(S+\alpha)} T_-^{2(T+\beta)} O_{\alpha\beta}\,|[f^*]\{ST\}\rangle \\
&= (-1)^{1+\eta([f],S,T)} N^{[f]}_{\alpha\beta}(S, T)\,|[f^*] \\
&\quad \times (S+\alpha, M_S = -S-\alpha) \\
&\quad \times (T+\beta, M_T = -T-\beta)\rangle,
\end{aligned} \tag{28}
$$

where the conjugation phase factor $\eta([f], S, T)$ is defined by Eq. (19). From Eq. (28) it can be seen that

$$\eta([f], S, T) = \eta([f], S = P, T = P') + p + q + r, \tag{29}$$

where the phase factor for the highest-weight state $\{S = P,\ T = P'\}$ can be chosen quite arbitrarily. In this investigation we have made our choice such that the single-nucleon creation and annihilation operators as well as the operators **S** and **T** have conjugation properties standard for ordinary spherical tensor operators, Eqs. (20) and (21). These requirements are satisfied by setting $\eta([f], P, P') = 0$ for all representations except $[y11]$. In this case it is convenient to set $\eta([y11], \tfrac{1}{2}y, \tfrac{1}{2}y) = -(y + 1)$. Results giving the full (y, S, T)-dependence of the phase factors are collected in Table VIII.

It should be pointed out that, besides the general y dependence, there is an additional arbitrariness in the phase factor η in those irreducible representations in which $y - S - T$ can take on both even and odd values. In the representation $[y\,y-1\,0]$, for example, states with $y - S - T =$ odd integer are constructed from neighboring states with $y - S - T =$ even by means of the step operator O_{-10}, in the prescription of Table VI. According to Eq. (28), the single-

TABLE VII. The normalization factors.

$SU(4)$ representation	$N_{\alpha\beta}(S, T)$ Algebraic factor	
$[yy0]$	$N_{-1-1}(S, T) = \left[\frac{ST(y+2-S-T)(y+2+S+T)}{(2S+1)(2T+1)}\right]^{\frac{1}{2}}$	
	$N_{-11}(S, T) = \left[\frac{S(T+1)(y+3-S+T)(y+1+S-T)}{(2S+1)(2T+3)}\right]^{\frac{1}{2}}$	
$[y\,y-1\;0]$	$N_{-10}(S, T) = \left[\frac{(2S-1)(y+2-S+T)(y+2+S+T)}{16S(T+1)^2}\right]^{\frac{1}{2}}$	for $(-1)^{y-S-T} = +1$
	$= \left[\frac{(2S-1)(y+1-S-T)(y+1+S-T)}{16S(T+1)^2}\right]^{\frac{1}{2}}$	for $(-1)^{y-S-T} = -1$
	$N_{-11}(S, T) = \left[\frac{(2S-1)(2T+1)(y+S-T)(y+2-S+T)}{16S(T+1)}\right]^{\frac{1}{2}}$	for $(-1)^{y-S-T} = +1$
	$= \left[\frac{(2S-1)(2T+1)(y+1+S-T)(y+3-S+T)}{16S(T+1)}\right]^{\frac{1}{2}}$	for $(-1)^{y-S-T} = -1$
$[y00]$	$N_{-1-1}(S, S) = \left[\frac{(2S-1)(y+2-2S)(y+2+2S)}{4(2S+1)}\right]^{\frac{1}{2}}$	
$[y10]$	$N_{-1-1}(S, S-1) = \left[\frac{(2S-3)(y+3-2S)(y+1+2S)}{4(2S-1)}\right]^{\frac{1}{2}}$	
	$N_{-11}(S, S-1) = \frac{\{y+1+4S^2\}}{2S(S+1)}$	
	$N_{-10}(S, S-1) = \frac{1}{2S}\left[\frac{(S-1)(y+1+2S)(y+1)}{S}\right]^{\frac{1}{2}}$	
$[y11]$	$N_{-1-1}(S, S) = \left[\frac{(S-1)(S+1)(2S-1)(y+2-2S)(y+2+2S)}{4S^2(2S+1)}\right]^{\frac{1}{2}}$	
	$N_{-10}(S, S) = N_{0-1}(S, S) = \left[\frac{(2S-1)(y+2+2S)(y+2)}{4S(S+1)(2S+1)}\right]^{\frac{1}{2}}$	

step operation O_{-10} implies a change in the phase $(-1)^{\eta}$. It would, of course, have been possible to construct the states with $y - S - T =$ odd from the even neighbors by means of the operator $O_{0-1}O_{-11}$ instead. This double-step operation would have implied no change in the phase $(-1)^{\eta}$. However, this arbitrariness in the phase factor η is no more bothersome than the arbitrariness of its y dependence. States with $y - S - T =$ even or odd fall into two distinct families. It will be seen that the algebraic structure of the $SU(4)$ Wigner coefficients is different for the two types of states and will depend on the parity of $y - S - T$. Since states with $y - S - T =$ even or odd must be treated separately, it is not surprising that their relative phase behavior under conjugation may be arbitrary. In this investigation, the choice of phase factors is that implied by the constructions of Table VI; the resultant phase factors to be used are those shown in Table VIII.

The irreducible tensor operators of greatest interest in the applications to nuclear problems transform according to the representations [1], $[1^3]$, [2], $[2^3]$, $[1^2]$, [211], [22], and [422] (Sec. 2.8). All but [422] are special cases of one of the representations enumerated in (26), so that their components are completely labeled by the spin–isospin quantum numbers. The reduction of the 84-dimensional representation [422] into representations of $SU(2) \times SU(2)$ leads to the following set of possible $\{ST\}$ values:

$$\begin{array}{lll} \{22\} & \{21\} & \{20\} \\ \{12\} & \{11\}^2 & \\ \{02\} & & \{00\}. \end{array}$$

TABLE VIII. Conjugation phase factors.

$[f]$	$(-1)^{\eta}$
$[yy0]$	$(-1)^{y-S} = (-1)^{T}$
$[y\,y-1\;0]$	$(-1)^{y-\frac{1}{2}-S}$
$[y00]$	$(-1)^{\frac{1}{2}y-S}$
$[y10]$	$(-1)^{\frac{1}{2}(y+1)-S}$
$[y11]$	$(-1)^{\frac{1}{2}y+1+\min(S,T)}$
[422]	$(-1)^{S+T+\mu(1,1)}$ [a]

[a] $\mu(1, 1) = \mu\delta_{S1}\delta_{T1}$, where $\mu = 0$ for the state $|\{11\}s\rangle$, $\mu = 1$ for the state $|\{11\}a\rangle$.

TABLE IX. Normalized state vectors for [422]. The state vectors $|[422]\, SM_S = S,\ TM_T = T\rangle$ are abbreviated by $|\{ST\}\rangle$.

$|\{21\}\rangle = (1/\sqrt{2})O_{0-1}|\{22\}\rangle$;	$|\{12\}\rangle = (1/\sqrt{2})O_{-10}|\{22\}\rangle$
$|\{20\}\rangle = (\sqrt{3}/4)(O_{0-1})^2|\{22\}\rangle$;	$|\{02\}\rangle = (\sqrt{3}/4)(O_{-10})^2|\{22\}\rangle$
$|\{11\}a\rangle = (1/\sqrt{2})O_{-1-1}|\{22\}\rangle$;	$|\{11\}s\rangle = (1/\sqrt{3})O_{0-1}O_{-10}|\{22\}\rangle$
$|\{00\}\rangle = (3/2\sqrt{10})(O_{-1-1})^2|\{22\}\rangle$	

There are two independent states with $\{ST\} = \{11\}$. The most natural way of constructing the full set of states $|\{ST\}\rangle$ by means of step-down operators acting on the highest-weight state $|\{22\}\rangle$ is shown in Table IX. States $|\{ST\}\rangle$ of this self-conjugate representation constructed by means of an even (or odd) number of step-down operations are symmetric (or antisymmetric), respectively, under conjugation. This gives a natural way of distinguishing the two independent states with $\{ST\} = \{11\}$. The state constructed by means of the single-step operator O_{-1-1} is antisymmetric, while the state constructed by means of the double-step operator $O_{0-1}O_{-10}$ is symmetric under conjugation. These two states, denoted by $|\{11\}a\rangle$ and $|\{11\}s\rangle$, respectively, are automatically orthogonal to each other. The symmetry label under conjugation thus forms a natural choice for the needed additional quantum number. It is interesting to note that neither of these states is an eigenvector of the operators Ω and Φ of Eq. (7).

4. $SU(4)$ WIGNER AND RACAH COEFFICIENTS

4.1. Definitions: Orthonormality

The $SU(4)$ Wigner coefficients are the elements of the matrix which reduces the Kronecker product of two irreducible representations of $SU(4)$. They are defined by

$$\begin{aligned} |([f^{(1)}][f^{(2)}])[f]\rho;\, \omega\varphi,\, SM_S TM_T\rangle &= \sum_{\substack{\omega_1\varphi_1 S_1 M_{S_1} T_1 M_{T_1}\\ \omega_2\varphi_2 S_2 M_{S_2} T_2 M_{T_2}}} |[f^{(1)}]\omega_1\varphi_1,\, S_1M_{S_1},\, T_1M_{T_1}\rangle \\ &\times |[f^{(2)}]\omega_2\varphi_2,\, S_2M_{S_2},\, T_2M_{T_2}\rangle \\ &\times \langle [f^{(1)}]\omega_1\varphi_1 S_1M_{S_1}T_1M_{T_1};\\ &\times [f^{(2)}]\omega_2\varphi_2 S_2M_{S_2}T_2M_{T_2} \,|\, [f]\omega\varphi SM_S TM_T\rangle_\rho . \end{aligned} \tag{30}$$

That is, the full $SU(4)$ Wigner coefficient can be considered as the scalar product of a coupled function with a product of uncoupled functions, the latter specified by the 12 quantum numbers $\omega_i\varphi_i$, $S_iM_{S_i}$, $T_iM_{T_i}$ with $i = 1$ and 2. Since a state $[f]$ may occur more than once in the product $[f^{(1)}] \times [f^{(2)}]$, the coupled state is not fully specified by the six quantum numbers $\omega\varphi$, SM_S, TM_T and the three irreducible representation labels for $[f]$. An additional label ρ is needed to distinguish between the various possible states with the same $[f]\omega\varphi$, SM_S, TM_T. (In principle, the labels ρ should be given by the eigenvalues of three additional operators. Such operators must lie outside the group $SU(4)$.[16] In practice the labels ρ are chosen through a set of canonical operators of irreducible tensor character $[f^{(2)}]$, for which only a specific reduced matrix element has a nonzero value, where these are defined in Sec. 4.2.

The full $SU(4)$ Wigner coefficient can be factored into a reduced $SU(4) \supset [SU(2) \times SU(2)]$ coefficient (to be denoted by a double bar) and two ordinary $SU(2)$ or angular momentum coupling coefficients for the spin and isospin spaces which carry the dependence on M_S and M_T:

$$\begin{aligned} &\langle [f^{(1)}]\omega_1\varphi_1 S_1M_{S_1}T_1M_{T_1};\, [f^{(2)}]\omega_2\varphi_2 \\ &\quad\times S_2M_{S_2}T_2M_{T_2} \,|\, [f]\omega\varphi SM_S TM_T\rangle_\rho \\ &= \langle [f^{(1)}]\omega_1\varphi_1 S_1T_1;\, [f^{(2)}]\omega_2\varphi_2 S_2T_2 \,\|\, [f]\omega\varphi ST\rangle_\rho \\ &\quad\times \langle S_1M_{S_1}S_2M_{S_2} \,|\, SM_S\rangle\langle T_1M_{T_1}T_2M_{T_2} \,|\, TM_T\rangle . \end{aligned} \tag{31}$$

The reduced or double-barred coefficients can be identified with the spin–isospin factor[2,8] of the fractional parentage coefficients which describe the coupling of n_1 nucleons of spin–isospin symmetry $[f^{(1)}]$ with n_2 nucleons of spin–isospin symmetry $[f^{(2)}]$ to a state of n nucleons of spin–isospin symmetry $[f]$. [From now on the term $SU(4)$ Wigner coefficient will be used to refer to these reduced (or double-barred) coefficients.] The reduced coefficients satisfy the orthonormality relations:

for *fixed* S, T:

$$\begin{aligned} \sum_{\substack{\omega_1\varphi_1 S_1 T_1\\ \omega_2\varphi_2 S_2 T_2}} &\langle [f^{(1)}]\omega_1\varphi_1 S_1T_1;\, [f^{(2)}]\omega_2\varphi_2 S_2T_2 \,\|\, [f]\omega\varphi ST\rangle_\rho \\ &\times \langle [f^{(1)}]\omega_1\varphi_1 S_1T_1;\, [f^{(2)}]\omega_2\varphi_2 S_2T_2 \,\|\, [f']\omega'\varphi' ST\rangle_{\rho'} \\ &= \delta_{[f][f']}\delta_{\omega\omega'}\delta_{\varphi\varphi'}\delta_{\rho\rho'}; \end{aligned} \tag{32a}$$

[16] L. C. Biedenharn, A. Giovannini, and J. D. Louck, J. Math. Phys. **8**, 691 (1967).

and again for *fixed S, T*:

$$\sum_{[f]\rho\omega\varphi} \langle [f^{(1)}]\omega_1\varphi_1 S_1 T_1; [f^{(2)}]\omega_2\varphi_2 S_2 T_2 \parallel [f]\omega\varphi ST\rangle_\rho \times \langle [f^{(1)}]\omega_1'\varphi_1' S_1' T_1'; [f^{(2)}]\omega_2'\varphi_2' S_2' T_2' \parallel [f]\omega\varphi ST\rangle_\rho = \delta_{\omega_1\omega_1'}\delta_{\varphi_1\varphi_1'}\delta_{S_1S_1'}\delta_{T_1T_1'}\delta_{\omega_2\omega_2'}\delta_{\varphi_2\varphi_2'}\delta_{S_2S_2'}\delta_{T_2T_2'}. \tag{32b}$$

In the applications to be considered in this investigation, the representations $[f]$ are restricted to the special classes for which the quantum numbers ω and φ are not needed and will henceforth be omitted in the expressions for the $SU(4)$ Wigner coefficients. (Unless labels ω and φ are explicitly shown, it will be understood that the representation belongs to one of the special classes for which S and T are sufficient for a complete classification of states.)

4.2. Matrix Elements of Tensor Operators; Wigner–Eckart Theorem

The matrix elements of an $SU(4)$ irreducible tensor operator $T^{[f]}_{(SM_S)(TM_T)}$ can be expressed in terms of a generalized Wigner–Eckart theorem by a product of factors involving the appropriate Wigner coefficients and reduced matrix elements which are independent of the quantum numbers S, T, M_S, M_T, (ω, φ if needed):

$$\langle [f'']S''M_S''T''M_T''|\, T^{[f]}_{(SM_S)(TM_T)} \,|[f']S'M_S'T'M_T'\rangle = \sum_\rho \langle [f''] \| T^{[f]} \| [f']\rangle_\rho \times \langle [f']S'T'; [f]ST \parallel [f'']S''T''\rangle_\rho \times \langle S'M_S'SM_S \mid S''M_S''\rangle\langle T'M_T'TM_T \mid T''M_T''\rangle. \tag{33}$$

If the representation $[f'']$ occurs only once in the reduction of the product $[f'] \times [f]$, the labels ρ and summations over ρ are not needed. The Wigner–Eckart theorem then takes its usual form. If the product $[f'] \times [f]$ is not simply reducible, the Wigner–Eckart theorem, Eq. (33), can be used to define the labels by a choice of canonical operators whose reduced matrix elements have special values.

In the applications to nuclear problems (Sec. 5), the multiplicity problem arises only in connection with the Kronecker products $[f'] \times [211]$ and $[f'] \times [422]$. The labels ρ are needed only for $[f'] = [f'']$, $[f] = [211]$ or $[422]$ in Eq. (33). In these two cases there is a straightforward choice for the canonical operators used to define ρ. Since the infinitesimal operators $\mathbf{E}$ transform according to the representation $[211]$, the matrix elements of these operators can be used to define the label ρ for $[f] \times [211] \to [f]$. The appropriately normalized matrix elements of the components of $\mathbf{E}$ can be identified with the $SU(4)$ Wigner coefficients labeled with $\rho = 1$. Specifically,

$$\langle [f] \| \mathbf{E} \| [f]\rangle_\rho = 0 \quad \text{for} \quad \rho \neq 1, \tag{34a}$$

$$\langle [f] \| \mathbf{E} \| [f]\rangle_{\rho=1} = [C(SU_4)]^{\frac{1}{2}}, \tag{34b}$$

and

$$\langle [f]S''M_S''T''M_T''|\, E_{ab} \,|[f]S'M_S'T'M_T'\rangle = (-1)^{\chi(a,b)}[C(SU_4)]^{\frac{1}{2}}\langle [f]S'T'; [211]11 \parallel [f]S''T''\rangle_{\rho=1} \times \langle S'M_S'1a \mid S''M_S''\rangle\langle T'M_T'1b \mid T''M_T''\rangle, \tag{35}$$

where the phase factor $(-1)^{\chi(a,b)}$ is given by the standard phase of the particular component E_{ab}, as indicated in the last column of Table I. Coefficients with $\rho = 2$, $\langle [f]S'T'; [211]ST \parallel [f]S''T''\rangle_{\rho=2}$, are then fixed by the orthogonality (32a). [For the special $SU(4)$ representations enumerated in Sec. 3, the multiplicity is never greater than two.]

The operators E_{ab} are one-body operators, $a^\dagger a$, with irreducible tensor character [211] which are complete space scalars. This suggests that the two-body operators, $a^\dagger a^\dagger aa$, with irreducible tensor character [422] which are complete space scalars, can be used to define the labels ρ for the coefficients which reduce $[f] \times [422]$ into $[f]$. The $S = M_S = T = M_T = 2$ component of such an operator, for example, would have the specific form

$$\sum_{\alpha\beta} a^\dagger_{\alpha\frac{1}{2}\frac{1}{2}} a^\dagger_{\beta\frac{1}{2}\frac{1}{2}} a_{\beta-\frac{1}{2}-\frac{1}{2}} a_{\alpha-\frac{1}{2}-\frac{1}{2}}. \tag{36}$$

The appropriately normalized coefficients which give the ST dependence of the matrix elements of such operators are to be identified with the coefficients $\langle [f]S'T'; [422]ST \parallel [f]S''T''\rangle_{\rho=1}$; that is, those with $\rho = 1$. Coefficients with $\rho = 2$ can again be constructed by means of the orthogonality requirement.

4.3. Phase Convention

The over-all phase of the $SU(4)$ Wigner coefficients is arbitrary. It is fixed by a generalized Condon–Shortley phase convention. The $SU(4)$ Wigner coefficients can be chosen to be real, and the so-called leading coefficient connecting the highest-weight state $S_1T_1 = P_1P_1'$ of the representation $[f^{(1)}]$ with the highest-weight state $S_3T_3 = P_3P_3'$ of the representation $[f^{(3)}]$ is chosen to be positive. For most of the simple products $[f^{(1)}] \times [f^{(2)}] \to [f^{(3)}]$ this choice of S_1T_1, S_3T_3 uniquely determines S_2T_2. If it does not, the leading coefficient is specified by a further choice of S_2T_2 (the specific S_2T_2 values to be singled out will be denoted by a bar: $\bar{S}_2\bar{T}_2$), such that

$$\langle [f^{(1)}]P_1P_1'; [f^{(2)}]\bar{S}_2\bar{T}_2 \parallel [f^{(3)}]P_3P_3'\rangle > 0. \tag{37}$$

In the case of the coefficients in the reduction $[f] \times [211] \to [f]$, it is convenient to set $\bar{S}_2\bar{T}_2 = 10$, or equally well 01 (rather than 11), so that the matrix elements of S_0 and T_0 have their conventional positive values. In all other cases, $\bar{S}_2\bar{T}_2$ will be chosen as the largest possible values of S_2T_2 consistent with the restriction $S_1T_1 = P_1P_1'$, $S_3T_3 = P_3P_3'$.

4.4. Symmetry Properties of the Wigner Coefficients

The symmetry properties of the $SU(4)$ Wigner coefficients may, in the most general case, be complicated by the state-labeling problem and the multiplicity problem. For those Wigner coefficients for which neither the quantum numbers ω and φ nor the labels ρ are needed, the symmetry properties can be derived by standard techniques.[17] They follow from the conjugation properties of the state vectors, Eq. (19). Combined with the well-known symmetry properties for the ordinary spin–isospin angular momentum coupling coefficients, the symmetry properties for the reduced $SU(4)$ Wigner coefficients follow from those of the full $SU(4)$ Wigner coefficients.

When *neither ω, φ, nor ρ are needed*, they are

$$\text{(I)}\ \langle[f^{(1)*}]S_1T_1;[f^{(2)*}]S_2T_2\parallel[f^{(3)*}]S_3T_3\rangle = (-1)^{\eta^{(1)}+\eta^{(2)}-\eta^{(3)}}\langle[f^{(1)}]S_1T_1;[f^{(2)}]S_2T_2\parallel[f^{(3)}]S_3T_3\rangle, \tag{38}$$

where the conjugation phase factors η are enumerated in Table VIII.

$$\text{(II)}\ \langle[f^{(1)}]S_1T_1;[f^{(2)}]S_2T_2\parallel[f^{(3)}]S_3T_3\rangle = (-1)^{\sigma+\eta^{(2)}+S_1+S_2-S_3+T_1+T_2-T_3} \times\left[\frac{\dim[f^{(3)}](2S_1+1)(2T_1+1)}{\dim[f^{(1)}](2S_3+1)(2T_3+1)}\right]^{\frac{1}{2}} \times\langle[f^{(3)}]S_3T_3;[f^{(2)*}]S_2T_2\parallel[f^{(1)}]S_1T_1\rangle, \tag{39}$$

where the over-all phase in this relation is fixed by the convention (37), giving

$$\sigma = P_1 - P_3 + P_1' - P_3' + \eta([f^{(2)}]\bar{S}_2\bar{T}_2) + \bar{S}_2 + \bar{T}_2, \tag{40}$$

and dim $[f^{(i)}]$ stands for the dimension of the irreducible representation $[f_1^{(i)}f_2^{(i)}f_3^{(i)}]$:

$$\dim[f_1f_2f_3] = \tfrac{1}{12}(f_1-f_2+1)(f_1-f_3+2) \times(f_2-f_3+1)(f_1+3)(f_2+2)(f_3+1). \tag{41}$$

A special case of relation (II) gives

$$\langle[f]ST;[f^*]ST\parallel[0]00\rangle = (-1)^{\sigma+\eta([f],S,T)}\left[\frac{(2S+1)(2T+1)}{\dim[f]}\right]^{\frac{1}{2}}, \tag{42}$$

where $\sigma = 0$ for all the representations enumerated in (26) (except $[y11]$ and its conjugate, for which $\sigma = y+1$). Finally, the symmetry property involving the simple interchange of representations (1) and (2) is given by

$$\text{(III)}\ \langle[f^{(1)}]S_1T_1;[f^{(2)}]S_2T_2\parallel[\ ^{(3)}]S_3T_3\rangle = (-1)^{\nu+S_1+S_2-S_3+T_1+T_2-T_3} \times\langle[f^{(2)}]S_2T_2;[f^{(1)}]S_1T_1\parallel[f^{(3)}]S_3T_3\rangle, \tag{43}$$

where the phase factor $(-1)^\nu$, through the convention (37), can be identified with the sign of the coefficient:

$$\langle[f^{(1)}]\bar{S}_1\bar{T}_1;[f^{(2)}]P_2P_2'\parallel[f^{(3)}]P_3P_3'\rangle.$$

The symmetry property (III) is not of very great interest, but the relations (I) and (II), as well as their combination, may be very useful in the applications to problems in nuclear physics and lead to a reduction in the number of coefficients which must be calculated (or tabulated).

If S and T are not sufficient to specify the states of a representation, the additional quantum numbers can always be chosen such that the symmetry relations (I)–(III) are satisfied. This requires that the state vectors have simple conjugation properties. For this purpose it may be convenient to choose quantum numbers other than ω, φ (as indicated in the case of the representation [422]). In the case of products which are not simply reducible, the symmetry relations may be dependent on the labels ρ. Only the coupling coefficients for the products $[f]\times[211]\to[f]$ and $[f]\times[422]\to[f]$ are of special interest in the applications to nuclear problems. With our choice of ρ, the symmetry relation (I) becomes

$$\text{(I)}\ \langle[f^{(1)*}]S_1T_1;[f^{(2)*}]S_2T_2\parallel[f^{(3)*}]S_3T_3\rangle_\rho = (-1)^{\rho+1+\eta^{(1)}+\eta^{(2)}-\eta^{(3)}} \times\langle[f^{(1)}]S_1T_1;[f^{(2)}]S_2T_2\parallel[f^{(3)}]S_3T_3\rangle_\rho, \tag{38'}$$

when $[f^{(2)}]$ is one of the self-conjugate representations [211] or [422]. The symmetry relations (II) and (III) are independent of ρ when $[f^{(2)}]$ is either of these two representations.

4.5. $SU(4)$ Racah Coefficients

The $SU(4)$ Racah coefficients are straightforward generalizations of the ordinary Racah coefficients and can be defined by a recoupling transformation for a coupled system built from the states of three irreducible representations $[f^{(i)}]$ with $i = 1, 2, 3$, and coupled to a resultant state of the representation $[f]$. Two ways of coupling such a system are illustrated in Fig. 2 by the type of diagrams introduced by French.[18]

[17] J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

[18] M. H. Macfarlane and J. B. French, Rev. Mod. Phys. 32, 567 (1960).

The recoupling process involves a unitary transformation whose matrix elements are the $SU(4)$ Racah or U coefficients:

$$
\begin{aligned}
&|(\{([f^{(1)}][f^{(2)}])[f^{(12)}]\rho_{12}\}[f^{(3)}])[f]\rho_{12,3};\ \omega\varphi SM_S TM_T\rangle \\
&\quad = \sum_{[f^{(23)}]\rho_{23}\rho_{1,23}} |([f^{(1)}]\{([f^{(2)}][f^{(3)}])[f^{(23)}]\rho_{23}\})[f]\rho_{1,23};\ \omega\varphi SM_S TM_T\rangle \\
&\qquad \times U([f^{(1)}][f^{(2)}][f][f^{(3)}];\ [f^{(12)}]\rho_{12}\rho_{12,3};\ [f^{(23)}]\rho_{23}\rho_{1,23}). \quad (44)
\end{aligned}
$$

The U coefficients satisfy the orthogonality relations

$$
\begin{aligned}
&\sum_{\alpha} U(\cdots;\alpha\mu)U(\cdots;\alpha\mu') = \delta_{\mu\mu'}, \\
&\sum_{\mu} U(\cdots;\alpha\mu)U(\cdots;\alpha'\mu) = \delta_{\alpha\alpha'}, \quad (45)
\end{aligned}
$$

where α is a short-hand notation for $[f^{(12)}]$, ρ_{12}, $\rho_{12,3}$, and where μ is a short-hand notation for $[f^{(23)}]$, ρ_{23}, $\rho_{1,23}$. The U coefficients can be related to the $SU(4)$ Wigner coefficients by the sum

$$
\begin{aligned}
&U([f^{(1)}][f^{(2)}][f][f^{(3)}];\ [f^{(12)}]\rho_{12}\rho_{12,3};\ [f^{(23)}]\rho_{23}\rho_{1,23}) \\
&\quad = \sum_{\substack{\epsilon_1\epsilon_2\epsilon_3\\ \epsilon_{12}\epsilon_{23}}} \langle[f^{(1)}]\epsilon_1;[f^{(2)}]\epsilon_2 \parallel [f^{(12)}]\epsilon_{12}\rangle_{\rho_{12}} \langle[f^{(12)}]\epsilon_{12};[f^{(3)}]\epsilon_3 \parallel [f]\epsilon\rangle_{\rho_{12,3}} \\
&\qquad \times \langle[f^{(2)}]\epsilon_2;[f^{(3)}]\epsilon_3 \parallel [f^{(23)}]\epsilon_{23}\rangle_{\rho_{23}} \langle[f^{(1)}]\epsilon_1;[f^{(23)}]\epsilon_{23} \parallel [f]\epsilon\rangle_{\rho_{1,23}} U(S_1S_2SS_3;S_{12}S_{23})U(T_1T_2TT_3;T_{12}T_{23}), \quad (46)
\end{aligned}
$$

where ϵ_i is a short-hand notation for S_iT_i (and $\omega_i\varphi_i$, if needed). The sums over M_{S_i} and M_{T_i} have been performed and expressed in terms of the angular momentum U coefficients (unitary or Jahn form of the S and T space Racah coefficients). The $SU(4)$ U coefficients are independent of ST, $(\omega\varphi)$, so that any convenient subgroup labeling can, in principle, be used in performing the sums over the subgroup quantum numbers. In principle, therefore, very general expressions can be given for the Racah coefficients. However, these would be unnecessarily complicated by the multiplicity labels ρ. In the actual applications, labels ρ_{12}, ρ_{23}, and $\rho_{12,3}$ are never needed; in those cases where they are needed, the label $\rho_{1,23}$ corresponds to a multiplicity of two only. The most useful equation relating the $SU(4)$ Racah and Wigner coefficients is given by

$$
\begin{aligned}
&\sum_{\rho_{1,23}} \langle[f^{(1)}]\epsilon_1;[f^{(23)}]\epsilon_{23} \parallel [f]\epsilon\rangle_{\rho_{1,23}} U([f^{(1)}][f^{(2)}][f][f^{(3)}];\ [f^{(12)}]\rho_{12}\rho_{12,3};\ [f^{(23)}]\rho_{23}\rho_{1,23}) \\
&\quad = \sum_{\epsilon_2\epsilon_3\epsilon_{12}} \langle[f^{(1)}]\epsilon_1;[f^{(2)}]\epsilon_2 \parallel [f^{(12)}]\epsilon_{12}\rangle_{\rho_{12}} \langle[f^{(12)}]\epsilon_{12};[f^{(3)}]\epsilon_3 \parallel [f]\epsilon\rangle_{\rho_{12,3}} \langle[f^{(2)}]\epsilon_2;[f^{(3)}]\epsilon_3 \parallel [f^{(23)}]\epsilon_{23}\rangle_{\rho_{23}} \\
&\qquad \times U(S_1S_2SS_3;S_{12}S_{23})U(T_1T_2TT_3;T_{12}T_{23}). \quad (47)
\end{aligned}
$$

Except for the summation over $\rho_{1,23}$ (when needed), this is again a straightforward generalization of a relation valid for ordinary angular momentum coefficients. This equation is to be used as the basis for a buildup process whereby relatively complicated $SU(4)$ Wigner coefficients are calculated from a knowledge of very simple ones.

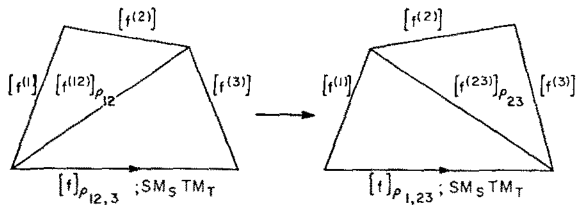

FIG. 2. Two ways of coupling states of 3 IR's to a resultant state.

Equation (47), together with Eqs. (42), (39), and (32a), also leads to the special value

$$
\begin{aligned}
&U([f][f^{(2)}][f][f^{(2)*}];\ [f^{(12)}];\ [0]) \\
&\quad = (-1)^{\sigma([f],[f^{(2)}],[f^{(12)}])+\sigma_2}\cdot\left[\frac{\dim [f^{(12)}]}{\dim [f]\dim [f^{(2)}]}\right]^{\frac{1}{2}}, \quad (48)
\end{aligned}
$$

where $\sigma([f], [f^{(2)}], [f^{(12)}])$ is given by Eq. (40), and $\sigma_2 = 0$ unless $[f^{(2)}] = [y11]$ or its conjugate, in which case $\sigma_2 = y + 1$, Eq. (42).

4.6. Method of Calculation

The calculation of the $SU(4)$ Wigner coefficients begins with the calculation of the matrix elements of the infinitesimal operators E_{ab}. These follow from the normalization coefficients of Table VII and the relations of Tables II and III (for the details, see Appendix A and Ref. 19). The matrix elements of E_{ab} are expressed in terms of reduced $SU(4)$ Wigner coefficients by means of Eq. (35). They can be read off from the tabulations of $\langle [f]S'T'; [211]11 \| [f]S''T''\rangle_{\rho=1}$ given in Appendix B.

The simplest Wigner coefficients are those involving a coupling of $[f]$ with the one-particle representation [1] (one-particle cfp's). These can be calculated by standard recursion techniques from a knowledge of the matrix elements of the infinitesimal operators. By operating with an operator $E_{ab} = E_{ab}(1) + E_{ab}(2)$ on the state of a coupled system built from systems 1 and 2, a recursion relation for the full $SU(4)$ Wigner coefficients is obtained. For example,

$$\begin{aligned}
\sum_{S'T'} \langle [f^{(1)}]S_1M_{S_1}T_1M_{T_1}; [1]\tfrac{1}{2}M_{S_2}\tfrac{1}{2}M_{T_2} \mid [f]S'(M_S+1)T'(M_T+1)\rangle& \\
\times \langle [f]S'(M_S+1)T'(M_T+1)|\, E_{11}\, |[f]SM_STM_T\rangle& \\
= \sum_{S_1'T_1'} \langle [f^{(1)}]S_1'(M_{S_1}-1)T_1'(M_{T_1}-1); [1]\tfrac{1}{2}M_{S_2}\tfrac{1}{2}M_{T_2} \mid [f]SM_STM_T\rangle& \\
\times \langle [f^{(1)}]S_1M_{S_1}T_1M_{T_1}|\, E_{11}\, |[f^{(1)}]S_1'(M_{S_1}-1)T_1'(M_{T_1}-1)\rangle& \\
+ \langle [f^{(1)}]S_1M_{S_1}T_1M_{T_1}; [1]\tfrac{1}{2}(M_{S_2}-1)\tfrac{1}{2}(M_{T_2}-1) \mid [f]SM_STM_T\rangle& \\
\times \langle [1]\tfrac{1}{2}M_{S_2}\tfrac{1}{2}M_{T_2}|\, E_{11}\, |[1]\tfrac{1}{2}(M_{S_2}-1)\tfrac{1}{2}(M_{T_2}-1)\rangle.& \qquad (49)
\end{aligned}$$

From such recursion relations, coefficients of the above simple type have been calculated for the cases when both $[f^{(1)}]$ and $[f]$ belong to the special representations of Sec. 3. Coefficients for the coupling of $[f^{(1)}]$ with more complicated representations are then calculated by a buildup process based on the recoupling relation, Eq. (47). By setting both $[f^{(2)}]$ and $[f^{(3)}]$ equal to the one-particle representation [1] in Eq. (47), $SU(4)$ Wigner coefficients with $[f^{(23)}] = [2]$ or $[1^2]$ (two-particle cfp's) can be calculated. In this case the products $[f^{(1)}] \times [f^{(23)}]$ are simply reducible. The ρ sum in Eq. (47) is not needed, and the $SU(4)$ U coefficient serves merely as a normalization factor for the Wigner coefficients. Coefficients with $[f^{(23)}] =$ [211], [22], and [422] can be calculated through Eq. (47) by setting $[f^{(2)}]$ and $[f^{(3)}]$ equal to $[1^3]$ and [1], $[1^2]$ and $[1^2]$, and $[2^3]$ and [2], respectively. In many of these cases the multiplicity in the product $[f^{(1)}] \times [f^{(23)}]$ requires the ρ sum for the left-hand side of (47), and the simultaneous calculation of both the Wigner and U coefficients requires the solution of a simple 2×2 linear system.

Algebraic expressions for both the Wigner and Racah coefficients are tabulated in Appendix B. This appendix is preceded by a table listing the cases covered and showing the arrangement of the tables of $SU(4)$ coefficients. Wigner coefficients involving the coupling with [22] and [422] include only the coefficients needed for diagonal matrix elements of the corresponding two-body operators. The tables of Racah coefficients are also restricted to those needed for the calculation of diagonal matrix elements—that is, those with $[f^{(1)}] = [f]$, and $[f^{(2)}]$ and $[f^{(3)}]$ equal to $[1^3]$ and [1], or $[1^2]$ and $[1^2]$, and $[2^3]$ and [2], needed for the evaluation of the matrix elements of one-body, or two-body operators.

5. APPLICATIONS

The recognition that the spin–isospin part of the fractional parentage coefficients can be identified with the reduced $SU(4)$ Wigner coefficients makes it possible to perform the spin–isospin sums in the cfp expansions of nuclear matrix elements by means of the Racah formalism of Sec. 4.

The cfp's needed for the decomposition of a totally antisymmetric n-nucleon wavefunction into totally antisymmetric functions for specific sets of n_1 and n_2 nucleons can be factored into a space and a spin–isospin part[2,8]:

$$\begin{aligned}
\langle [f^{(n_1)}]\alpha_{n_1}L_{n_1}, \beta_{n_1}S_{n_1}T_{n_1}; [f^{(n_2)}]\alpha_{n_2}L_{n_2}, \beta_{n_2}S_{n_2}T_{n_2} \,|\}\, [f^{(n)}]\alpha_nL_n, \beta_nS_nT_n\rangle& \\
= \left[\frac{\mathcal{N}_{[f^{(n_1)}]}\mathcal{N}_{[f^{(n_2)}]}}{\mathcal{N}_{[f^{(n)}]}}\right]^{\frac{1}{2}} \langle [\tilde{f}^{(n_1)}]\alpha_{n_1}L_{n_1}; [\tilde{f}^{(n_2)}]\alpha_{n_2}L_{n_2} \,|\}\, [\tilde{f}^{(n)}]\alpha_nL_n\rangle& \\
\times \langle [f^{(n_1)}]\beta_{n_1}S_{n_1}T_{n_1}; [f^{(n_2)}]\beta_{n_2}S_{n_2}T_{n_2} \,\|\, [f^{(n)}]\beta_nS_nT_n\rangle,& \qquad (50)
\end{aligned}$$

[19] S. C. Pang, "On Eight-Dimensional Quasispin," Ph.D. thesis, Univ. of Michigan, 1967.

where the spin–isospin factor has been written in the $SU(4)$ notation of Sec. 4, and where $\mathcal{N}_{[f^{(n_i)}]}$ is the dimension of the irreducible representation of the *symmetric* or *permutation* group on n_i objects described by the Young tableau $[f^{(n_i)}]$. The representation contragredient to $[f]$ under the symmetric group is denoted by $[\tilde{f}]$; that is, $[\tilde{f}]$ is obtained from $[f]$ by interchanging rows and columns in the Young tableau. (It should perhaps be pointed out that the symbol $[f]$ is usually used to denote the symmetry of the space part of the wavefunction, while $[\tilde{f}]$ is used for the spin–isospin part. In this investigation the role of the two has been interchanged for economy in writing. The tilde still implies interchange of rows and columns of the Young tableau.) The full set of space quantum numbers other than L is abbreviated by the label α; quantum numbers such as ω, φ are replaced by β. [If needed, it will be understood that these quantum numbers will be chosen such that the $SU(4)$ Wigner coefficients satisfy the symmetry relations (I)–(III).] Matrix elements of one- and two-body operators can be expressed in terms of these cfp's by the usual expansions.

5.1. One-Body Operators

It will be assumed that the one-body operator

$$\mathcal{O} = \sum_{i=1}^{n} o_i = \sum_{\alpha' \cdots m_t} \langle \alpha' l' m_l' m_s' m_t' |\, o\, | \alpha l m_l m_s m_t \rangle\, a^\dagger_{\alpha' l' m_l' m_s' m_t'} a_{\alpha l m_l m_s m_t} \tag{51}$$

has a definite $SU(4)$ irreducible tensor character $[f_{\rm op}]$ with components $(\mathcal{S}\mathcal{M}_\mathcal{S}\mathcal{T}\mathcal{M}_\mathcal{T})$, and spherical tensor character $\mathcal{L}$ with component $\mathcal{M}_\mathcal{L}$. The matrix element of the one-particle operator can then be factored:

$$\begin{aligned}\langle \alpha' l' m_l' m_s' m_t' |\, o\, | \alpha l m_l m_s m_t \rangle = \langle \alpha' l' \|\, o\, \| \alpha l \rangle \langle [1]\tfrac{1}{2}\tfrac{1}{2};\, [f_{\rm op}]\mathcal{S}\mathcal{T} \,\|\, [1]\tfrac{1}{2}\tfrac{1}{2} \rangle \\ \times \langle \tfrac{1}{2} m_s \mathcal{S}\mathcal{M}_\mathcal{S} \mid \tfrac{1}{2} m_s' \rangle \langle \tfrac{1}{2} m_t \mathcal{T}\mathcal{M}_\mathcal{T} \mid \tfrac{1}{2} m_t' \rangle \langle l m_l \mathcal{L}\mathcal{M}_\mathcal{L} \mid l' m_l' \rangle.\end{aligned} \tag{52}$$

The matrix element of the one-body operator between n-nucleon states can be expressed by the cfp expansion

$$\langle [f']\alpha' L' M_L', \beta' S' M_S' T' M_T' |\, \mathcal{O}\, | [f]\alpha L M_L, \beta S M_S T M_T \rangle = n \sum_{[f^{(n-1)}]} \frac{\mathcal{N}_{[f^{(n-1)}]}}{[\mathcal{N}_{[f]}\mathcal{N}_{[f']}]^{\frac{1}{2}}} \mathcal{F}(\text{space}) \mathcal{F}(SU_4), \tag{53}$$

where $\mathcal{F}$(space) and $\mathcal{F}(SU_4)$ are the space and spin–isospin parts of the cfp expansion which are given by

$$\begin{aligned}\mathcal{F}(\text{space}) = \sum_{\substack{\alpha_{n-1}L_{n-1}\\ \alpha_1\alpha_1' l l'}} &\langle [\tilde{f}^{(n-1)}]\alpha_{n-1}L_{n-1};\, [1]\alpha_1 l \,|\}\, [\tilde{f}]\alpha L \rangle \langle [\tilde{f}^{(n-1)}]\alpha_{n-1}L_{n-1};\, [1]\alpha_1' l' \,|\}\, [\tilde{f}']\alpha' L' \rangle \\ &\times \langle \alpha_1' l' \|\, o\, \| \alpha_1 l \rangle \left[\frac{(2L+1)(2l'+1)}{(2L_{n-1}+1)(2\mathcal{L}+1)}\right]^{\frac{1}{2}} (-1)^{\mathcal{L}+L_{n-1}-l'-L} U(LlL'l';\, L_{n-1}\mathcal{L}) \langle L M_L \mathcal{L}\mathcal{M}_\mathcal{L} \mid L' M_L' \rangle\end{aligned} \tag{54}$$

and

$$\begin{aligned}\mathcal{F}(SU_4) = \sum_{\beta_{n-1}S_{n-1}T_{n-1}} &\langle [f^{(n-1)}]\beta_{n-1}S_{n-1}T_{n-1};\, [1]\tfrac{1}{2}\tfrac{1}{2} \,\|\, [f]\beta S T \rangle \\ &\times \langle [f^{(n-1)}]\beta_{n-1}S_{n-1}T_{n-1};\, [1]\tfrac{1}{2}\tfrac{1}{2} \,\|\, [f']\beta' S' T' \rangle \langle [1]\tfrac{1}{2}\tfrac{1}{2};\, [f_{\rm op}]\mathcal{S}\mathcal{T} \,\|\, [1]\tfrac{1}{2}\tfrac{1}{2} \rangle \\ &\times \left[\frac{(2S+1)2}{(2S_{n-1}+1)(2\mathcal{S}+1)}\right]^{\frac{1}{2}} (-1)^{\mathcal{S}+S_{n-1}-\frac{1}{2}-S} U(S\tfrac{1}{2}S'\tfrac{1}{2};\, S_{n-1}\mathcal{S}) \langle S M_S \mathcal{S}\mathcal{M}_\mathcal{S} \mid S' M_S' \rangle \\ &\times \left[\frac{(2T+1)2}{(2T_{n-1}+1)(2\mathcal{T}+1)}\right]^{\frac{1}{2}} (-1)^{\mathcal{T}+T_{n-1}-\frac{1}{2}-T} U(T\tfrac{1}{2}T'\tfrac{1}{2};\, T_{n-1}\mathcal{T}) \langle T M_T \mathcal{T}\mathcal{M}_\mathcal{T} \mid T' M_T' \rangle.\end{aligned} \tag{55}$$

With the aid of the symmetry relations (I)–(III) for the $SU(4)$ Wigner coefficients, the latter can be written

$$\begin{aligned}\mathcal{F}(SU_4) = &\left\{(-1)^{\sigma([f^{(n-1)}],[1],[f])} \left[\frac{\dim[1]\dim[f]}{\dim[f^{(n-1)}]}\right]^{\frac{1}{2}}\right\} \frac{(-1)^{\sigma_{\rm op}}}{[\dim[f_{\rm op}]]^{\frac{1}{2}}} \langle S M_S \mathcal{S}\mathcal{M}_\mathcal{S} \mid S' M_S' \rangle \langle T M_T \mathcal{T}\mathcal{M}_\mathcal{T} \mid T' M_T' \rangle \\ &\times \sum_{\beta_{n-1}S_{n-1}T_{n-1}} \langle [f]\beta S T;\, [1^3]\tfrac{1}{2}\tfrac{1}{2} \,\|\, [f^{(n-1)}]\beta_{n-1}S_{n-1}T_{n-1} \rangle \langle [f^{(n-1)}]\beta_{n-1}S_{n-1}T_{n-1};\, [1]\tfrac{1}{2}\tfrac{1}{2} \,\|\, [f']\beta' S' T' \rangle \\ &\times \langle [1^3]\tfrac{1}{2}\tfrac{1}{2};\, [1]\tfrac{1}{2}\tfrac{1}{2} \,\|\, [f_{\rm op}]\mathcal{S}\mathcal{T} \rangle U(S\tfrac{1}{2}S'\tfrac{1}{2};\, S_{n-1}\mathcal{S}) U(T\tfrac{1}{2}T'\tfrac{1}{2};\, T_{n-1}\mathcal{T}),\end{aligned} \tag{56}$$

where $\sigma_{\rm op} = \delta_{[f_{\rm op}][211]}$; that is, $(-1)^{\sigma_{\rm op}} = -1$ for $[f_{\rm op}] = [211]$, $(-1)^{\sigma_{\rm op}} = +1$ for $[f_{\rm op}] = [0]$.

The coefficients in the spin–isospin sum of the last factor are now in a form in which they can be summed by means of Eq. (47). Although the first factor (enclosed by curly brackets) is made up solely of trivial dimensional and phase factors, it is convenient to write it in terms of the $SU(4)$ Racah coefficient with

$[f^{(23)}] = [0]$, Eq. (48). This makes it possible to express the resultant of the spin–isospin sum in the cfp expansion in terms of ratios of $SU(4)$ U coefficients which are independent of the particular phase conventions [such as (37)] chosen for the $SU(4)$ coefficients. The resultant expression is

$$\mathcal{F}(SU_4) = \frac{(-1)^{\delta[f_{\rm op}][211]}}{[\dim\,[f_{\rm op}]]^{\frac{1}{2}}} \sum_\rho \frac{U([f][1^3][f'][1];[f^{(n-1)}];[f_{\rm op}]\rho)}{U([f][1^3][f][1];[f^{(n-1)}];[0])}$$
$$\times \langle [f]\beta ST;[f_{\rm op}]\mathcal{S}\mathcal{T} \,\|\, [f']\beta' S'T'\rangle_\rho \langle SM_S\mathcal{S}\mathcal{M}_{\mathcal{S}} \mid S'M'_S\rangle\langle TM_T\mathcal{T}\mathcal{M}_{\mathcal{T}} \mid T'M'_T\rangle. \quad (57)$$

The $SU(4)$ Wigner and Racah coefficients needed for the evaluation of this expression are given in Tables A.4 and A.7 of Appendix B. If the one-body operator is a *complete* space scalar (if the reduced matrix elements of $\mathscr{o}$ are independent of α and l), the sum over the spatial quantum numbers is trivial, and the full matrix element has the simple form

$$\langle [f']\alpha' L' M_L, \beta' S' M'_S T' M'_T |\, \mathcal{O}^{[f_{\rm op}],\text{space-scalar}}_{(\mathcal{S}\mathcal{M}_\mathcal{S})(\mathcal{T}\mathcal{M}_\mathcal{T})} \,|[f]\alpha L M_L, \beta S M_S T M_T\rangle$$
$$= \delta_{[f][f']}\delta_{\alpha\alpha'}\delta_{LL'}\langle [1] \| \mathscr{o}^{[f]\rm op} \| [1]\rangle \frac{(-1)^{\delta[f_{\rm op}][211]}}{[\dim\,[f_{\rm op}]]^{\frac{1}{2}}}\, n \sum_{[f^{(n-1)}]} \frac{\mathcal{N}_{[f^{(n-1)}]}}{\mathcal{N}_{[f]}} \sum_\rho \frac{U([f][1^3][f][1];[f^{(n-1)}];[f_{\rm op}]\rho)}{U([f][1^3][f][1];[f^{(n-1)}];[0])}$$
$$\times \langle [f]\beta ST;[f_{\rm op}]\mathcal{S}\mathcal{T} \,\|\, [f]\beta' S'T'\rangle_\rho \langle SM_S\mathcal{S}\mathcal{M}_{\mathcal{S}} \mid S'M'_S\rangle\langle TM_T\mathcal{T}\mathcal{M}_{\mathcal{T}} \mid T'M'_T\rangle. \quad (58)$$

The only nontrivial case involves operators of $SU(4)$ tensor character [211]. In this case the sums over the possible $(n-1)$-particle symmetries have the very simple value

$$n \sum_{[f^{(n-1)}]} \frac{\mathcal{N}_{[f^{(n-1)}]}}{\mathcal{N}_{[f]}} \frac{U([f][1^3][f][1];[f^{(n-1)}];[211]\rho)}{U([f][1^3][f][1];[f^{(n-1)}];[0])} \begin{aligned} &= 0 && \text{for } \rho \neq 1, \\ &= -2[C(SU_4)]^{\frac{1}{2}} && \text{for } \rho = 1, \end{aligned} \quad (59)$$

where the Casimir invariant $C(SU_4)$ is given by Eq. (23). The S, T-dependence of the matrix element of a space-scalar one-body operator of $SU(4)$ tensor character [211] is thus given by the single $SU(4)$ Wigner coefficient with $\rho = 1$, that is, by the matrix element of the corresponding infinitesimal operator. The only nontrivial operators of this type are the infinitesimal generators **E**, Eq. (3), which give the Gamow–Teller matrix elements in beta decay. The tables of Appendix B can thus be used to read off Gamow–Teller matrix elements for a wide class of Wigner supermultiplets.

5.2. Particle–Hole Interaction; Space-Scalar Approximation

It has been shown that a space-scalar approximation to the particle–hole interaction may give a good estimate of the full particle–hole interaction energy in nuclei near the beginning of the $2s$, $1d$ shell.[10] The matrix elements of such an interaction can be written down at once in terms of the results of Eqs. (58) and (59) for particle–hole configurations described by the weak-coupling model. In the space-scalar approximation the particle–hole interaction can be represented by

$$V_{ph} = \sum_{i,j} (-a_{00} + a_{01}\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j + a_{10}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + a_{11}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)), \quad (60)$$

where $a_{\mathcal{S}\mathcal{T}}$ are constants, and where the summation indices i and j refer to nucleons in different major shells such as the $1p$ and $2s$, $1d$ shell. Zamick[20] has pointed out that the first two terms of Eq. (60) may be used to give a rough idea of the location of the particle–hole states. The matrix elements of the first three terms of Eq. (60) can be calculated by ordinary angular-momentum calculus. The last term is more complicated. Moreover, it may lead to important contributions to the particle–hole interaction energy in many cases.[10] It may give rise not only to important J-dependent contributions to the diagonal matrix element of the interaction, but may also give an estimate of the often significant mixing of particle–hole states with the same space structure but with different spin–isospin quantum numbers for the particle and hole configurations. The last three terms of Eq. (60) are built from space-scalar one-body operators for each shell. Each has $SU(4)$ tensor character [211] with $\mathcal{S}\mathcal{T}$ components of 01, 10, and 11, respectively. The full tensor character of each is of the form

$$a_{\mathcal{S}\mathcal{T}}(\mathcal{O}^{[211]}_{\mathcal{S}\mathcal{T}} \cdot\cdot\, \mathcal{O}^{[211]}_{\mathcal{S}\mathcal{T}}), \quad (61)$$

where the double dot refers to the scalar product in $\mathcal{S}$ and $\mathcal{T}$ space. It is assumed that the particle–hole state can be described in the weak-coupling model in which the Wigner supermultiplet quantum numbers for both the particle and hole configurations are good

[20] L. Zamick, Phys. Letters 19, 580 (1965).

quantum numbers, to be denoted by $[f_p]$ and $[f_h]$, respectively. If the number of particles and holes are denoted by n_p and n_h, we shall take $[f_p]$ and $[f_h]$ to be Young tableaux describing the symmetry of the spin–isospin functions for n_p and $(N - n_h)$ nucleons, respectively, where $N =$ number of nucleons in the closed shell. The basic form of the wavefunction is chosen to be

$$|([f_h]\alpha_h L_h S_h J_h T_h,\ [f_p]\alpha_p L_p S_p J_p T_p) J M_J T M_T\rangle, \tag{62}$$

where the subscripts p and h refer to the particle and hole configurations. In the weak-coupling description there is no further coupling of the supermultiplets $[f_p]$, $[f_h]$; but the angular momenta and isospins of the particle and hole configurations are coupled to total angular momentum J and total isospin T. Matrix elements of the operators (61) follow from Eqs. (58) and (59) {with $\langle[1]\| o^{[211]} \|[1]\rangle = (15)^{\frac{1}{2}}$}, leading to

$$\begin{aligned}
&\langle([f_h]\alpha_h L_h S_h' J_h' T_h',\ [f_p]\alpha_p L_p S_p' J_p' T_p') J M_J T M_T|\, a_{\mathcal{S}\mathcal{T}} \mathcal{O}^{[211]}_{\mathcal{S}\mathcal{T}} \cdot \cdot\, \mathcal{O}^{[211]}_{\mathcal{S}\mathcal{T}}\, |([f_h]\alpha_h L_h S_h J_h T_h,\ [f_p]\alpha_p L_p S_p J_p T_p) J M_J T M_T\rangle \\
&= a_{\mathcal{S}\mathcal{T}} (-1)^{J+J_h+J_p'} \begin{Bmatrix} J_h & J_p & J \\ J_p' & J_h' & \mathcal{S} \end{Bmatrix} (-1)^{T+T_h+T_p'} \begin{Bmatrix} T_h & T_p & T \\ T_p' & T_h' & \mathcal{T} \end{Bmatrix} \\
&\quad \times [(2J_h+1)(2J_h'+1)(2J_p+1)(2J_p'+1)]^{\frac{1}{2}} (-1)^{J_h'+S_h+L_h+\mathcal{S}} \begin{Bmatrix} S_h & J_h & L_h \\ J_h' & S_h' & \mathcal{S} \end{Bmatrix} (-1)^{J_p'+S_p+L_p+\mathcal{S}} \begin{Bmatrix} S_p & J_p & L_p \\ J_p' & S_p' & \mathcal{S} \end{Bmatrix} \\
&\quad \times [(2S_p'+1)(2T_p'+1)]^{\frac{1}{2}} 2[C(SU_4)_p]^{\frac{1}{2}} \langle [f_p] S_p T_p;\ [211]\mathcal{S}\mathcal{T} \,\|\, [f_p] S_p' T_p' \rangle_{\rho=1} \\
&\quad \times [(2S_h'+1)(2T_h'+1)]^{\frac{1}{2}} 2[C(SU_4)_h]^{\frac{1}{2}} \langle [f_h] S_h T_h;\ [211]\mathcal{S}\mathcal{T} \,\|\, [f_h] S_h' T_h' \rangle_{\rho=1},
\end{aligned} \tag{63}$$

where it is convenient to express the ordinary angular-momentum Racah coefficients in their 6-j symbol form. The $SU(4)$ Wigner coefficients for the operators with $\mathcal{S}\mathcal{T} = 10$, 01 are given by the simple matrix elements of **S** and **T**, and have the values

$$\begin{aligned}
[C(SU_4)]^{\frac{1}{2}} \langle [f] ST;\ [211]10 \,\|\, [f] S'T' \rangle_{\rho=1} &= \delta_{SS'} \delta_{TT'} [S(S+1)]^{\frac{1}{2}}, \\
[C(SU_4)]^{\frac{1}{2}} \langle [f] ST;\ [211]01 \,\|\, [f] S'T' \rangle_{\rho=1} &= \delta_{SS'} \delta_{TT'} [T(T+1)]^{\frac{1}{2}}.
\end{aligned} \tag{64}$$

In these two cases, therefore, Eq. (63) reduces to a simple result of ordinary angular-momentum calculus. The diagonal-matrix elements of the full interaction (60) have been given in Ref. 10. The operator with $\mathcal{S}\mathcal{T} = 11$ can give important contributions to both the diagonal and off-diagonal matrix elements. From the symmetry relation (38′), however, it can be seen that matrix elements with $S' = S$, $T' = T$ are zero for all self-conjugate representations such as $[yy0]$ or $[211]$. For configurations with an even number of particles (or holes) the most important symmetries for the spin–isospin functions are likely to be those belonging to $SU(4)$ representations such as $[0]$, $[11]$, $[22]$, $\cdots$, or $[211]$, for which the diagonal-matrix element (63) is zero. The last term of (60) is therefore important mainly for configurations with an odd number of both particles and holes. It can, however, lead to matrix elements off-diagonal in both $S_p T_p$ and $S_h T_h$ for all $SU(4)$ representations, and the last term of (60) may be a major contributor to the mixing of different particle–hole states with the same space structure. The $SU(4)$ Wigner coefficients needed for the evaluation of (63) are given in Tables 4.1–4.6 of Appendix B.

5.3. Two-Body Operators

The techniques used in Sec. 5.1 can also be used to simplify the expressions for the matrix elements of a two-body operator, such as the two-body interaction

$$H = \sum_{i<j} h_{ij}. \tag{65}$$

Such operators can be decomposed into their $SU(4)$ irreducible-tensor parts with components $(\mathcal{S}\mathcal{M}_{\mathcal{S}})(\mathcal{T}\mathcal{M}_{\mathcal{T}})$ and spherical-tensor character $\mathcal{L}$ (for orbital space) with components $\mathcal{M}_{\mathcal{L}}$. To be invariant under rotations in ordinary three-dimensional space such operators must be of the form

$$H^{[f_{\mathrm{op}}]}_{\mathcal{S};\mathcal{T}\mathcal{M}_{\mathcal{T}}} = \sum_{\mathcal{M}_{\mathcal{S}}} (-1)^{\mathcal{M}_{\mathcal{S}}} H^{[f_{\mathrm{op}}];\mathcal{L}=\mathcal{S}}_{(\mathcal{S}\mathcal{M}_{\mathcal{S}})(\mathcal{T}\mathcal{M}_{\mathcal{T}});\mathcal{M}_{\mathcal{L}}=-\mathcal{M}_{\mathcal{S}}}. \tag{66}$$

The reduced matrix elements of the two-particle operators are defined by the relation

$$\begin{aligned}
&\langle [f^{(2)}]\alpha_2' L_2' M_{L_2}' S_2' M_{S_2}' T_2' M_{T_2}'|\, h^{[f_{\mathrm{op}}];\mathcal{L}}_{(\mathcal{S}\mathcal{M}_{\mathcal{S}})(\mathcal{T}\mathcal{M}_{\mathcal{T}});\mathcal{M}_{\mathcal{L}}}\, |[f^{(2)}]\alpha_2 L_2 M_{L_2} S_2 M_{S_2} T_2 M_{T_2}\rangle \\
&\qquad = \langle [f^{(2)}]\alpha_2' L_2' \|\, h^{[f_{\mathrm{op}}];\mathcal{L}} \,\| [f^{(2)}]\alpha_2 L_2 \rangle \langle [f^{(2)}] S_2 T_2;\ [f_{\mathrm{op}}]\mathcal{S}\mathcal{T} \,\|\, [f^{(2)}] S_2' T_2' \rangle \\
&\qquad\quad \times \langle S_2 M_{S_2} \mathcal{S}\mathcal{M}_{\mathcal{S}} \mid S_2' M_{S_2}' \rangle \langle T_2 M_{T_2} \mathcal{T}\mathcal{M}_{\mathcal{T}} \mid T_2' M_{T_2}' \rangle \langle L_2 M_{L_2} \mathcal{L}\mathcal{M}_{\mathcal{S}} \mid L_2' M_{L_2}' \rangle.
\end{aligned} \tag{67}$$

(For simplicity, operators antisymmetric in *both* the space and spin–isospin variables will be excluded so that the two-particle matrix elements to be considered will be restricted to those with $[f^{(2)\prime}] = [f^{(2)}]$.) The spin–isospin sums in the cfp expansion for the matrix elements of such operators can be carried out by techniques similar to those of Sec. 5.1. The matrix element between n-nucleon states can then be given by the expression

$$\langle [f']\alpha'\beta'L'S'JM_JT'M'_T|\, H^{[f_{\rm op}]}_{\mathcal{S};\mathcal{T}\mathcal{M}_\mathcal{T}}\,|[f]\alpha\beta LSJM_JTM_T\rangle$$

$$= \tfrac{1}{2}n(n-1)\sum_{[f^{(n-2)}][f^{(2)}]}\frac{\mathcal{N}_{[f^{(n-2)}]}}{[\mathcal{N}_{[f]}\mathcal{N}_{[f']}]^{\frac{1}{2}}}\sum_{\substack{\alpha_{n-2}L_{n-2}\\ \alpha_2L_2\\ \alpha_2'L_2'}}\frac{\langle [f^{(2)}]\alpha_2'L_2'\|\, h^{[f_{\rm op}];\mathcal{S}}\,\|[f^{(2)}]\alpha_2L_2\rangle}{[\dim [f_{\rm op}]]^{\frac{1}{2}}}$$

$$\times \langle [\tilde f^{(n-2)}\alpha_{n-2}L_{n-2};[\tilde f^{(2)}]\alpha_2L_2\,|\}\,[\tilde f]\alpha L\rangle\langle [\tilde f^{(n-2)}]\alpha_{n-2}L_{n-2};[\tilde f^{(2)}]\alpha_2'L_2'\,|\}\,[\tilde f']\alpha'L'\rangle$$

$$\times (-1)^{L_{n-2}+L_2+L+\mathcal{S}+L'+S'+J}[(2L+1)(2L_2'+1)(2S'+1)(2L'+1)]^{\frac{1}{2}}\begin{Bmatrix} L & L_2 & L_{n-2}\\ L_2' & L' & \mathcal{S}\end{Bmatrix}\begin{Bmatrix} L & S & J\\ S' & L' & \mathcal{S}\end{Bmatrix}$$

$$\times \sum_\rho \frac{U([f][f^{(2)*}][f'][f^{(2)}];[f^{(n-2)}];[f_{\rm op}]\rho)}{U([f][f^{(2)*}][f][f^{(2)}];[f^{(n-2)}];[0])}\langle [f]\beta ST;[f_{\rm op}]\mathcal{S}\mathcal{T}\,\|\,[f']\beta'S'T'\rangle_\rho\langle TM_T\mathcal{T}\mathcal{M}_\mathcal{T}\,|\,T'M'_T\rangle. \quad (68)$$

If the two-body operator is a *complete* space scalar (that is, if the reduced matrix elements are independent of α_2 and L_2 and if $\mathcal{L} = \mathcal{S} = 0$), the matrix element again has a very simple form which can be evaluated completely with the aid of the tabulations of $SU(4)$ Wigner and Racah coefficients of Appendix B:

$$\langle [f']\alpha'\beta'L'S'JM_JT'M'_T|\, H^{[f_{\rm op}]\text{space-scalar}}_{0;\mathcal{T}\mathcal{M}_\mathcal{T}}|[f]\alpha\beta LSJM_JTM_T\rangle$$

$$= \tfrac{1}{2}n(n-1)\delta_{[f][f']}\delta_{\alpha\alpha'}\delta_{SS'}\delta_{LL'}\sum_{[f^{(n-2)}][f^{(2)}]}\frac{\mathcal{N}_{[f^{(n-2)}]}}{\mathcal{N}_{[f]}}\frac{\langle [f^{(2)}]\|\, h^{[f_{\rm op}]}\|[f^{(2)}]\rangle}{[\dim [f_{\rm op}]]^{\frac{1}{2}}}$$

$$\times \sum_\rho \frac{U([f][f^{(2)*}][f][f^{(2)}];[f^{(n-2)}];[f_{\rm op}]\rho)}{U([f][f^{(2)*}][f][f^{(2)}];[f^{(n-2)}];[0])}\langle [f]\beta ST;[f_{\rm op}]0\mathcal{T}\,\|\,[f]\beta'ST'\rangle_\rho\langle TM_T\mathcal{T}\mathcal{M}_\mathcal{T}\,|\,T'M'_T\rangle. \quad (69)$$

If the operator is also an isoscalar (charge-independent)—that is, if $\mathcal{S} = 0$ *and* $\mathcal{T} = 0$—then the $SU(4)$ tensor character is restricted to $[f_{\rm op}] = [0]$, [22], or [422]. For the special $SU(4)$ representations of Sec. 3 the multiplicity label ρ is needed only for the case $[f_{\rm op}] = [422]$. In this case the quantum number was chosen such that the (S, T)-dependence of the matrix element is given solely by the $SU(4)$ Wigner coefficients with $\rho = 1$; that is,

$$\sum_{[f^{(n-2)}]}\frac{\mathcal{N}_{[f^{(n-2)}]}}{\mathcal{N}_{[f]}}\frac{U([f][2^3][f][2];[f^{(n-2)}];[422]\rho)}{U([f][2^3][f][2];[f^{(n-2)}];[0])} = 0 \quad \text{for} \quad \rho \neq 1. \quad (70)$$

In the case of complete space-scalar, charge-independent operators, however, the matrix elements (69) have a very simple form which can be derived by much more elementary techniques. A complete space-scalar, charge-independent two-body operator can be expressed in terms of the operators

$$\sum_{i<j}\tfrac{1}{2}(1 \pm P_{ij}^{(\rm space)}), \quad \sum_{i<j}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \quad \sum_{i<j}(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j), \quad (71)$$

where $P_{ij}^{(\rm space)}$ is the Majorana or space exchange operator. These have the corresponding well-known eigenvalues

$$n_\pm, \quad 2[S(S+1) - \tfrac{3}{4}n], \quad 2[T(T+1) - \tfrac{3}{4}n], \quad (72)$$

where n_+ (n_-) are the number of *spacially* symmetrically (or antisymmetrically) coupled pairs of nucleons in the n-nucleon state, where[21]

$$n_\pm = \tfrac{1}{4}n(n-1) \mp \tfrac{1}{4}[\tfrac{1}{4}n^2 - 4n + C(SU_4)]. \quad (73)$$

The $SU(4)$ irreducible tensor form of these operators is given by

$$\begin{aligned} T^{[0]} &= \sum_{i<j}[(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j) + (\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) + (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]; \\ &\qquad\qquad\qquad \text{or} \quad \sum_{i<j}1, \\ T^{[22]} &= \sum_{i<j}[(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) - (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)], \\ T^{[422]} &= \sum_{i<j}[\tfrac{2}{3}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j) - (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j) - (\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)]. \end{aligned} \quad (74)$$

If the two-body operator includes the Coulomb interaction so that it can have isovector ($\mathcal{T} = 1$) and isotensor ($\mathcal{T} = 2$) components, the matrix elements (68) and (69) are much more complicated, and their evaluation in general form involves the full $SU(4)$ machinery. The isovector part has $SU(4)$ tensor character [211], and both $SU(4)$ Wigner and Racah coefficients with $\rho = 1$ and $\rho \neq 1$ make a contribution to the matrix elements of the n-nucleon system. The isotensor part will receive contributions from operators with $SU(4)$ tensor character [22] and [422].

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[21] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1960), p. 239.

5.4. Coulomb Interaction

The Coulomb energy in nuclei seems to show only a relatively mild dependence on the spatial quantum numbers, and it may be a good approximation to replace the full Coulomb interaction energy by the diagonal matrix elements of the complete space-scalar part of the interaction,[9] particularly if the motivation focuses on a study of the dependence of the Coulomb interaction on nucleon number and the spin–isospin, and Wigner supermultiplet quantum numbers.[9]

The diagonal matrix element of the full Coulomb interaction

$$V_C = \sum_{i<j} \frac{e^2}{r_{ij}} [\tfrac{1}{3}(\tfrac{3}{4} + \mathbf{t}_i \cdot \mathbf{t}_j) - \tfrac{1}{2}(t_{z_i} + t_{z_j}) + \tfrac{1}{3}(3t_{z_i}t_{z_j} - \mathbf{t}_i \cdot \mathbf{t}_j)] \quad (75)$$

leads to the Coulomb energy formula

$$E_C = E_C^{(0)} - M_T E_C^{(1)} + [3M_T^2 - T(T+1)]E_C^{(2)}. \quad (76)$$

The Coulomb interaction can be decomposed into irreducible tensor operators of the type $H^{[f_{\text{op}}]}_{\mathcal{S};\mathcal{T}\mathcal{M}_\mathcal{T}} = H^{[f_{\text{op}}]}_{0;\mathcal{T}0}$, defined in Eq. (66). The full decomposition is given in Ref. 9. The complete space-scalar part of the Coulomb interaction can be expressed as

$$\begin{aligned} V_C^{\text{space-scalar}} = \alpha\Big\{ & \frac{1}{\sqrt{6}}(H'^{[0]}_{0;00} + H'^{[22]}_{0;00}) - \frac{1}{\sqrt{2}} H'^{[211]}_{0;10} - \frac{1}{\sqrt{6}} H'^{[22]}_{0;20}\Big\} \\ & + \beta\Big\{\frac{1}{\sqrt{10}}(3H''^{[0]}_{0;00} + H''^{[422]}_{0;00}) + \sqrt{\tfrac{3}{2}}\, H''^{[211]}_{0;10} + \frac{1}{\sqrt{2}} H''^{[422]}_{0;20}\Big\}, \end{aligned} \quad (77)$$

where the coefficients α and β must be calculated for each major oscillator shell. [Results for the $1p$ and $2s$, $1d$ shells are given in Ref. 9. Equation (6b) of Ref. 9 should read $\alpha = 127/96$, $\beta = 7/6$.] The two-body operators H' (characterized by a single prime) are built from pair operators $a^\dagger a^\dagger$, (aa), with $SU(4)$ tensor character [11], while the two-body operators H'' (characterized by a double prime) are built from pair operators $a^\dagger a^\dagger$, (aa), with $SU(4)$ tensor character [2], ([2^3]). These operators have two-particle reduced matrix elements

$$\langle[11]\| h'^{[f_{\text{op}}]} \|[11]\rangle = \left[\frac{\dim [f_{\text{op}}]}{\dim [11]}\right]^{\frac{1}{2}},$$

$$\langle[2]\| h''^{[f_{\text{op}}]} \|[2]\rangle = \left[\frac{\dim [f_{\text{op}}]}{\dim [2]}\right]^{\frac{1}{2}}. \quad (78)$$

In the approximation in which the full Coulomb energy is replaced by its complete space-scalar part, the coefficients $E_C^{(0)}$, $E_C^{(1)}$, and $E_C^{(2)}$ of Eq. (76) can now be evaluated with the aid of Eqs. (69), (77), and (78). The isoscalar coefficient $E_C^{(0)}$ can also be evaluated by more elementary techniques, Eqs. (71)–(74). It has the value

$$\begin{aligned} E_C^{(0)} = {} & \frac{(\alpha + 3\beta)}{16} \tfrac{1}{2}n(n-1) + \frac{(\alpha + 3\beta)}{24}[T(T+1) - \tfrac{3}{4}n] \\ & - \frac{(\alpha - \beta)}{24}[C(SU_4) + 2S(S+1) - T(T+1) - \tfrac{9}{2}n] + \tfrac{3}{2}na_c. \end{aligned} \quad (79)$$

The isovector and isotensor coefficients can be calculated with the aid of the expressions for the $SU(4)$ Wigner and Racah coefficients needed for the evaluation of (69). These are given in Appendix B, which includes tabulations of the diagonal coefficients

$$\langle[f]ST; [f_{\text{op}}]0\mathcal{T} \,\|\, [f]ST\rangle_\rho$$

with $[f_{\text{op}}] = [211]$, [22], and [422], and the needed Racah coefficients, including the sums

$$\Sigma_2 = \tfrac{1}{2}n(n-1) \sum_{[f^{(n-2)}]} \frac{\mathcal{N}_{[f^{(n-2)}]}}{\mathcal{N}_{[f]}} \times \frac{U([f][f^{(2)*}][f][f^{(2)}]; [f^{(n-2)}]; [f_{\text{op}}]\rho)}{U([f][f^{(2)*}][f][f^{(2)}]; [f^{(n-2)}]; [0])}. \quad (80)$$

Results for the isovector and isotensor Coulomb energy coefficients $E_C^{(1)}$ and $E_C^{(2)}$ are collected in Table X. Some of these results have been given previously[9] in a somewhat different form. It is convenient to express the Coulomb energy coefficients in terms of the parameters

$$b = \tfrac{1}{24}(\alpha + 3\beta), \quad c = \tfrac{1}{48}(\alpha - \beta), \quad (81)$$

and a parameter a_c which gives the contribution from the interaction of the n nucleons in the partially filled major oscillator shell with those of the core.[9] The coefficients c are of the order of 5 to 10 kev for the $1p$ and $2s$, $1d$ shells, while b is of the order of 50–100 kev.[22] Since the dependence on the spin–isospin and Wigner supermultiplet quantum numbers is given entirely by the c terms, it can be seen that the Coulomb energy shows only a mild dependence on the quantum numbers y, S, T. If the integers y are related to nucleon number n, it can be seen that the nature of

[22] J. Jänecke, in *Isospin in Nuclear Physics*, D. H. Wilkinson, Ed. (North-Holland Publishing Co., to be published).

TABLE X. Isovector and isotensor Coulomb energy coefficients.[a]

$SU(4)$ Rep. $[f]$	$(PP'P'')$	$E_C^{(1)}$	$E_C^{(2)}$
$[yy0]$	$(y00)$	$3a_c + 3b(n-1) + 18c$	$b - c + c\,\dfrac{[(2y+5)(2y+3) - 4S(S+1)]}{(2T-1)(2T+3)}$
$[y\,y-1\,0]$ $[yy1]$	$(y - \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2})$ $(y - \frac{1}{2}\,\frac{1}{2}\,-\frac{1}{2})$	$3a_c + 3b(n-1) + 18c$ $-6cz\,\dfrac{[(2y+3) + (-1)^{y-S-T}(2S+1)(2T+1)]}{4T(T+1)}$	$b - c + c\,\dfrac{[(y+2)(y+1) - S(S+1)]}{T(T+1)}$
$[y00]$ $[yyy]$	$(\frac{1}{2}y\,\frac{1}{2}y\,\frac{1}{2}y)$ $(\frac{1}{2}y\,\frac{1}{2}y\,-\frac{1}{2}y)$	$3a_c + 3b(n-1) + 18c$ $-6cz(y+2)$	$b - 2c$
$[y11]$ $[y\,y-1\,y-1]$	$(\frac{1}{2}y\,\frac{1}{2}y\,\frac{1}{2}y - 1)$ $(\frac{1}{2}y\,\frac{1}{2}y\,-(\frac{1}{2}y - 1))$	$3a_c + 3b(n-1) + 18c + 6cz\begin{cases} \dfrac{(y+2)}{T(T+1)} - y & \leftarrow S = T \\ \dfrac{(y+2)}{T} - (y+2) & \leftarrow S = T-1 \\ \dfrac{-(y+2)}{T+1} - (y+2) & \leftarrow S = T+1 \end{cases}$ for	$b - 2c + 2c\begin{cases} \dfrac{(y+2)}{T(T+1)} & S = T \\ -\dfrac{[(y+2) - 2T]}{(2T-1)T} & S = T-1 \\ -\dfrac{[(y+2) + 2(T+1)]}{(2T+3)(T+1)} & S = T+1 \end{cases}$ for

[a] $z = (P''/|P''|)$. n = number of nucleons in a major oscillator shell.

the (n, T)-dependent terms for the Wigner supermultiplet scheme are very similar to those predicted for the low v limit of the seniority scheme.[9] The results of Table X thus seem to indicate that the major (n, T)-dependent effects in the systematics of Coulomb displacement energies are quite insensitive to the exact nature of the wavefunctions of the n-nucleon system.

6. CONCLUDING REMARKS

In principle, it is possible to extend the techniques used in this investigation for $SU(4)$ to the unitary groups needed to classify the space parts of the wavefunctions, such as $SU(3)$ and $SU(6)$ for the $1p$ and $2s$, $1d$ shells, for example. In principle, therefore, the full cfp expansions can be summed in general, and the matrix elements of one- and two-body operators can be expressed entirely in terms of Wigner and Racah coefficients for the special unitary groups. In the most general case, however, the algebraic nature of such coefficients is again very complicated,[23] and the expressions for the matrix elements are severely complicated by the multiplicity problem and the sums over the multiplicity labels. It may, however, again be possible to single out certain simple representations of special interest for which the summations over both the spin-isospin and space quantum numbers can be carried out in the cfp expansions for the matrix elements. The resultant interplay between the Wigner supermultiplet and the spacial quantum numbers may lead to interesting studies.

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APPENDIX A. CALCULATION OF NORMALIZATION COEFFICIENTS

To illustrate the technique used in the calculation of the normalization coefficients associated with the step operators $O_{\alpha\beta}$, Eqs. (11) and Table VII, the details of the calculation will be sketched for two of the representations of Sec. 3, viz., $[yy0]$ and $[y\,y-1\,0]$. In order to evaluate the normalization constants for the step operators $O_{\alpha\beta}$ it is necessary to evaluate matrix elements of the type

$$\langle [f]\{ST\}|\, E_{-\alpha-\beta}E_{\alpha\beta}\,|[f]\{ST\}\rangle \equiv (\overline{E_{-\alpha-\beta}E_{\alpha\beta}}). \quad \text{(A1)}$$

The curly bracket is again used to denote states with $M_s = S$, $M_T = T$. There are altogether five independent types of such matrix elements, those with $\alpha\beta = 11$, $1-1$, 10, 01, and 00, respectively. One relation among the five can be obtained from the expression for the quadratic Casimir operator, Eq. (22b), which gives

$$2[(\overline{E_{-1-1}E_{11}}) + (\overline{E_{-11}E_{1-1}}) + (\overline{E_{-10}E_{10}}) + (\overline{E_{0-1}E_{01}})] + (\overline{E_{00}E_{00}}) = C(SU_4) - S^2 - T^2 - 4S - 2T. \quad \text{(A2)}$$

The further evaluation of the matrix elements proceeds differently for the different irreducible representations.

The Representation $[yy0]$

In this representation the possible ST values (Table IV) are such that $y - S - T$ = even integer. Neighbor states thus have the property $|\Delta S| + |\Delta T| = 2$, so that

$$O_{\alpha\beta}\,|[yy0]\{ST\}\rangle = 0 \quad \text{if} \quad |\alpha| + |\beta| = 1. \quad \text{(A3)}$$

[23] J. D. Vergados, Nucl. Phys. A111, 681 (1968).

This implies

$$\langle [yy0]\{ST\}|\, O_{-\alpha-\beta}O_{\alpha\beta}\,|[yy0]\{ST\}\rangle = 0 \quad \text{for } |\alpha| + |\beta| = 1. \quad \text{(A4)}$$

With the relations of Tables II and III and the commutation properties of the infinitesimal operators, the four equations (A4) lead to four relations among the matrix elements (A1):

$$(\overline{E_{-10}E_{10}}) = (\overline{E_{-1-1}E_{11}})\frac{1}{(T+1)},$$

$$(\overline{E_{0-1}E_{01}}) = (\overline{E_{-1-1}E_{11}})\frac{1}{(S+1)},$$

$$(\overline{E_{-10}E_{10}}) + (\overline{E_{0-1}E_{01}})\frac{1}{S(T+1)} - (\overline{E_{-11}E_{1-1}})\frac{1}{(T+1)} - (\overline{E_{00}E_{00}})\frac{1}{S} + \frac{T(S+1)}{(T+1)} = 0,$$

$$(\overline{E_{-10}E_{10}})\frac{1}{T(S+1)} + (\overline{E_{0-1}E_{01}}) - (\overline{E_{-11}E_{1-1}})\frac{1}{(S+1)} - (\overline{E_{00}E_{00}})\frac{1}{T} + T = 0. \quad \text{(A5)}$$

Together with (A2) these furnish the five equations needed to evaluate the matrix elements (A1) as functions of y, S, and T. In particular,

$$(\overline{O_{-1-1}O_{11}}) = (\overline{E_{-1-1}E_{11}}) = \frac{(S+1)(T+1)(y-S-T)(y+S+T+4)}{(2S+3)(2T+3)}, \quad \text{(A6)}$$

$$(\overline{O_{-11}O_{1-1}}) = (\overline{E_{-11}E_{1-1}}) - \frac{(\overline{E_{-1-1}E_{11}})}{(T+1)(2T+1)} = \frac{T(S+1)(y+3+S-T)(y+1-S+T)}{(2S+3)(2T+1)}, \quad \text{(A7)}$$

which lead to the normalization coefficients of Table VII.

The Representation $[y\, y-1\, 0]$

The unit step operators $O_{\alpha 0}$, $O_{0\alpha}$ do not give zero when acting on the states of the representation $[y\, y-1\, 0]$, so that the construction of the state vectors is more complicated than in the previous example. However, from the relations

$$\langle [y\, y-1\, 0]\{ST\}|\, O_{-\alpha-\beta}O_{\alpha\beta}\,|[y\, y-1\, 0]\{ST\}\rangle = \langle [y\, y-1\, 0]\{S+\alpha, T+\beta\}| \times O_{\alpha\beta}O_{-\alpha-\beta}\,|[y\, y-1\, 0]\{S+\alpha, T+\beta\}\rangle \quad \text{(A8)}$$

with the four $\alpha\beta$ values 11, $1-1$, 10, and 01, four equations are obtained, which, together with (A2), are sufficient to determine the five needed matrix elements. Equations (A8) are essentially recursion equations relating matrix elements of states with $S+T = y-(k-1)$ (states shown in the kth column of Table V) to matrix elements of states with $S+T = y-(k-1)+\alpha+\beta$ [states in the $(k-\alpha-\beta)$th column of Table V]. General expressions for the matrix elements must thus be evaluated through recursion techniques. For this purpose it is convenient to expand the shorthand notation of Eq. (A1) with a subscript k for states with $S+T = y-(k-1)$, identifying the corresponding kth column of Table V. The recursive process is sketched in this section.

Matrix Elements for States with $k = 1$: $S+T = y$

Since E_{11}, E_{10}, E_{01} give zero when operating on states with $k = 1$,

$$(\overline{E_{-1-1}E_{11}})_1 = (\overline{E_{-10}E_{10}})_1 = (\overline{E_{0-1}E_{01}})_1 = 0. \quad \text{(A9a)}$$

Also, E_{00} commutes with E_{-11}, which is equivalent to O_{-11} when acting on states with $k = 1$. The matrix element $(\overline{E_{00}E_{00}})_1$ is thus independent of S and T and can be evaluated from (A2) applied to the highest weight state. This gives $(\overline{E_{00}E_{00}})_1 = \frac{1}{4}$. With these four matrix elements (A2) can be used to evaluate

$$(\overline{E_{-11}E_{1-1}})_1 = (S+\tfrac{1}{2})(T-\tfrac{1}{2}) \quad \text{with} \quad S+T = y. \quad \text{(A9b)}$$

With the relations of Tables II and III these lead to

$$(\overline{O_{-11}O_{1-1}})_1 = (S+\tfrac{1}{2})(T-\tfrac{1}{2}),$$

$$(\overline{O_{11}O_{-1-1}})_1 = \frac{(S-\frac{1}{2})(T-\frac{1}{2})(S+T+1)}{ST},$$

$$(\overline{O_{10}O_{-10}})_1 = \frac{(S-\frac{1}{2})(S+T+1)}{2S(T+1)}, \quad \text{(A10)}$$

$$(\overline{O_{01}O_{0-1}})_1 = \frac{(T-\frac{1}{2})(S+T+1)}{2T(S+1)}.$$

Matrix Elements for States with $k = 2$: $S+T = y-1$

The basic recursion relation (A8) gives

$$(\overline{O_{0-1}O_{01}})_2 = \langle\{S, T+1\}|\, O_{01}O_{0-1}\,|\{S, T+1\}\rangle_{k=1} = \frac{(T+\frac{1}{2})(S+T+2)}{2(T+1)(S+1)}, \quad \text{(A11a)}$$

$$(\overline{O_{-10}O_{10}})_2 = \langle\{S+1, T\}|\, O_{10}O_{-10}\,|\{S+1, T\}\rangle_{k=1} = \frac{(S+\frac{1}{2})(S+T+2)}{2(S+1)(T+1)}, \quad \text{(A11b)}$$

where the right-hand side follows from (A10). Also, since states with $S+T = y+1$ do not exist,

$$(\overline{O_{-1-1}O_{11}})_2 = 0. \quad \text{(A11c)}$$

Finally, setting $\alpha\beta = 1 - 1$ in the recursion relation (A8), and using the relations of Table II together with the commutation properties of the operators and the results (A11), a recursion relation is obtained for the matrix elements $\overline{(E_{-11}E_{1-1})}_2$:

$$\begin{aligned}\langle\{ST\}|\, E_{-11}E_{1-1}\,|\{ST\}\rangle_{k=2} &= \langle\{S+1, T-1\}|\, E_{-11}E_{1-1}\,|\{S+1, T-1\}\rangle_{k=2} \\ &+ \frac{(y+1)(y+\frac{1}{2})(S-T+1)}{2(S+1)(S+2)T(T+1)} + (S-T+2).\end{aligned} \quad \text{(A12)}$$

With the initial term

$$\langle\{y-\tfrac{3}{2},\tfrac{1}{2}\}|\, E_{-11}E_{1-1}\,|\{y-\tfrac{3}{2},\tfrac{1}{2}\}\rangle_2 = \frac{2(y-1)(y+1)}{3(y-\frac{1}{2})}, \quad \text{(A13)}$$

which follows from $O_{1-1}\,|\{y-\frac{3}{2},\frac{1}{2}\}\rangle = 0$, the result of the recursion process gives

$$\overline{(E_{-11}E_{1-1})}_2 = \frac{(S+\frac{1}{2})(T+\frac{1}{2})(S+T+2)}{(S+1)(T+1)} + (S+\tfrac{1}{2})(T-\tfrac{1}{2}). \quad \text{(A14)}$$

This result, together with Eqs. (A11) and (A2) leads to the five basic matrix elements $\overline{(E_{-\alpha-\beta}E_{\alpha\beta})}_2$ and the remaining matrix elements $\overline{(O_{-\alpha-\beta}O_{\alpha\beta})}_2$ with $\alpha\beta = -1-1, -10, 0-1, -11$.

Matrix Elements for States of Arbitrary k

The recursion equation (A8) relates the matrix elements $\overline{(O_{-1-1}O_{11})}_k$ to those of type $\overline{(O_{11}O_{-1-1})}_{k-2}$, and the matrix elements $\overline{(O_{-10}O_{10})}_k$ and $\overline{(O_{0-1}O_{01})}_k$ to those of type $\overline{(O_{10}O_{-10})}_{k-1}$ and $\overline{(O_{01}O_{0-1})}_{k-1}$. Since matrix elements for states in the $(k-1)$th and $(k-2)$th column of Table V are known, the matrix elements $\overline{(E_{-\alpha-\beta}E_{\alpha\beta})}_k$ with $\alpha\beta = 11, 10$, and 01 can be evaluated. Matrix elements $\overline{(E_{-11}E_{1-1})}_k$ are evaluated from the recursion equation which is the analog of (A12), while those with $\alpha\beta = 00$ then follow from Eq. (A2). From the five basic matrix elements all others follow.

APPENDIX B. TABLES OF $SU(4)$ WIGNER AND RACAH COEFFICIENTS

The $SU(4)$ Wigner coefficients tabulated are those involving products of the special $SU(4)$ representations enumerated in Sec. 3 with the representations [1], and [2] or [11], needed for one- and two-particle cfp's, as well as products with [211], [22], and [422]. Wigner coefficients involving the coupling with [22] and [422] include only the coefficients for the diagonal matrix elements of the corresponding two-body operators used in the applications. In those special cases where the $SU(4)$ Wigner coefficients coincide with the numerical tabulations of Jahn and coworkers,[2,24] there are differences in the phases of the coefficients. Unfortunately, there is no simple relationship between the phase conventions used in this work and the earlier ones of Ref. 2 (which involve many arbitrary choices of sign).

The tables of $SU(4)$ Wigner coefficients are preceded by Table A.0 listing all of the cases covered in the subsequent tables. Other coefficients can be obtained from these through the symmetry properties (I)–(III), Eqs. (38)–(40), and (38′).

The $SU(4)$ U coefficients tabulated are those needed to evaluate diagonal matrix elements of one- and two-body operators; that is,

$$U([f][f^{(2)*}][f][f^{(2)}];\,[f^{(12)}];\,[f_{\rm op}]\rho)$$

with $[f^{(2)}] = [1]$, and [2], or [11], Tables VII and VIII, respectively. The tables include the sums

$$\Sigma_1 = n \sum_{[f^{(n-1)}]} \frac{\mathcal{N}_{[f^{(n-1)}]}}{\mathcal{N}_{[f]}} \times \frac{U([f][1^3][f][1];\,[f^{(n-1)}];\,[f_{\rm op}]\rho)}{U([f][1^3][f][1];\,[f^{(n-1)}];\,[0])} \quad \text{(B1)}$$

and

$$\Sigma_2 = \tfrac{1}{2}n(n-1) \sum_{[f^{(n-2)}]} \frac{\mathcal{N}_{[f^{(n-2)}]}}{\mathcal{N}_{[f]}} \times \frac{U([f][f^{(2)*}][f][f^{(2)}];\,[f^{(n-2)}];\,[f_{\rm op}]\rho)}{U([f][f^{(2)*}][f][f^{(2)}];\,[f^{(n-2)}][0])} \quad \text{(B2)}$$

with $[f^{(2)}] = [2]$ or [11]. The summations are over all possible values of $[f^{(n-1)}]$ or $[f^{(n-2)}]$, that is, over all possible rows of Tables VII and VIII. $\mathcal{N}_{[f]}$ denotes the dimension of the irreducible representation of the symmetric group on n objects described by the Young tableau $[f]$.

The sums Σ_1 are expressed in general form in Eq. (59). From the nature of the operators with irreducible tensor character [422] and [22], Eqs. (36) and (74), it can be seen that the sums Σ_2 with $[f_{\rm op}] = [422]$ or [22] can be functions only of the $SU(4)$ quantum numbers and must be independent of nucleon number n. With $[f_{\rm op}] = [211]$, on the other hand, the sums Σ_2 are functions of the full $U(4)$ representation labels. In Tables VIII, the $SU(4)$ irreducible representation labels $[f_1 - f_4, f_2 - f_4, f_3 - f_4]$ are expressed in terms of the integers y. The label f_4 is replaced by the integer x; the full $U(4)$ tableau is assumed to include x columns of 4.

[24] J. P. Elliott, J. Hope, and H. A. Jahn, Phil. Trans. Roy. Soc. London A246, 241 (1953).

TABLE A.0. Organization of tables of $SU(4)$ Wigner coefficients. $[f^{(1)}] \times [f^{(2)}] \rightarrow [f]$.

$[f^{(1)}]$	$[f^{(2)}]$	$[f]$	Table number
$[yy0]$	$[1]$	$[y+1\ y0]$, $[yy1]$	A1.1
$[y\ y-1\ 0]$	$[1]$	$[yy0]$	A1.2
$[yy1]$	$[1]$	$[y-1\ y-1\ 0]$	A1.3
$[y00]$	$[1]$	$[y+1\ 00]$, $[y10]$	A1.4
$[y10]$	$[1]$	$[y+1\ 10]$, $[y11]$	A1.5
$[y11]$	$[1]$	$[y-1\ 00]$, $[y+1\ 11]$	A1.6
$[yy0]$	$[11]$	$[y+1\ y+1\ 0]$, $[y-1\ y-1\ 0]$	A2.1
$[y00]$	$[11]$	$[y+1\ 10]$, $[y11]$	A2.2
$[y\ y-1\ 0]$	$[11]$	$[y+1\ y0]$, $[yy1]$, $[y-1\ y-2\ 0]$	A2.3
$[y11]$	$[11]$	$[y-1\ 10]$, $[y00]$	A2.4
$[y\ y-1\ 0]$	$[2]$	$[y+1\ y0]$, $[yy1]$	A3.1
$[yy1]$	$[2]$	$[y\ y-1\ 0]$, $[y-1\ y-1\ 1]$	A3.2
$[y00]$	$[2]$	$[y+1\ 10]$, $[y+2\ 00]$	A3.3
$[y-1\ 10]$	$[2]$	$[y+1\ 10]$, $[y11]$	A3.4
$[y11]$	$[2]$	$[y+2\ 11]$, $[y-1\ 10]$, $[y00]$	A3.5
$[yy0]$	$[211]$	$[yy0]$	A4.1
$[y00]$	$[211]$	$[y00]$	A4.2
$[y\ y-1\ 0]$	$[211]$	$[y\ y-1\ 0]$ $\rho = 1, 2,$	A4.3
$[y\ y-1\ 0]$	$[211]$	$[y+1\ y+1\ 1]$	A4.4
$[y10]$	$[211]$	$[y10]$, $\rho = 1, 2,$	A4.5
$[y11]$	$[211]$	$[y11]$, $\rho = 1, 2$	A4.6
$[yy0]$	$[22]$, $[422]$	$[yy0]$	A5.1, 5.2
$[y00]$	$[422]$	$[y00]$	A5.3
$[y\ y-1\ 0]$	$[22]$, $[422]$	$[y\ y-1\ 0]$ $\rho = 1, 2$	A6.1, 6.2
$[y11]$	$[22]$, $[422]$	$[y11]$ $\rho = 1, 2$	A6.3, 6.4

TABLE A1.1.

S_1T_1	$\left\langle \begin{matrix} [yy0]\ [100] \\ S_1T_1;\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\Vert \begin{matrix} [y+1\ y0] \\ ST \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [yy0]\ [100] \\ S_1T_1;\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\Vert \begin{matrix} [yy1] \\ ST \end{matrix} \right\rangle$
$S+\frac{1}{2}\ T+\frac{1}{2}$	$\left[\frac{(y-S-T+1)}{2(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+S+T+3)}{2(y+2)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T-\frac{1}{2}$	$\left[\frac{(y+S+T+3)}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-S-T+1)}{2(y+2)}\right]^{\frac{1}{2}}$
$S+\frac{1}{2}\ T-\frac{1}{2}$	$\left[\frac{(y-S+T+2)}{2(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+S-T+2)}{2(y+2)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T+\frac{1}{2}$	$\left[\frac{(y+S-T+2)}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-S+T+2)}{2(y+2)}\right]^{\frac{1}{2}}$

TABLE A1.2.

S_1T_1	$\left\langle \begin{matrix} [y\ y-1\ 0]\ [100] \\ S_1T_1;\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\Vert \begin{matrix} [yy0] \\ ST \end{matrix} \right\rangle$
$S+\frac{1}{2}\ T+\frac{1}{2}$	$-\left[\frac{(y-S-T)(S+1)(T+1)}{y(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T-\frac{1}{2}$	$-\left[\frac{(y+S+T+2)ST}{y(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S+\frac{1}{2}\ T-\frac{1}{2}$	$\left[\frac{(y-S+T+1)(S+1)T}{y(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T+\frac{1}{2}$	$\left[\frac{(y+S-T+1)S(T+1)}{y(2S+1)(2T+1)}\right]^{\frac{1}{2}}$

TABLE A1.3.

S_1T_1	$\left\langle \begin{matrix} [yy1]\ [100] \\ S_1T_1;\ \frac{1}{2}\frac{1}{2} \end{matrix} \middle\Vert \begin{matrix} [y-1\ y-1\ 0] \\ ST \end{matrix} \right\rangle$
$S+\frac{1}{2}\ T+\frac{1}{2}$	$\left[\frac{(y+S+T+3)(S+1)(T+1)}{(y+3)(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T-\frac{1}{2}$	$-\left[\frac{(y-S-T+1)ST}{(y+3)(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S+\frac{1}{2}\ T-\frac{1}{2}$	$-\left[\frac{(y+S-T+2)(S+1)T}{(y+3)(2S+1)(2T+1)}\right]^{\frac{1}{2}}$
$S-\frac{1}{2}\ T+\frac{1}{2}$	$\left[\frac{(y-S+T+2)S(T+1)}{(y+3)(2S+1)(2T+1)}\right]^{\frac{1}{2}}$

TABLE A1.4.

S_1T_1	$\left\langle \begin{matrix} [y00]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y+1\ 00] \\ ST=S \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [y00]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y10] \\ ST=S \end{matrix} \right\rangle$
$S+\frac12\ S+\frac12$	$\left[\frac{(S+1)(y+1-2S)}{(y+1)(2S+1)}\right]^{\frac12}$	$-\left[\frac{S(y+3+2S)}{(y+1)(2S+1)}\right]^{\frac12}$
$S-\frac12\ S-\frac12$	$\left[\frac{S(y+3+2S)}{(y+1)(2S+1)}\right]^{\frac12}$	$\left[\frac{(S+1)(y+1-2S)}{(y+1)(2S+1)}\right]^{\frac12}$

$$\left\langle \begin{matrix} [y00]\ [100] \\ S-\frac12\ S-\frac12;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y10] \\ S(S-1) \end{matrix} \right\rangle = \left\langle \begin{matrix} [y00]\ [100] \\ S+\frac12\ S+\frac12;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y10] \\ S(S+1) \end{matrix} \right\rangle = +1.$$

TABLE A1.5.

S_1T_1	ST	$\left\langle \begin{matrix} [y10]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y+1\ 10] \\ ST \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [y10]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y11] \\ ST \end{matrix} \right\rangle$
$S+\frac12\ S-\frac12$	SS	$\frac{1}{(2S+1)}\left[\frac{y+1}{y}\right]^{\frac12}$	$\frac{1}{\sqrt2}$
$S-\frac12\ S+\frac12$	SS	$\frac{1}{(2S+1)}\left[\frac{y+1}{y}\right]^{\frac12}$	$-\frac{1}{\sqrt2}$
$S-\frac12\ S-\frac12$	SS	$\frac{[(2S-1)(S+1)(y+2+2S)]^{\frac12}}{(2S+1)[y]^{\frac12}}$	0
$S+\frac12\ S+\frac12$	SS	$\frac{[(2S+3)S(y-2S)]^{\frac12}}{(2S+1)[y]^{\frac12}}$	0
$S-\frac12\ S-\frac32$	$S\ (S-1)$	$\left[\frac{(S-1)(y+2+2S)(y+1)}{(2S-1)y(y+2)}\right]^{\frac12}$	$\left[\frac{(S-1)(y+2-2S)}{2(2S-1)(y+2)}\right]^{\frac12}$
$S-\frac12\ S-\frac12$	$S\ (S-1)$	$-\left[\frac{(y+2+2S)(y+2-2S)}{(2S-1)(2S+1)y(y+2)}\right]^{\frac12}$	$\frac{-S[2(y+1)]^{\frac12}}{[(2S-1)(2S+1)(y+2)]^{\frac12}}$
$S+\frac12\ S-\frac12$	$S\ (S-1)$	$\left[\frac{(S+1)(y+2-2S)(y+1)}{(2S+1)y(y+2)}\right]^{\frac12}$	$-\left[\frac{(S+1)(y+2+2S)}{2(2S+1)(y+2)}\right]^{\frac12}$

$$\left\langle \begin{matrix} [y10]\ [100] \\ T_1S_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y+1\ 10] \\ (S-1)S \end{matrix} \right\rangle = \left\langle \begin{matrix} [y10]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y+1\ 10] \\ S(S-1) \end{matrix} \right\rangle, \left\langle \begin{matrix} [y10]\ [100] \\ T_1S_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y11] \\ (S-1)S \end{matrix} \right\rangle = -\left\langle \begin{matrix} [y10]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [y11] \\ S(S-1) \end{matrix} \right\rangle.$$

TABLE A1.6.

S_1T_1	ST	$\left\langle \begin{matrix} [y11]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [(y-1)\ 00] \\ ST \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [y11]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [(y+1)\ 11] \\ ST \end{matrix} \right\rangle$
$S+\frac12\ S+\frac12$	SS	$\frac{1}{(2S+1)}\left[\frac{(S+1)(2S+3)(y+3+2S)}{3(y+3)}\right]^{\frac12}$	$\frac{1}{(2S+1)}\left[\frac{S(2S+3)(y+1-2S)(y+2)}{y(y+3)}\right]^{\frac12}$
$S-\frac12\ S-\frac12$	SS	$\frac{-1}{(2S+1)}\left[\frac{S(2S-1)(y+1-2S)}{3(y+3)}\right]^{\frac12}$	$\frac{1}{(2S+1)}\left[\frac{(S+1)(2S-1)(y+3+2S)(y+2)}{y(y+3)}\right]^{\frac12}$
$S+\frac12\ S-\frac12$	SS	$\frac{2}{(2S+1)}\left[\frac{S(S+1)(y+2)}{3(y+3)}\right]^{\frac12}$	$\frac{-1}{(2S+1)}\left[\frac{(y+1-2S)(y+3+2S)}{y(y+3)}\right]^{\frac12}$
$S-\frac12\ S+\frac12$	SS	$\frac{2}{(2S+1)}\left[\frac{S(S+1)(y+2)}{3(y+3)}\right]^{\frac12}$	$\frac{-1}{(2S+1)}\left[\frac{(y+1-2S)(y+3+2S)}{y(y+3)}\right]^{\frac12}$
$S+\frac12\ S-\frac12$	$S\ (S-1)$		$\left[\frac{(S+1)(y+1-2S)}{(2S+1)y}\right]^{\frac12}$
$S-\frac12\ S-\frac12$	$S\ (S-1)$		$\left[\frac{(y+2)}{y(2S-1)(2S+1)}\right]^{\frac12}$
$S-\frac12\ S-\frac32$	$S\ (S-1)$		$\left[\frac{(S-1)(y+1+2S)}{(2S-1)y}\right]^{\frac12}$

$$\left\langle \begin{matrix} [y11]\ [100] \\ T_1S_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [(y+1)11] \\ (S-1)S \end{matrix} \right\rangle = \left\langle \begin{matrix} [y11]\ [100] \\ S_1T_1;\ \frac12\frac12 \end{matrix} \middle\| \begin{matrix} [(y+1)11] \\ S(S-1) \end{matrix} \right\rangle.$$

TABLE A2.1.

$S_1T_1;\ S_2T_2$	$\left\langle \begin{matrix}[yy0]\ [110] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y+1\ y+1\ 0] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[yy0]\ [110] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y-1\ y-1\ 0] \\ ST\end{matrix} \right\rangle$
$S+1\ T;\ 10$	$-\left[\frac{(S+1)(y-S+T+2)(y-S-T+1)}{2(y+1)(y+2)(2S+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(S+1)(y+S-T+2)(y+S+T+3)}{2(y+2)(y+3)(2S+1)}\right]^{\frac{1}{2}}$
$S\ T+1;\ 01$	$-\left[\frac{(T+1)(y+S-T+2)(y-S-T+1)}{2(y+1)(y+2)(2T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(T+1)(y-S+T+2)(y+S+T+3)}{2(y+2)(y+3)(2T+1)}\right]^{\frac{1}{2}}$
$S\ T-1;\ 01$	$\left[\frac{T(y+S+T+3)(y-S+T+2)}{2(y+1)(y+2)(2T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{T(y-S-T+1)(y+S-T+2)}{2(y+2)(y+3)(2T+1)}\right]^{\frac{1}{2}}$
$S-1\ T;\ 10$	$\left[\frac{S(y+S+T+3)(y+S-T+2)}{2(y+1)(y+2)(2S+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{S(y-S-T+1)(y-S+T+2)}{2(y+2)(y+3)(2S+1)}\right]^{\frac{1}{2}}$

TABLE A2.2.

$S_1T_1;$	$S_2T_2\ ST$	$\left\langle \begin{matrix}[y00]\ [110] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y+1\ 10] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[y00]\ [110] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y11] \\ ST\end{matrix} \right\rangle$
$SS;$	$10\ SS$	$-1/\sqrt{2}$	$1/\sqrt{2}$
$SS;$	$01\ SS$	$-1/\sqrt{2}$	$-1/\sqrt{2}$
$S-1\ S-1;$	$10\ S(S-1)$	$\left[\frac{y+2+2S}{2(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{y+2-2S}{2(y+2)}\right]^{\frac{1}{2}}$
$SS;$	$01\ S(S-1)$	$-\left[\frac{y+2-2S}{2(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{y+2+2S}{2(y+2)}\right]^{\frac{1}{2}}$
$S+1\ S+1;$	$10\ S(S+1)$	$-\left[\frac{y-2S}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{y+4+2S}{2(y+2)}\right]^{\frac{1}{2}}$
$SS;$	$01\ S(S+1)$	$\left[\frac{y+4+2S}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{y-2S}{2(y+2)}\right]^{\frac{1}{2}}$

TABLE A2.3. $\left\langle \begin{matrix}[y\;y-1\;0] & [110] \\ S_1T_1; & S_2T_2\end{matrix} \middle\| \begin{matrix}[f'] \\ ST\end{matrix}\right\rangle$.

$S_1T_1;\ S_2T_2$	$[f'] = [y+1\;y0]$	$[f'] = [yy1]$	$[f'] = [y-1\;y-2\;0]$
		(a) $y - S - T =$ even integer	
$S+1\;T;\,10$	$-\left[\frac{(2S+3)(y-S-T)(y-S+T+2)}{8y(y+2)(S+1)}\right]^{1/2}$	$-\left[\frac{(2S+3)(y-S-T)(y+S-T+2)}{16(y+1)(y+2)(S+1)}\right]^{1/2}$	$\left[\frac{(2S+3)(y+S+T+2)(y+S-T+2)}{8(y+1)(y+3)(S+1)}\right]^{1/2}$
$ST;\,10$	$-\left[\frac{(y-S+T+2)(y+S-T+2)}{8y(y+2)S(S+1)}\right]^{1/2}$	$\frac{\{(y+1)(2S+1)+S+T+1\}}{4[(y+1)(y+2)S(S+1)]^{1/2}}$	$\left[\frac{(y+S+T+2)(y-S-T)}{8(y+1)(y+3)S(S+1)}\right]^{1/2}$
$S-1\;T;\,10$	$\left[\frac{(2S-1)(y+S+T+2)(y+S-T+2)}{8y(y+2)S}\right]^{1/2}$	$-\left[\frac{(2S-1)(y+S+T+2)(y-S+T+2)}{16(y+1)(y+2)S}\right]^{1/2}$	$-\left[\frac{(2S-1)(y-S-T)(y-S+T+2)}{8(y+1)(y+3)S}\right]^{1/2}$
$S\;T+1;\,01$	$-\left[\frac{(2T+3)(y-S-T)(y+S-T+2)}{8y(y+2)(T+1)}\right]^{1/2}$	$\left[\frac{(2T+3)(y-S-T)(y-S+T+2)}{16(y+1)(y+2)(T+1)}\right]^{1/2}$	$-\left[\frac{(2T+3)(y+S+T+2)(y-S+T+2)}{8(y+1)(y+3)(T+1)}\right]^{1/2}$
$ST;\,01$	$-\left[\frac{(y+S-T+2)(y-S+T+2)}{8y(y+2)T(T+1)}\right]^{1/2}$	$\frac{-\{(y+1)(2T+1)+S+T+1\}}{4[(y+1)(y+2)T(T+1)]^{1/2}}$	$-\left[\frac{(y-S-T)(y+S+T+2)}{8(y+1)(y+3)T(T+1)}\right]^{1/2}$
$S\;T-1;\,01$	$\left[\frac{(2T-1)(y+S+T+2)(y-S+T+2)}{8y(y+2)T}\right]^{1/2}$	$\left[\frac{(2T-1)(y+S+T+2)(y+S-T+2)}{16(y+1)(y+2)T}\right]^{1/2}$	$\left[\frac{(2T-1)(y-S-T)(y+S-T+2)}{8(y+1)(y+3)T}\right]^{1/2}$
		(b) $y - S - T =$ odd integer	
$S+1\;T;\,10$	$-\left[\frac{(2S+3)(y-S-T+1)(y-S+T+1)}{8y(y+2)(S+1)}\right]^{1/2}$	$-\left[\frac{(2S+3)(y-S+T+1)(y+S+T+3)}{16(y+1)(y+2)(S+1)}\right]^{1/2}$	$\left[\frac{(2S+3)(y+S+T+3)(y+S-T+1)}{8(y+1)(y+3)(S+1)}\right]$
$ST;\,10$	$-\left[\frac{(y+S+T+3)(y-S-T+1)}{8y(y+2)S(S+1)}\right]^{1/2}$	$\frac{\{(y+1)(2S+1)+S-T\}}{4[(y+1)(y+2)S(S+1)]^{1/2}}$	$\left[\frac{(y-S+T+1)(y+S-T+1)}{8(y+1)(y+3)S(S+1)}\right]^{1/2}$
$S-1\;T;\,10$	$\left[\frac{(2S-1)(y+S+T+3)(y+S-T+1)}{8y(y+2)S}\right]^{1/2}$	$-\left[\frac{(2S-1)(y+S-T+1)(y-S-T+1)}{16(y+1)(y+2)S}\right]^{1/2}$	$-\left[\frac{(2S-1)(y-S-T+1)(y-S+T+1)}{8(y+1)(y+3)S}\right]^{1/2}$
$S\;T+1;\,01$	$-\left[\frac{(2T+3)(y-S-T+1)(y+S-T+1)}{8y(y+2)(T+1)}\right]^{1/2}$	$-\left[\frac{(2T+3)(y+S-T+1)(y+S+T+3)}{16(y+1)(y+2)(T+1)}\right]^{1/2}$	$-\left[\frac{(2T+3)(y-S+T+1)(y+S+T+3)}{8(y+1)(y+3)(T+1)}\right]^{1/2}$
$ST;\,01$	$-\left[\frac{(y+S+T+3)(y-S-T+1)}{8y(y+2)T(T+1)}\right]^{1/2}$	$\frac{\{(y+1)(2T+1)-S+T\}}{4[(y+1)(y+2)T(T+1)]^{1/2}}$	$-\left[\frac{(y+S-T+1)(y-S+T+1)}{8(y+1)(y+3)T(T+1)}\right]^{1/2}$
$S\;T-1;\,01$	$\left[\frac{(2T-1)(y+S+T+3)(y-S+T+1)}{8y(y+2)T}\right]^{1/2}$	$-\left[\frac{(2T-1)(y-S+T+1)(y-S-T+1)}{16(y+1)(y+2)T}\right]^{1/2}$	$\left[\frac{(2T-1)(y-S-T+1)(y+S-T+1)}{8(y+1)(y+3)T}\right]^{1/2}$

TABLE A2.4.

S_1T_1;	S_2T_2	ST	$\left\langle \begin{matrix} [y11] & [110] \\ S_1T_1; & S_2T_2 \end{matrix} \right\Vert \left. \begin{matrix} [y-1\ 10] \\ ST \end{matrix} \right\rangle$	$\left\langle \begin{matrix} [y11] & [110] \\ S_1T_1; & S_2T_2 \end{matrix} \right\Vert \left. \begin{matrix} [y00] \\ ST \end{matrix} \right\rangle$
$S+1\ S$; $S\ S+1$;	10 01	SS SS	$\left[\frac{(2S+3)S(y+2+2S)(y+2)}{4(S+1)(2S+1)y(y+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S+3)(y-2S)}{6y(2S+1)}\right]^{\frac{1}{2}}$
$S\ S$; $S\ S$;	10 01	SS SS	$\left[\frac{(y-2S)(y+2+2S)}{4S(S+1)y(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{y+2}{6y}\right]^{\frac{1}{2}}$
$S-1\ S$; $S\ S-1$;	10 01	SS SS	$\left[\frac{(2S-1)(S+1)(y-2S)(y+2)}{4S(2S+1)y(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)(y+2+2S)}{6y(2S+1)}\right]^{\frac{1}{2}}$
$S\ S-1$; $S-1\ S$;	10 01	$S(S-1)$ $(S-1)S$	$-\left[\frac{(y+2)(S+1)}{4S(y+3)}\right]^{\frac{1}{2}}$	
$S-1\ S-1$; $S-1\ S-1$;	10 01	$S(S-1)$ $(S-1)S$	$\left[\frac{(S-1)(y+2-2S)}{4S(y+3)}\right]^{\frac{1}{2}}$	
$S\ S$; $S\ S$;	01 10	$S(S-1)$ $(S-1)S$	$\left[\frac{(S+1)(y+2+2S)}{4S(y+3)}\right]^{\frac{1}{2}}$	
$S\ S-1$; $S-1\ S$;	01 10	$S(S-1)$ $(S-1)S$	$\left[\frac{(y+2)(S-1)}{4S(y+3)}\right]^{\frac{1}{2}}$	

TABLE A3.1.

$S_1T_1;\quad S_2T_2$	$\left\langle \begin{matrix}[y\,y-1\,0]\,[200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[y+1\,y\,0] \\ ST\end{matrix}\right\rangle$	$\left\langle \begin{matrix}[y\,y-1\,0]\,[200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[yy1] \\ ST\end{matrix}\right\rangle$
	(a) $y - S - T =$ even integer	
$S+1\ T+1;\ 11$	0	0
$S+1\ T;\ 11$	$-\left[\frac{(2S+3)(T+1)(y-S-T)(y-S+T+2)}{24y(y+2)(S+1)T}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S+3)(T+1)(y-S-T)(y+S-T+2)}{16(y-1)(y+2)(S+1)T}\right]^{\frac{1}{2}}$
$S+1\ T-1;\ 11$	$\left[\frac{(2S+3)(2T-1)(y-S+T)(y-S+T+2)}{24y(y+2)(S+1)T}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)(2T-1)(y-S+T)(y+S-T+2)}{16(y-1)(y+2)(S+1)T}\right]^{\frac{1}{2}}$
$S-1\ T+1;\ 11$	$\left[\frac{(2S-1)(2T+3)(y+S-T)(y+S-T+2)}{24y(y+2)S(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S-1)(2T+3)(y+S-T)(y-S+T+2)}{16(y-1)(y+2)S(T+1)}\right]^{\frac{1}{2}}$
$S-1\ T;\ 11$	$-\left[\frac{(2S-1)T(y+S+T+2)(y+S-T+2)}{24y(y+2)S(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)T(y+S+T+2)(y-S+T+2)}{16(y-1)(y+2)S(T+1)}\right]^{\frac{1}{2}}$
$S-1\ T-1;\ 11$	0	0
$S\ T+1;\ 11$	$-\left[\frac{(S+1)(2T+3)(y-S-T)(y+S-T+2)}{24y(y+2)S(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(S+1)(2T+3)(y-S-T)(y-S+T+2)}{16(y-1)(y+2)S(T+1)}\right]^{\frac{1}{2}}$
$S\ T;\ 11$	$\{2ST+S+T\} \times \left[\frac{(y+S-T+2)(y-S+T+2)}{24y(y+2)S(S+1)T(T+1)}\right]^{\frac{1}{2}}$	$\frac{(S-T)\{2ST+S+T+y+2\}}{[16(y-1)(y+2)S(S+1)T(T+1)]^{\frac{1}{2}}}$
$S\ T-1;\ 11$	$-\left[\frac{S(2T-1)(y+S+T+2)(y-S+T+2)}{24y(y+2)(S+1)T}\right]^{\frac{1}{2}}$	$-\left[\frac{S(2T-1)(y+S+T+2)(y+S-T+2)}{16(y-1)(y+2)(S+1)T}\right]^{\frac{1}{2}}$
$S\ T;\ 00$	$\left[\frac{(y+S-T+2)(y-S+T+2)}{6y(y+2)}\right]^{\frac{1}{2}}$	$\frac{(S-T)}{[4(y-1)(y+2)]^{\frac{1}{2}}}$
	(b) $y - S - T =$ odd integer	
$S+1\ T+1;\ 11$	$-\left[\frac{(2S+3)(2T+3)(y-S-T-1) \times (y-S-T+1)}{24y(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S+3)(2T+3)(y-S-T-1) \times (y+S+T+3)}{16(y-1)(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S+1\ T;\ 11$	$\left[\frac{(2S+3)T(y-S+T+1)(y-S-T+1)}{24y(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)T(y-S+T+1)(y+S+T+3)}{16(y-1)(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S+1\ T-1;\ 11$	0	0
$S-1\ T+1;\ 11$	0	0
$S-1\ T;\ 11$	$\left[\frac{(2S-1)(T+1)(y+S-T+1) \times (y+S+T+3)}{24y(y+2)ST}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S-1)(T+1)(y+S-T+1) \times (y-S-T+1)}{16(y-1)(y+2)ST}\right]^{\frac{1}{2}}$
$S-1\ T-1;\ 11$	$-\left[\frac{(2S-1)(2T-1)(y+S+T+1) \times (y+S+T+3)}{24y(y+2)ST}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)(2T-1)(y+S+T+1) \times (y-S-T+1)}{16(y-1)(y+2)ST}\right]^{\frac{1}{2}}$
$S\ T+1;\ 11$	$\left[\frac{S(2T+3)(y+S-T+1)(y-S-T+1)}{24y(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{S(2T+3)(y+S-T+1)(y+S+T+3)}{16(y-1)(y+2)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S\ T;\ 11$	$-\{2ST+S+T+1\} \times \left[\frac{(y+S+T+3)(y-S-T+1)}{24y(y+2)S(S+1)T(T+1)}\right]^{\frac{1}{2}}$	$\frac{(S+T+1)\{y+1-S-T-2ST\}}{[16(y-1)(y+2)S(S+1)T(T+1)]^{\frac{1}{2}}}$
$S\ T-1;\ 11$	$\left[\frac{(S+1)(2T-1)(y-S+T+1) \times (y+S+T+3)}{24y(y+2)ST}\right]^{\frac{1}{2}}$	$-\left[\frac{(S+1)(2T-1)(y-S-T+1) \times (y-S+T+1)}{16(y-1)(y+2)ST}\right]^{\frac{1}{2}}$
$S\ T;\ 00$	$\left[\frac{(y+S+T+3)(y-S-T+1)}{6y(y+2)}\right]^{\frac{1}{2}}$	$\frac{(S+T+1)}{[4(y-1)(y+2)]^{\frac{1}{2}}}$

TABLE A3.2.

$S_1T_1;\quad S_2T_2$	$\left\langle \begin{matrix}[yy1]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y\,y-1\,0] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[yy1]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y-1\,y-1\,1] \\ ST\end{matrix} \right\rangle$
	(a) $y-S-T=$ even integer	
$S+1\ T+1;\ 11$	$\left[\frac{(2S+3)(2T+3)(y-S-T)(y+S+T+4)}{16(y+1)(y+4)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)(2T+3)(y+S+T+2)\times(y+S+T+4)}{24(y+1)(y+3)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S+1\ T;\ 11$	$-\left[\frac{(2S+3)T(y-S-T)(y+S-T+2)}{16(y+1)(y+4)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S+3)T(y+S+T+2)(y+S-T+2)}{24(y+1)(y+3)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S+1\ T-1;\ 11$	0	0
$S-1\ T+1;\ 11$	0	0
$S-1\ T;\ 11$	$\left[\frac{(2S-1)(T+1)(y-S+T+2)\times(y+S+T+2)}{16(y+1)(y+4)ST}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S-1)(T+1)(y-S-T)(y-S+T+2)}{24(y+1)(y+3)ST}\right]^{\frac{1}{2}}$
$S-1\ T-1;\ 11$	$-\left[\frac{(2S-1)(2T-1)(y-S-T+2)\times(y+S+T+2)}{16(y+1)(y+4)ST}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)(2T-1)(y-S-T)(y-S-T+2)}{24(y+1)(y+3)ST}\right]^{\frac{1}{2}}$
$S\ T+1;\ 11$	$\left[\frac{S(2T+3)(y-S-T)(y-S+T+2)}{16(y+1)(y+4)(S+1)(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{S(2T+3)(y-S+T+2)(y+S+T+2)}{24(y+1)(y+3)(S+1)(T+1)}\right]^{\frac{1}{2}}$
$S\ T;\ 11$	$\frac{(S+T+1)\{y+2+S+T+2ST\}}{[16(y+1)(y+4)S(S+1)T(T+1)]^{\frac{1}{2}}}$	$-\{2ST+S+T+1\}\times\left[\frac{(y-S-T)(y+S+T+2)}{24(y+1)(y+3)S(S+1)T(T+1)}\right]^{\frac{1}{2}}$
$S\ T-1;\ 11$	$-\left[\frac{(S+1)(2T-1)(y+S+T+2)\times(y+S-T+2)}{16(y+1)(y+4)ST}\right]^{\frac{1}{2}}$	$\left[\frac{(S+1)(2T-1)(y-S-T)(y+S-T+2)}{24(y+1)(y+3)ST}\right]^{\frac{1}{2}}$
$S\ T;\ 00$	$\frac{-(S+T+1)}{[4(y+1)(y+4)]^{\frac{1}{2}}}$	$\left[\frac{(y-S-T)(y+S+T+2)}{6(y+1)(y+3)}\right]^{\frac{1}{2}}$
	(b) $y-S-T=$ odd integer	
$S+1\ T+1;\ 11$	0	0
$S+1\ T;\ 11$	$\left[\frac{(2S+3)(T+1)(y-S+T+1)\times(y+S+T+3)}{16(y+1)(y+4)(S+1)T}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)(T+1)(y+S+T+3)\times(y+S-T+1)}{24(y+1)(y+3)(S+1)T}\right]^{\frac{1}{2}}$
$S+1\ T-1;\ 11$	$-\left[\frac{(2S+3)(2T-1)(y-S+T+1)\times(y+S-T+3)}{16(y+1)(y+4)(S+1)T}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S+3)(2T-1)(y+S-T+3)\times(y+S-T+1)}{24(y+1)(y+3)(S+1)T}\right]^{\frac{1}{2}}$
$S-1\ T+1;\ 11$	$\left[\frac{(2S-1)(2T+3)(y+S-T+1)\times(y-S+T+3)}{16(y+1)(y+4)S(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S-1)(2T+3)(y-S+T+3)\times(y-S+T+1)}{24(y+1)(y+3)S(T+1)}\right]^{\frac{1}{2}}$
$S-1\ T;\ 11$	$-\left[\frac{(2S-1)T(y-S-T+1)(y+S-T+1)}{16(y+1)(y+4)S(T+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)T(y-S-T+1)(y-S+T+1)}{24(y+1)(y+3)S(T+1)}\right]^{\frac{1}{2}}$
$S-1\ T-1;\ 11$	0	0
$S\ T+1;\ 11$	$\left[\frac{(S+1)(2T+3)(y+S+T+3)\times(y+S-T+1)}{16(y+1)(y+4)S(T+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(S+1)(2T+3)(y+S+T+3)\times(y-S+T+1)}{24(y+1)(y+3)S(T+1)}\right]^{\frac{1}{2}}$
$S\ T;\ 11$	$\frac{(S-T)\{y+1-S-T-2ST\}}{[16(y+1)(y+4)S(S+1)T(T+1)]^{\frac{1}{2}}}$	$\{2ST+S+T\}\times\left[\frac{(y+S-T+1)(y-S+T+1)}{24(y+1)(y+3)S(S+1)T(T+1)}\right]^{\frac{1}{2}}$
$S\ T-1;\ 11$	$-\left[\frac{S(2T-1)(y-S-T+1)(y-S+T+1)}{16(y+1)(y+4)(S+1)T}\right]^{\frac{1}{2}}$	$-\left[\frac{S(2T-1)(y-S-T+1)(y+S-T+1)}{24(y+1)(y+3)(S+1)T}\right]^{\frac{1}{2}}$
$S\ T;\ 00$	$\frac{-(S-T)}{[4(y+1)(y+4)]^{\frac{1}{2}}}$	$\left[\frac{(y+S-T+1)(y-S+T+1)}{6(y+1)(y+3)}\right]^{\frac{1}{2}}$

TABLE A3.3.

$S_1T_1;$	S_2T_2	ST	$\left\langle \begin{matrix}[y00]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y+1\ 10] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[y00]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y+2\ 00] \\ ST\end{matrix} \right\rangle$
$S+1\ S+1;$	11	SS	$-\left[\frac{S(2S+3)(y-2S)(y+4+2S)}{2y(y+2)(S+1)(2S+1)}\right]^{1/2}$	$\left[\frac{(2S+3)(y-2S)(y+2-2S)}{4(y+1)(y+2)(2S+1)}\right]^{1/2}$
$S-1\ S-1;$	11	SS	$\left[\frac{(S+1)(2S-1)(y+2+2S)(y+2-2S)}{2y(y+2)S(2S+1)}\right]^{1/2}$	$\left[\frac{(2S-1)(y+2+2S)(y+4+2S)}{4(y+1)(y+2)(2S+1)}\right]^{1/2}$
$S\ S;$	11	SS	$\frac{\{y+2-2S(S+1)\}}{[2y(y+2)S(S+1)]^{1/2}}$	$\left[\frac{(y+2-2S)(y+4+2S)}{4(y+1)(y+2)}\right]^{1/2}$
$S\ S;$	00	SS	$-\left[\frac{2S(S+1)}{y(y+2)}\right]^{1/2}$	$\left[\frac{(y+2-2S)(y+4+2S)}{4(y+1)(y+2)}\right]^{1/2}$
$S\ S;$	11	$S(S-1)$	$\left[\frac{(S+1)(y+2-2S)}{2yS}\right]^{1/2}$	
$S\ S;$	11	$(S-1)S$		
$S-1\ S-1;$	11	$S(S-1)$	$\left[\frac{(S-1)(y+2+2S)}{2yS}\right]^{1/2}$	
$S-1\ S-1;$	11	$(S-1)S$		

TABLE A3.4.

$S_1T_1;$	S_2T_2	ST	$\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y+1\ 10] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y11] \\ ST\end{matrix} \right\rangle$
$S+1\ S+1;$	11	SS	$\frac{1}{2(S+1)}\left[\frac{S(S+2)(2S+3)(y-2-2S)(y-2S)}{y(y-1)(2S+1)}\right]^{1/2}$	0
$S+1\ S;$	11	SS	$\frac{1}{2(S+1)}\left[\frac{(2S+3)(y-2S)}{(y-1)(2S+1)}\right]^{1/2}$	$\pm\left[\frac{(2S+3)(y-2S)}{4(y-1)(2S+1)}\right]^{1/2}$
$S\ S+1;$	11	SS		
$S\ S;$	11	SS	$\frac{\{S^2+S-1\}}{2S(S+1)}\left[\frac{(y-2S)(y+2+2S)}{y(y-1)}\right]^{1/2}$	0
$S\ S-1;$	11	SS	$\frac{1}{2S}\left[\frac{(2S-1)(y+2+2S)}{(y-1)(2S+1)}\right]^{1/2}$	$\pm\left[\frac{(2S-1)(y+2+2S)}{4(y-1)(2S+1)}\right]^{1/2}$
$S-1\ S;$	11	SS		
$S-1\ S-1;$	11	SS	$\frac{1}{2S}\left[\frac{(S-1)(S+1)(2S-1)(y+2+2S)(y+2S)}{y(y-1)(2S+1)}\right]^{1/2}$	0
$S\ S;$	00	SS	$\left[\frac{(y-2S)(y+2+2S)}{4y(y-1)}\right]^{1/2}$	0
$S+1\ S;$	11	$S\ S-1$	$\left[\frac{(2S+3)(y-2S)(y+2-2S)}{4(y-1)(y+2)(2S+1)}\right]^{1/2}$	$-\left[\frac{(2S+3)(y-2S)(y+2+2S)}{4(y-1)(y+2)(2S+1)}\right]^{1/2}$
$S\ S;$	11	$S\ S-1$	$\frac{-1}{2S}\left[\frac{(y-2S)(y+2+2S)(y+2-2S)}{y(y-1)(y+2)}\right]^{1/2}$	$-\left[\frac{y(y-2S)}{4(y-1)(y+2)}\right]^{1/2}$
$S\ S-1;$	11	$S\ S-1$	$\frac{1}{2S}\left[\frac{(S-1)(S+1)(y+2+2S)(y+2-2S)}{(y-1)(y+2)}\right]^{1/2}$	$-\left[\frac{(S-1)(S+1)}{(y-1)(y+2)}\right]^{1/2}$
$S-1\ S;$	11	$S\ S-1$	$\frac{-1}{2S}\left[\frac{(y+2+2S)(y+2-2S)}{(y-1)(y+2)(2S+1)(2S-1)}\right]^{1/2}$	$\frac{-(y+1)}{[(y-1)(y+2)(2S-1)(2S+1)]^{1/2}}$
$S-1\ S-1;$	11	$S\ S-1$	$\frac{-1}{2S}\left[\frac{(y+2S)(y+2+2S)(y+2-2S)}{y(y-1)(y+2)}\right]^{1/2}$	$-\left[\frac{y(y+2S)}{4(y-1)(y+2)}\right]^{1/2}$
$S-1\ S-2;$	11	$S\ S-1$	$\left[\frac{(2S-3)(y+2S)(y+2+2S)}{4(2S-1)(y-1)(y+2)}\right]^{1/2}$	$\left[\frac{(2S-3)(y+2S)(y+2-2S)}{4(2S-1)(y-1)(y+2)}\right]^{1/2}$
$S\ S-1;$	00	$S\ S-1$	$\left[\frac{(y+2+2S)(y+2-2S)}{4(y-1)(y+2)}\right]^{1/2}$	$\frac{-S}{[(y-1)(y+2)]^{1/2}}$

$$\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ T_1S_1;\ T_2S_2\end{matrix} \right\Vert \left. \begin{matrix}[y+1\ 10] \\ (S-1)S\end{matrix} \right\rangle = +\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y+1\ 10] \\ S(S-1)\end{matrix} \right\rangle;$$
$$\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ T_1S_1;\ T_2S_2\end{matrix} \right\Vert \left. \begin{matrix}[y11] \\ (S-1)S\end{matrix} \right\rangle = -\left\langle \begin{matrix}[y-1\ 10]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \right\Vert \left. \begin{matrix}[y11] \\ S(S-1)\end{matrix} \right\rangle.$$

Table A3.5.

S_1T_1;	S_2T_2	ST	$\left\langle \begin{matrix}[y11]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[y+2\ 11] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[y11]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[y-1\ 10] \\ ST\end{matrix} \right\rangle$	$\left\langle \begin{matrix}[y11]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[y00] \\ SS\end{matrix} \right\rangle$
$S+1\ S+1$;	11	SS	$\left[\frac{S(S+2)(2S+3)(y+2-2S)(y-2S)(y+2)}{4y(y+1)(y+4)(2S+1)(S+1)^2}\right]^{1/2}$	$-\left[\frac{S(S+2)(2S+3)(y+2+2S)(y+4+2S)}{8y(y+3)(2S+1)(S+1)^2}\right]^{1/2}$	$\left[\frac{(S+2)(2S+3)(y-2S)(y+4+2S)}{6y(y+4)(S+1)(2S+1)}\right]^{1/2}$
$S+1\ S$; $S\ S+1$;	11 11	SS SS	$-\left[\frac{(2S+3)(y-2S)(y+2-2S)(y+4+2S)}{4y(y+1)(y+4)(2S+1)(S+1)^2}\right]^{1/2}$	$-\left[\frac{S^2(2S+3)(y+2+2S)(y+2)}{8y(y+3)(2S+1)(S+1)^2}\right]^{1/2}$	$\left[\frac{S(2S+3)(y-2S)(y+2)}{6y(y+4)(S+1)(2S+1)}\right]^{1/2}$
$S\ S$;	11	SS	$\frac{\{S(S+1)-1\}}{2S(S+1)}\left[\frac{(y+2-2S)(y+4+2S)(y+2)}{y(y+1)(y+4)}\right]^{1/2}$	$\frac{\{S(S+1)+1\}}{2S(S+1)}\left[\frac{(y-2S)(y+2+2S)}{2y(y+3)}\right]^{1/2}$	$\frac{\{2S(S+1)+(y+2)\}}{[6y(y+4)S(S+1)]^{1/2}}$
$S\ S-1$; $S-1\ S$;	11 11	SS SS	$-\left[\frac{(2S-1)(y+2-2S)(y+2+2S)(y+4+2S)}{4y(y+1)(y+4)(2S+1)S^2}\right]^{1/2}$	$\left[\frac{(S+1)^2(2S-1)(y-2S)(y+2)}{8y(y+3)(2S+1)S^2}\right]^{1/2}$	$\left[\frac{(S+1)(2S-1)(y+2+2S)(y+2)}{6y(y+4)S(2S+1)}\right]^{1/2}$
$S-1\ S-1$;	11	SS	$\left[\frac{(S-1)(S+1)(2S-1)(y+2+2S)(y+4+2S)(y+2)}{4y(y+1)(y+4)(2S+1)S^2}\right]^{1/2}$	$-\left[\frac{(S-1)(S+1)(2S-1)(y+2-2S)(y-2S)}{8y(y+3)(2S+1)S^2}\right]^{1/2}$	$-\left[\frac{(S-1)(2S-1)(y+2+2S)(y+2-2S)}{6y(y+4)S(2S+1)}\right]^{1/2}$
$S\ S$;	00	SS	$\left[\frac{(y+2-2S)(y+4+2S)(y+2)}{4y(y+1)(y+4)}\right]^{1/2}$	$\left[\frac{(y-2S)(y+2+2S)}{8y(y+3)}\right]^{1/2}$	$\left[\frac{2S(S+1)}{3y(y+4)}\right]^{1/2}$
$S+1\ S$;	11	$S(S-1)$	$\left[\frac{(2S+3)(y+2-2S)(y-2S)}{4y(y+1)(2S+1)}\right]^{1/2}$	0	
$S\ S$;	11	$S(S-1)$	$\frac{1}{2S}\left[\frac{(y+2)(y+2-2S)}{y(y+1)}\right]^{1/2}$	$\frac{(S+1)}{2S}\left[\frac{(y+2+2S)}{2(y+3)}\right]^{1/2}$	
$S\ S-1$;	11	$S(S-1)$	$\frac{1}{2S}\left[\frac{(S+1)(S-1)(y+2+2S)(y+2-2S)}{y(y+1)}\right]^{1/2}$	$\frac{1}{2S}\left[\frac{(S+1)(S-1)(y+2)}{2(y+3)}\right]^{1/2}$	
$S-1\ S$;	11	$S(S-1)$	$\frac{-1}{2S}\left[\frac{(y+2+2S)(y+2-2S)}{y(y+1)(2S-1)(2S+1)}\right]^{1/2}$	$\frac{1}{2S}\left[\frac{(2S-1)(2S+1)(y+2)}{2(y+3)}\right]^{1/2}$	
$S-1\ S-1$;	11	$S(S-1)$	$\frac{1}{2S}\left[\frac{(y+2)(y+2+2S)}{y(y+1)}\right]^{1/2}$	$-\frac{(S-1)}{2S}\left[\frac{(y+2-2S)}{2(y+3)}\right]^{1/2}$	
$S-1\ S-2$;	11	$S(S-1)$	$\left[\frac{(2S-3)(y+2+2S)(y+2S)}{4y(y+1)(2S-1)}\right]^{1/2}$	0	
$S\ S-1$;	00	$S(S-1)$	$\left[\frac{(y+2+2S)(y+2-2S)}{4y(y+1)}\right]^{1/2}$	$-\left[\frac{(y+2)}{8(y+3)}\right]^{1/2}$	

$$\left\langle \begin{matrix}[y11]\ [200] \\ T_1S_1;\ T_2S_2\end{matrix} \middle\Vert \begin{matrix}[f'] \\ (S-1)S\end{matrix} \right\rangle = \left\langle \begin{matrix}[y11]\ [200] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\Vert \begin{matrix}[f'] \\ S(S-1)\end{matrix} \right\rangle \quad \text{for both} \quad \begin{matrix}[f'] = [y+2\ 11], \\ [f'] = [y-1\ 10].\end{matrix}$$

TABLE A4.1.

S_1T_1; S_2T_2	$\left\langle \begin{matrix} [yy0] & [211] \\ S_1T_1; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [yy0] \\ ST \end{matrix} \right\rangle$
$S+1\ T+1$; 11	$-\left[\frac{(S+1)(T+1)(y-S-T)(y+S+T+4)}{(2S+1)(2T+1)y(y+4)}\right]^{\frac{1}{2}}$
$S+1\ T-1$; 11	$\left[\frac{(S+1)T(y-S+T+1)(y+S-T+3)}{(2S+1)(2T+1)y(y+4)}\right]^{\frac{1}{2}}$
$S-1\ T+1$; 11	$\left[\frac{S(T+1)(y+S-T+1)(y-S+T+3)}{(2S+1)(2T+1)y(y+4)}\right]^{\frac{1}{2}}$
$S-1\ T-1$; 11	$-\left[\frac{ST(y-S-T+2)(y+S+T+2)}{(2S+1)(2T+1)y(y+4)}\right]^{\frac{1}{2}}$
$S\ T$; 10	$\left[\frac{S(S+1)}{y(y+4)}\right]^{\frac{1}{2}}$
$S\ T$; 01	$\left[\frac{T(T+1)}{y(y+4)}\right]^{\frac{1}{2}}$

TABLE A4.2.

S_1T_1; S_2T_2	$\left\langle \begin{matrix} [y00] & [211] \\ S_1T_1; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [y00] \\ ST \end{matrix} \right\rangle$
$S+1\ S+1$; 11	$-\left[\frac{(2S+3)(y-2S)(y+4+2S)}{(2S+1)3y(y+4)}\right]^{\frac{1}{2}}$
$S\ S$; 11	$\frac{-(y+2)}{[3y(y+4)]^{\frac{1}{2}}}$
$S-1\ S-1$; 11	$-\left[\frac{(2S-1)(y+2-2S)(y+2+2S)}{(2S+1)3y(y+4)}\right]^{\frac{1}{2}}$
$S\ S$; 10 $S\ S$; 01	$\left[\frac{4S(S+1)}{3y(y+4)}\right]^{\frac{1}{2}}$

TABLE A4.3.

$$\left\langle \begin{matrix} [y\, y-1\, 0] & [211] \\ S_1T_1; & S_2T_2 \end{matrix} \right\| \left. \begin{matrix} [y\, y-1\, 0] \\ ST \end{matrix} \right\rangle$$

S_1T_1; S_2T_2	$\rho = 1$	$\rho = 2$
	(a) $y - S - T =$ even integer	
$S+1\ T+1$; 11	$-\left[\frac{(2S+3)(2T+3)(y-S-T)(y+S+T+4)}{4(S+1)(T+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{(y+2)(2y-7)[(2S+3)(2T+3)(y-S-T)(y+S+T+4)]^{\frac{1}{2}}}{4[6(S+1)(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S+1\ T$; 11	$-\left[\frac{(2S+3)(y-S-T)(y+S-T+2)}{4(S+1)T(T+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{(2y+13)(y+1)+T(4y^2+12y-1)\}[(2S+3)(y-S-T)(y+S-T+2)]^{\frac{1}{2}}}{4[6(S+1)T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S+1\ T-1$; 11	$\left[\frac{(2S+3)(2T-1)(y-S+T)(y+S-T+2)}{4(S+1)T(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{(2y+13)(y+1)[(2S+3)(2T-1)(y-S+T)(y+S-T+2)]^{\frac{1}{2}}}{4[6(S+1)T(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S\ T+1$; 11	$-\left[\frac{(2T+3)(y-S-T)(y-S+T+2)}{4S(S+1)(T+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{(2y+13)(y+1)+S(4y^2+12y-1)\}[(2T+3)(y-S-T)(y-S+T+2)]^{\frac{1}{2}}}{4[6S(S+1)(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S\ T$; 11	$-\frac{\{(y+\frac{3}{2})+2(S+\frac{1}{2})(T+\frac{1}{2})\}}{[4S(S+1)T(T+1)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{\left\{\begin{matrix}\frac{1}{2}(16y^2+48y+41)-[S(S+1)+T(T+1)](4y^2+12y-1) \\ -2(S+\frac{1}{2})(T+\frac{1}{2})(2y+3)(2y^2+6y-5)\end{matrix}\right\}}{4[6S(S+1)T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S\ T-1$; 11	$\left[\frac{(2T-1)(y+S-T+2)(y+S+T+2)}{4S(S+1)T(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{(2y-7)(y+2)+S(4y^2+12y-1)\}[(2T-1)(y+S-T+2)(y+S+T+2)]^{\frac{1}{2}}}{4[6ST(S+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S-1\ T+1$; 11	$\left[\frac{(2S-1)(2T+3)(y+S-T)(y-S+T+2)}{4S(T+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{(2y+13)(y+1)[(2S-1)(2T+3)(y+S-T)(y-S+T+2)]^{\frac{1}{2}}}{4[6S(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S-1\ T$; 11	$\left[\frac{(2S-1)(y+S+T+2)(y-S+T+2)}{4ST(T+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{(2y-7)(y+2)+T(4y^2+12y-1)[(2S-1)(y+S+T+2)(y-S+T+2)]^{\frac{1}{2}}}{4[6ST(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S-1\ T-1$; 11	$-\left[\frac{(2S-1)(2T-1)(y-S-T+2)(y+S+T+2)}{4ST(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{(y+2)(2y-7)[(2S-1)(2T-1)(y-S-T+2)(y+S+T+2)]^{\frac{1}{2}}}{4[6ST(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S+1\ T$; 10	0	$\frac{[(2S+3)(y-S-T)(y+S-T+2)(4y^2+12y-1)]^{\frac{1}{2}}}{4[6(S+1)(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}$
$S\ T$; 10	$\left[\frac{4S(S+1)}{(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{[(4y^2+12y-1)][(2y+3)+(2S+1)(2T+1)]-36(2y+3)S(S+1)\}}{8[6S(S+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S-1\ T$; 10	0	$-\frac{[(2S-1)(y+S+T+2)(y-S+T+2)(4y^2+12y-1)]^{\frac{1}{2}}}{4[6S(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}$
$S\ T+1$; 01	0	$\frac{[(2T+3)(y-S-T)(y-S+T+2)(4y^2+12y-1)]^{\frac{1}{2}}}{4[6(T+1)(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}$
$S\ T$; 01	$\left[\frac{4T(T+1)}{(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{\{[(4y^2+12y-1)][(2y+3)+(2S+1)(2T+1)]-36(2y+3)T(T+1)\}}{8[6T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$
$S\ T-1$; 01	0	$-\frac{[(2T-1)(y+S+T+2)(y+S-T+2)(4y^2+12y-1)]^{\frac{1}{2}}}{4[6T(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}$

(b) $y - S - T =$ odd integer

$S+1\ T+1;\ 11$	$-\left[\frac{(2S+3)(2T+3)(y-S-T-1)(y+S+T+3)}{4(S+1)(T+1)(4y^2+12y-1)}\right]^{\frac12}$	$\frac{(2y+13)(y+1)[(2S+3)(2T+3)(y-S-T-1)(y+S+T+3)]^{\frac12}}{4[6(S+1)(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S+1\ T;\ 11$	$-\left[\frac{(2S+3)(y-S+T+1)(y+S+T+3)}{4(S+1)T(T+1)(4y^2+12y-1)}\right]^{\frac12}$	$-\frac{\{(2y-7)(y+2)+T(4y^2+12y-1)\}[(2S+3)(y-S+T+1)(y+S+T+3)]^{\frac12}}{4[6(S+1)T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S+1\ T-1;\ 11$	$\left[\frac{(2S+3)(2T-1)(y+S-T+3)(y-S+T+1)}{4(S+1)T(4y^2+12y-1)}\right]^{\frac12}$	$\frac{(2y-7)(y+2)[(2S+3)(2T-1)(y+S-T+3)(y-S+T+1)]^{\frac12}}{4[6(S+1)T(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S\ T+1;\ 11$	$-\left[\frac{(2T+3)(y+S-T+1)(y+S+T+3)}{4S(S+1)(T+1)(4y^2+12y-1)}\right]^{\frac12}$	$-\frac{\{(2y-7)(y+2)+S(4y^2+12y-1)\}[(2T+3)(y+S-T+1)(y+S+T+3)]^{\frac12}}{4[6S(S+1)(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S\ T;\ 11$	$-\frac{\{(y+\frac32)-2(S+\frac12)(T+\frac12)\}}{[4S(S+1)T(T+1)(4y^2+12y-1)]^{\frac12}}$	$\frac{\left\{\frac12(16y^2+48y+41)-[S(S+1)+T(T+1)](4y^2+12y-1)+2(S+\frac12)(T+\frac12)(2y+3)(2y^2+6y-5)\right\}}{4[6S(S+1)T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S\ T-1;\ 11$	$\left[\frac{(2T-1)(y-S-T+1)(y-S+T+1)}{4S(S+1)T(4y^2+12y-1)}\right]^{\frac12}$	$-\frac{\{(2y+13)(y+1)+S(4y^2+12y-1)\}[(2T-1)(y-S+T+1)(y-S-T+1)]^{\frac12}}{4[6ST(S+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S-1\ T+1;\ 11$	$\left[\frac{(2S-1)(2T+3)(y+S-T+1)(y-S+T+3)}{4S(T+1)(4y^2+12y-1)}\right]^{\frac12}$	$\frac{(2y-7)(y+2)[(2S-1)(2T+3)(y+S-T+1)(y-S+T+3)]^{\frac12}}{4[6S(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S-1\ T;\ 11$	$\left[\frac{(2S-1)(y-S-T+1)(y+S-T+1)}{4ST(T+1)(4y^2+12y-1)}\right]^{\frac12}$	$-\frac{\{(2y+13)(y+1)+T(4y^2+12y-1)\}[(2S-1)(y+S-T+1)(y-S-T+1)]^{\frac12}}{4[6ST(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S-1\ T-1;\ 11$	$-\left[\frac{(2S-1)(2T-1)(y-S-T+1)(y+S+T+1)}{4ST(4y^2+12y-1)}\right]^{\frac12}$	$\frac{(2y+13)(y+1)[(2S-1)(2T-1)(y+S+T+1)(y-S-T+1)]^{\frac12}}{4[6ST(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S+1\ T;\ 10$	0	$\frac{[(2S+3)(y-S+T+1)(y+S+T+3)(4y^2+12y-1)]^{\frac12}}{4[6(S+1)(y-1)(y+1)(y+2)(y+4)]^{\frac12}}$
$S\ T;\ 10$	$\left[\frac{4S(S+1)}{(4y^2+12y-1)}\right]^{\frac12}$	$\frac{\{[(4y^2+12y-1)][(2y+3)-(2S+1)(2T+1)]-36(2y+3)S(S+1)\}}{8[6S(S+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S-1\ T;\ 10$	0	$-\frac{[(2S-1)(y+S-T+1)(y-S-T+1)(4y^2+12y-1)]^{\frac12}}{4[6S(y-1)(y+1)(y+2)(y+4)]^{\frac12}}$
$S\ T+1;\ 01$	0	$\frac{[(2T+3)(y+S-T+1)(y+S+T+3)(4y^2+12y-1)]^{\frac12}}{4[6(T+1)(y-1)(y+1)(y+2)(y+4)]^{\frac12}}$
$S\ T;\ 01$	$\left[\frac{4T(T+1)}{(4y^2+12y-1)}\right]^{\frac12}$	$\frac{\{[(4y^2+12y-1)][(2y+3)-(2S+1)(2T+1)]-36(2y+3)T(T+1)\}}{8[6T(T+1)(y-1)(y+1)(y+2)(y+4)(4y^2+12y-1)]^{\frac12}}$
$S\ T-1;\ 01$	0	$-\frac{[(2T-1)(y-S+T+1)(y-S-T+1)(4y^2+12y-1)]^{\frac12}}{4[6T(y-1)(y+1)(y+2)(y+4)]^{\frac12}}$

TABLE A4.4.

$$\left\langle \begin{matrix} [y\ y-1\ 0]\ [211] \\ S_1T_1;\ S_2T_2 \end{matrix} \right\| \left. \begin{matrix} [y+1\ y+1\ 1] \\ ST \end{matrix} \right\rangle$$

S_1T_1; S_2T_2	$y - S - T =$ odd integer	$y - S - T =$ even integer
$S+1\ T+1;\ 11$	$\left[\frac{(2S+3)(2T+3)(y-S-T+1)(y-S-T-1)}{32(S+1)(T+1)y(y+2)}\right]^{\frac{1}{2}}$	0
$S+1\ T;\ 11$	$-\left[\frac{(2S+3)T(y-S-T+1)(y-S+T+1)}{32(S+1)(T+1)y(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)(T+1)(y-S-T)(y-S+T+2)}{32(S+1)Ty(y+2)}\right]^{\frac{1}{2}}$
$S+1\ T-1;\ 11$	0	$-\left[\frac{(2S+3)(2T-1)(y-S+T)(y-S+T+2)}{32(S+1)Ty(y+2)}\right]^{\frac{1}{2}}$
$S\ T+1;\ 11$	$-\left[\frac{S(2T+3)(y-S-T+1)(y+S-T+1)}{32(S+1)(T+1)y(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(S+1)(2T+3)(y-S-T)(y+S-T+2)}{32S(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S\ T;\ 11$	$-(S+T+1)\left[\frac{(y+S+T+3)(y-S-T+1)}{32S(S+1)T(T+1)y(y+2)}\right]^{\frac{1}{2}}$	$-(S-T)\left[\frac{(y+S-T+2)(y-S+T+2)}{32S(S+1)T(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S\ T-1;\ 11$	$\left[\frac{(S+1)(2T-1)(y-S+T+1)(y+S+T+3)}{32STy(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{S(2T-1)(y+S+T+2)(y-S+T+2)}{32(S+1)Ty(y+2)}\right]^{\frac{1}{2}}$
$S-1\ T+1;\ 11$	0	$\left[\frac{(2S-1)(2T+3)(y+S-T)(y+S-T+2)}{32S(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S-1\ T;\ 11$	$\left[\frac{(2S-1)(T+1)(y+S+T+3)(y+S-T+1)}{32STy(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2S-1)T(y+S+T+2)(y+S-T+2)}{32S(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S-1\ T\ 0;\ -11$	$-\left[\frac{(2S-1)(2T-1)(y+S+T+1)(y+S+T+3)}{32STy(y+2)}\right]^{\frac{1}{2}}$	0
$S+1\ T;\ 10$	$\left[\frac{(2S+3)(y-S-T+1)(y-S+T+1)}{32(S+1)y(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S+3)(y-S-T)(y-S+T+2)}{32(S+1)y(y+2)}\right]^{\frac{1}{2}}$
$ST;\ 10$	$-(2S+1)\left[\frac{(y-S-T+1)(y+S+T+3)}{32S(S+1)y(y+2)}\right]^{\frac{1}{2}}$	$-(2S+1)\left[\frac{(y+S-T+2)(y-S+T+2)}{32S(S+1)y(y+2)}\right]^{\frac{1}{2}}$
$S-1\ T;\ 10$	$\left[\frac{(2S-1)(y+S-T+1)(y+S+T+3)}{32Sy(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(2S-1)(y+S+T+2)(y+S-T+2)}{32Sy(y+2)}\right]^{\frac{1}{2}}$
$S\ T+1;\ 01$	$\left[\frac{(2T+3)(y-S-T+1)(y+S-T+1)}{32(T+1)y(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2T+3)(y-S-T)(y+S-T+2)}{32(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S\ T;\ 01$	$-(2T+1)\left[\frac{(y-S-T+1)(y+S+T+3)}{32T(T+1)y(y+2)}\right]^{\frac{1}{2}}$	$(2T+1)\left[\frac{(y-S+T+2)(y+S-T+2)}{32T(T+1)y(y+2)}\right]^{\frac{1}{2}}$
$S\ T-1;\ 01$	$\left[\frac{(2T-1)(y-S+T+1)(y+S+T+3)}{32Ty(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(2T-1)(y+S+T+2)(y-S+T+2)}{32Ty(y+2)}\right]^{\frac{1}{2}}$

TABLE A4.5. $\left\langle \begin{matrix}[y10]\ [211] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y10] \\ ST\end{matrix}\right\rangle_\rho$.

S_1T_1; S_2T_2	ST	$\rho = 1$	$\rho = 2$
$S+1\ S+1$; 11	SS	$-\left[\frac{S(S+2)(2S+3)(y-1-2S)(y+3+2S)}{(2S+1)(S+1)^2(y+1)(3y+7)}\right]^{1/2}$	$-\frac{(y+9)}{2(S+1)}\left[\frac{S(S+2)(2S+3)(y-1-2S)(y+3+2S)}{2(2S+1)(y-1)(y+4)(y+1)(3y+7)}\right]^{1/2}$
$S+1\ S$; 11 $S\ S+1$; 11	SS SS	$\frac{-1}{(S+1)}\left[\frac{(2S+3)(y+3+2S)}{(2S+1)(3y+7)}\right]^{1/2}$	$\frac{\{(y-11)+S(3y+7)\}}{4(S+1)}\left[\frac{(2S+3)(y+3+2S)}{2(2S+1)(y-1)(y+4)(3y+7)}\right]^{1/2}$
$S\ S$; 11	SS	$\frac{\{(y+1)-(y+3)S(S+1)\}}{S(S+1)[(y+1)(3y+7)]^{1/2}}$	$\frac{\{(y+1)(y+9)+2(y^2-y-10)S(S+1)\}}{2S(S+1)[2(y-1)(y+4)(y+1)(3y+7)]^{1/2}}$
$S\ S-1$; 11 $S-1\ S$; 11	SS SS	$\frac{-1}{S}\left[\frac{(2S-1)(y+1-2S)}{(2S+1)(3y+7)}\right]^{1/2}$	$\frac{-\{2(y+9)+S(3y+7)\}}{4S}\left[\frac{(2S-1)(y+1-2S)}{2(2S+1)(y-1)(y+4)(3y+7)}\right]^{1/2}$
$S-1\ S-1$; 11	SS	$\frac{-1}{S}\left[\frac{(S-1)(S+1)(2S-1)(y+1+2S)(y+1-2S)}{(2S+1)(y+1)(3y+7)}\right]^{1/2}$	$-\frac{(y+9)}{2S}\left[\frac{(S-1)(S+1)(2S-1)(y+1+2S)(y+1-2S)}{2(2S+1)(y-1)(y+4)(y+1)(3y+7)}\right]^{1/2}$
$S+1\ S$; 10 $S\ S+1$; 01	SS SS	0	$\left[\frac{S(2S+3)(y+3+2S)(3y+7)}{32(S+1)(2S+1)(y-1)(y+4)}\right]^{1/2}$
$S-1\ S$; 10 $S\ S-1$; 01	SS SS	0	$\left[\frac{(S+1)(2S-1)(y+1-2S)(3y+7)}{32S(2S+1)(y-1)(y+4)}\right]^{1/2}$
$S\ S$; 10 $S\ S$; 01	SS SS	$\left[\frac{4S(S+1)}{(y+1)(3y+7)}\right]^{1/2}$	$-\frac{\{(y+1)(3y+7)-4(y+9)S(S+1)\}}{4[2S(S+1)(y-1)(y+4)(y+1)(3y+7)]^{1/2}}$
$S+1\ S$; 11	$S\ S-1$	$-\left[\frac{(2S+3)(y+3+2S)(y+1-2S)}{(2S+1)(y+1)(3y+7)}\right]^{1/2}$	$(y-11)\left[\frac{(2S+3)(y+3+2S)(y+1-2S)}{32(2S+1)(y-1)(y+4)(y+1)(3y+7)}\right]^{1/2}$
$S\ S$; 11	$S\ S-1$	$\frac{1}{S}\left[\frac{y+1-2S}{(3y+7)}\right]^{1/2}$	$\frac{\{2(y+9)+S(3y+7)\}}{4S}\left[\frac{(y+1-2S)}{2(y-1)(y+4)(3y+7)}\right]^{1/2}$
$S\ S-1$; 11	$S\ S-1$	$-\frac{1}{S}\left[\frac{(S-1)(S+1)(y+1)}{(3y+7)}\right]^{1/2}$	$-\frac{(y+9)}{2S}\left[\frac{(S-1)(S+1)(y+1)}{2(y-1)(y+4)(3y+7)}\right]^{1/2}$
$S-1\ S$; 11	$S\ S-1$	$\frac{\{y+1+4S^2\}}{S[(2S-1)(2S+1)(y+1)(3y+7)]}$	$\frac{\{(y+1)(y+9)-2S^2(3y-1)(y+4)\}}{2S[2(2S-1)(2S+1)(y-1)(y+4)(y+1)(3y+7)]^{1/2}}$
$S-1\ S-1$; 11	$S\ S-1$	$\frac{1}{S}\left[\frac{y+1+2S}{3y+7}\right]^{1/2}$	$\frac{\{2(y+9)-S(3y+7)\}}{4S}\left[\frac{(y+1+2S)}{2(y-1)(y+4)(3y+7)}\right]^{1/2}$
$S-1\ S-2$; 11	$S\ S-1$	$-\left[\frac{(2S-3)(y+3-2S)(y+1+2S)}{(2S-1)(y+1)(3y+7)}\right]^{1/2}$	$(y-11)\left[\frac{(2S-3)(y+3-2S)(y+1+2S)}{32(2S-1)(y-1)(y+4)(y+1)(3y+7)}\right]^{1/2}$
$S\ S-1$; 10	$S\ S-1$	$\left[\frac{4S(S+1)}{(y+1)(3y+7)}\right]^{1/2}$	$-\frac{\{(y+1)(3y+7)+2(y-11)S\}(S+1)}{4[2S(S+1)(y-1)(y+4)(y+1)(3y+7)]^{1/2}}$
$S-1\ S-1$; 10	$S\ S-1$	0	$-\left[\frac{(S-1)(y+1+2S)(3y+7)}{32S(y-1)(y+4)}\right]^{1/2}$
$S\ S$; 10	$S\ S-1$	0	$-\left[\frac{(S+1)(y+1-2S)(3y+7)}{32S(y-1)(y+4)}\right]^{1/2}$
$S\ S-1$; 01	$S\ S-1$	$\left[\frac{4(S-1)S}{(y+1)(3y+7)}\right]^{1/2}$	$\frac{\{(y+1)(3y+7)-2(y-11)S\}(S-1)}{4[2(S-1)S(y-1)(y+4)(y+1)(3y+7)]^{1/2}}$

$$\left\langle \begin{matrix}[y10]\ [211] \\ T_1S_1;\ T_2S_2\end{matrix} \middle\| \begin{matrix}[y10] \\ (S-1)S\end{matrix}\right\rangle_\rho = \left\langle \begin{matrix}[y10]\ [211] \\ S_1T_1;\ S_2T_2\end{matrix} \middle\| \begin{matrix}[y10] \\ S(S-1)\end{matrix}\right\rangle_\rho .$$

TABLE A4.6. $\left\langle \begin{matrix}[y11] & [211] \\ S_1T_1; & S_2T_2\end{matrix} \middle\| \begin{matrix}[y11] \\ ST\end{matrix} \right\rangle_\rho$.

S_1T_1; S_2T_2	ST	$\rho = 1$	$\rho = 2$
$S+1\ S+1$; 11	SS	$-\left[\frac{S(2S+3)(S+2)(y-2S)(y+4+2S)}{(2S+1)(S+1)^2(y+2)(3y+2)}\right]^{1/2}$	$-\frac{(y-6)}{2(S+1)}\left[\frac{S(S+2)(2S+3)(y-2S)(y+4+2S)}{2(2S+1)(y-1)(y+4)(y+2)(3y+2)}\right]^{1/2}$
$S+1\ S$; 11 $S\ S+1$; 11	SS SS	$\frac{1}{(S+1)}\left[\frac{(2S+3)(y-2S)}{(2S+1)(3y+2)}\right]^{1/2}$	$\frac{\{2(y-6)-(S+1)(3y+2)\}}{4(S+1)}\left[\frac{(2S+3)(y-2S)}{2(2S+1)(y-1)(y+4)(3y+2)}\right]^{1/2}$
$S\ S$; 11	SS	$\frac{\{(y+2)-yS(S+1)\}}{S(S+1)[(y+2)(3y+2)]^{1/2}}$	$\frac{\{(y+2)(y-6)+2(y^2+7y+2)S(S+1)\}}{2S(S+1)[2(y-1)(y+4)(y+2)(3y+2)]^{1/2}}$
$S\ S-1$; 11 $S-1\ S$; 11	SS SS	$\frac{1}{S}\left[\frac{(2S-1)(y+2+2S)}{(2S+1)(3y+2)}\right]^{1/2}$	$\frac{\{2(y-6)+S(3y+2)\}}{4S}\left[\frac{(2S-1)(y+2+2S)}{2(2S+1)(y-1)(y+4)(3y+2)}\right]^{1/2}$
$S-1\ S-1$; 11	SS	$-\frac{1}{S}\left[\frac{(S-1)(S+1)(2S-1)(y+2+2S)(y+2-2S)}{(2S+1)(y+2)(3y+2)}\right]^{1/2}$	$-\frac{(y-6)}{2S}\left[\frac{(S-1)(S+1)(2S-1)(y+2+2S)(y+2-2S)}{2(2S+1)(y-1)(y+4)(y+2)(3y+2)}\right]^{1/2}$
$S+1\ S$; 10 $S\ S+1$; 01	SS SS	0	$\left[\frac{S(2S+3)(y-2S)(3y+2)}{32(S+1)(2S+1)(y-1)(y+4)}\right]^{1/2}$
$S\ S$; 10 $S\ S$; 01	SS SS	$\left[\frac{4S(S+1)}{(y+2)(3y+2)}\right]^{1/2}$	$\frac{\{(y+2)(3y+2)+4(y-6)S(S+1)\}}{4[2S(S+1)(y-1)(y+4)(y+2)(3y+2)]^{1/2}}$
$S-1\ S$; 10 $S\ S-1$; 01	SS SS	0	$\left[\frac{(S+1)(2S-1)(y+2+2S)(3y+2)}{32S(2S+1)(y-1)(y+4)}\right]^{1/2}$
$S+1\ S$; 11	$S\ S-1$	$-\left[\frac{(2S+3)(y-2S)(y+2+2S)}{(2S+1)(y+2)(3y+2)}\right]^{1/2}$	$\frac{(y+14)}{4}\left[\frac{(2S+3)(y-2S)(y+2+2S)}{2(2S+1)(y-1)(y+4)(y+2)(3y+2)}\right]^{1/2}$
$S\ S$; 11	$S\ S-1$	$-\frac{1}{S}\left[\frac{(y+2+2S)}{(3y+2)}\right]^{1/2}$	$-\frac{\{2(y-6)+S(3y+2)\}}{4S}\left[\frac{(y+2+2S)}{2(y-1)(y+4)(3y+2)}\right]^{1/2}$
$S\ S-1$; 11	$S\ S-1$	$-\frac{1}{S}\left[\frac{(S-1)(S+1)(y+2)}{(3y+2)}\right]^{1/2}$	$-\frac{(y-6)}{2S}\left[\frac{(S-1)(S+1)(y+2)}{2(y-1)(y+4)(3y+2)}\right]^{1/2}$
$S-1\ S$; 11	$S\ S-1$	$\frac{\{(y+2)-4S^2\}}{S[(2S-1)(2S+1)(y+2)(3y+2)]^{1/2}}$	$\frac{\{(y+2)(y-6)-2S^2(y-1)(3y+10)\}}{2S[2(2S-1)(2S+1)(y-1)(y+4)(y+2)(3y+2)]^{1/2}}$
$S-1\ S-1$; 11	$S\ S-1$	$-\frac{1}{S}\left[\frac{(y+2-2S)}{(3y+2)}\right]^{1/2}$	$\frac{\{S(3y+2)-2(y-6)\}}{4S}\left[\frac{(y+2-2S)}{2(y-1)(y+4)(3y+2)}\right]^{1/2}$
$S-1\ S-2$; 11	$S\ S-1$	$-\left[\frac{(2S-3)(y+2S)(y+2-2S)}{(2S-1)(y+2)(3y+2)}\right]^{1/2}$	$\frac{(y+14)}{4}\left[\frac{(2S-3)(y+2S)(y+2-2S)}{2(2S-1)(y-1)(y+4)(y+2)(3y+2)}\right]^{1/2}$
$S\ S-1$; 10	$S\ S-1$	$\left[\frac{4S(S+1)}{(y+2)(3y+2)}\right]^{1/2}$	$\frac{\{(y+2)(3y+2)-2S(y+14)\}[(S+1)]^{1/2}}{4[2S(y-1)(y+4)(y+2)(3y+2)]^{1/2}}$
$S-1\ S-1$; 10	$S\ S-1$	0	$-\left[\frac{(S-1)(y+2-2S)(3y+2)}{32S(y-1)(y+4)}\right]^{1/2}$
$S\ S-1$; 01	$S\ S-1$	$\left[\frac{4(S-1)S}{(y+2)(3y+2)}\right]^{1/2}$	$-\frac{\{(y+2)(3y+2)+2S(y+14)\}[(S-1)]^{1/2}}{4[2S(y-1)(y+4)(y+2)(3y+2)]^{1/2}}$
$S\ S$; 01	$S\ S-1$	0	$-\left[\frac{(S+1)(y+2+2S)(3y+2)}{32S(y-1)(y+4)}\right]^{1/2}$

$$\left\langle \begin{matrix}[y11] & [211] \\ T_1S_1; & T_2S_2\end{matrix} \middle\| \begin{matrix}[y11] \\ (S-1)S\end{matrix} \right\rangle_\rho = +\left\langle \begin{matrix}[y11] & [211] \\ S_1T_1; & S_2T_2\end{matrix} \middle\| \begin{matrix}[y11] \\ S(S-1)\end{matrix} \right\rangle_\rho.$$

TABLE A5.1.

S_2T_2	$\left\langle \begin{matrix} [yy0] & [22] \\ ST; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [yy0] \\ ST \end{matrix} \right\rangle$
00	$\dfrac{\{T(T+1) - S(S+1)\}}{[2y(y+1)(y+3)(y+4)]^{\frac{1}{2}}}$
02	$-\{(y+1)(y+3) + T(T+1) - S(S+1)\}\left[\dfrac{T(T+1)}{2(2T-1)(2T+3)y(y+1)(y+3)(y+4)}\right]^{\frac{1}{2}}$
20	$\{(y+1)(y+3) + S(S+1) - T(T+1)\}\left[\dfrac{S(S+1)}{2(2S-1)(2S+3)y(y+1)(y+3)(y+4)}\right]^{\frac{1}{2}}$

TABLE A5.2.

S_2T_2	$\left\langle \begin{matrix} [yy0] & [422] \\ ST; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [yy0] \\ ST \end{matrix} \right\rangle$
00	$\dfrac{\{5S(S+1) + 5T(T+1) - 2y(y+4)\}}{3[6y(y-1)(y+4)(y+5)]^{\frac{1}{2}}}$
02	$\frac{1}{3}\{7T(T+1) + S(S+1) - y(y+4) - 9\}\left[\dfrac{5T(T+1)}{6(2T-1)(2T+3)y(y-1)(y+4)(y+5)}\right]^{\frac{1}{2}}$
20	$\frac{1}{3}\{7S(S+1) + T(T+1) - y(y+4) - 9\}\left[\dfrac{5S(S+1)}{6(2S-1)(2S+3)y(y-1)(y+4)(y+5)}\right]^{\frac{1}{2}}$

TABLE A5.3.

S_2T_2	$\left\langle \begin{matrix} [y00] & [422] \\ SS; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [y00] \\ SS \end{matrix} \right\rangle$
00	$\dfrac{\{20S(S+1) - 3y(y+4)\}}{12[y(y-1)(y+4)(y+5)]^{\frac{1}{2}}}$
02 20	$\dfrac{1}{3}\left[\dfrac{5S(S+1)(2S+3)(2S-1)}{y(y-1)(y+4)(y+5)}\right]^{\frac{1}{2}}$

TABLE A6.1.

S_2T_2	$\left\langle \begin{matrix} [y\ y-1\ 0] & [22] \\ ST; & S_2T_2 \end{matrix} \middle\| \begin{matrix} [y\ y-1\ 0] \\ ST \end{matrix} \right\rangle$
00	$\dfrac{\{T(T+1) - S(S+1)\}}{[2(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}$
20	$\dfrac{\{(y+1)(y+2) + S(S+1) - T(T+1)\}}{4[2(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}\left[\dfrac{(2S-1)(2S+3)}{S(S+1)}\right]^{\frac{1}{2}}$
02	$-\dfrac{\{(y+1)(y+2) + T(T+1) - S(S+1)\}}{4[2(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}}\left[\dfrac{(2T-1)(2T+3)}{T(T+1)}\right]^{\frac{1}{2}}$

TABLE A6.2.

S_2T_2	ρ	$\left\langle \begin{matrix} [y\,y-1\,0]\ [422] \\ ST;\ S_2T_2 \end{matrix} \middle\| \begin{matrix} [y\,y-1\,0] \\ ST \end{matrix} \right\rangle^{\frac{1}{2}}$
00	1	$\dfrac{\{(4y^2+12y-1)-10S(S+1)-10T(T+1)\}}{6[2(y-1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}}$
00	2	$\dfrac{\sqrt{5}\{(2y+3)[y(y+3)-4S(S+1)-4T(T+1)]+(-1)^{y-S-T}(3y^2+9y-2)(S+\frac{1}{2})(T+\frac{1}{2})\}}{3[3(y-1)(y+1)(y-2)(y+2)(y+4)(y+5)(3y^2+9y-2)]^{\frac{1}{2}}}$
02	1	$\dfrac{\{(y+1)(y+2)-S(S+1)-7T(T+1)\}}{12[2(y-1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}}\left[\dfrac{5(2T-1)(2T+3)}{T(T+1)}\right]^{\frac{1}{2}}$
02	2	$\dfrac{\left\{\begin{matrix}(2y+3)[(7y^2+21y+6)-4S(S+1)-28T(T+1)] \\ +(-1)^{y-S-T}(3y^2+9y-2)(2S+1)(2T+1)\end{matrix}\right\}}{12[3(y-1)(y+1)(y-2)(y+2)(y+4)(y+5)(3y^2+9y-2)]^{\frac{1}{2}}}\left[\dfrac{(2T-1)(2T+3)}{T(T+1)}\right]^{\frac{1}{2}}$
20	1	$\dfrac{\{(y+1)(y+2)-T(T+1)-7S(S+1)\}}{12[2(y-1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}}\left[\dfrac{5(2S-1)(2S+3)}{S(S+1)}\right]^{\frac{1}{2}}$
20	2	$\dfrac{\left\{\begin{matrix}(2y+3)[(7y^2+21y+6)-4(TT+1)-28S(S+1)] \\ +(-1)^{y-S-T}(3y^2+9y-2)(2S+1)(2T+1)\end{matrix}\right\}}{12[3(y-1)(y+1)(y-2)(y+2)(y+4)(y+5)(3y^2+9y-2)]^{\frac{1}{2}}}\left[\dfrac{(2S-1)(2S+3)}{S(S+1)}\right]^{\frac{1}{2}}$

TABLE A6.3.

S_2T_2	ST	$\left\langle \begin{matrix} [y11]\ [22] \\ ST;\ S_2T_2 \end{matrix} \middle\| \begin{matrix} [y11] \\ ST \end{matrix} \right\rangle$
00	$S\,S$	0
20 02	$S\,S$	$\pm\dfrac{1}{4}\left[\dfrac{(2S-1)(2S+3)(y+2)}{2S(S+1)(y-1)}\right]^{\frac{1}{2}}$
11	$S\,S$	0
00 00	$S\,S-1$ $S-1\,S$	$\mp\dfrac{S}{[2(y-1)(y+2)]^{\frac{1}{2}}}$
20 02	$S\,S-1$ $S-1\,S$	$\mp\dfrac{(y+2-2S)}{4}\left[\dfrac{(2S+3)(S+1)}{2S(2S-1)(y-1)(y+2)}\right]^{\frac{1}{2}}$
02 20	$S\,S-1$ $S-1\,S$	$\pm\dfrac{(y+2+2S)}{4}\left[\dfrac{(2S-3)(S-1)}{2S(2S+1)(y-1)(y+2)}\right]^{\frac{1}{2}}$
11 11	$S\,S-1$ $S-1\,S$	0

TABLE A6.4.

S_2T_2	ST	ρ	$\left\langle \begin{matrix}[y11] & [422] \\ ST; & S_2T_2\end{matrix} \middle\| \begin{matrix}[y11] \\ ST\end{matrix} \right\rangle_\rho$
00	ST	1	$\frac{\{(3y+2)(y+2)-10S(S+1)-10T(T+1)\}}{4[3(y-1)(y+4)(y+2)(3y+1)]^{1/2}}$
00	SS	2	$-\frac{\{3(y+2)(5y+4)+4(2y-11)S(S+1)\}}{2[6(y-1)(y-2)(y+2)(y+4)(y+5)(3y+1)]^{1/2}}$
00	$S\,S-1$ $S-1\,S$	2	$-\frac{\{3(y+2)(5y+4)-4(y+12)S^2\}}{2[6(y-1)(y-2)(y+2)(y+4)(y+5)(3y+1)]^{1/2}}$
02 20	SS	1	$\frac{\{(y+2)-4S(S+1)\}}{4}\left[\frac{5(2S+3)(2S-1)}{3S(S+1)(y-1)(y+4)(y+2)(3y+1)}\right]^{1/2}$
02 20	SS	2	$\frac{\{(y+2)^2(3y-13)-8(2y-11)S(S+1)\}}{4} \times\left[\frac{(2S+3)(2S-1)}{30S(S+1)(y-1)(y-2)(y+2)(y+4)(y+5)(3y+1)}\right]^{1/2}$
02 20	$S\,S-1$ $S-1\,S$	1	$-\frac{\{y+2+8S^2+6S\}}{4}\left[\frac{5(2S-3)(S-1)}{3S(2S+1)(y-1)(y+4)(y+2)(3y+1)}\right]^{1/2}$
02 20	$S\,S-1$ $S-1\,S$	2	$-\frac{\{(y+2)^2(3y-13)-6S(3y^2+9y+26)-16S^2(y+12)\}[(2S-3)(S-1)]^{1/2}}{4[30S(2S+1)(y-1)(y-2)(y+2)(y+4)(y+5)(3y+1)]^{1/2}}$
02 20	$S-1\,S$ $S\,S-1$	1	$-\frac{\{y+2+8S^2-6S\}}{4}\left[\frac{5(2S+3)(S+1)}{3S(2S-1)(y-1)(y+4)(y+2)(3y+1)}\right]^{1/2}$
02 20	$S-1\,S$ $S\,S-1$	2	$-\frac{\{(y+2)^2(3y-13)+6S(3y^2+9y+26)-16S^2(y+12)\}[(2S+3)(S+1)]^{1/2}}{4[30S(2S-1)(y-1)(y-2)(y+2)(y+4)(y+5)(3y+1)]^{1/2}}$

TABLE A7.1. $U([yy0][1^3][yy0][1];\ [f^{(12)}][f^{(23)}])$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]
$[y\,y-1\,0]$	$-\left[\frac{y}{2(y+2)}\right]^{1/2}$	$\left[\frac{(y+4)}{2(y+2)}\right]^{1/2}$
$[y+1\,y+1\,1]$	$\left[\frac{(y+4)}{2(y+2)}\right]^{1/2}$	$\left[\frac{y}{2(y+2)}\right]^{1/2}$
Σ_1	n	$-2[y(y+4)]^{1/2}$

TABLE A7.2. $U([y\,y-1\,0][1^3][y\,y-1\,0][1];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$
$[y-1\,y-1\,0]$	$\left[\frac{y+1}{8(y+3)}\right]^{1/2}$	$\frac{-(y+1)(2y+13)}{[8(y+1)(y+3)(4y^2+12y-1)]^{1/2}}$	$-\left[\frac{3(y-1)(y+2)(y+4)}{(y+3)(4y^2+12y-1)}\right]^{1/2}$
$[y\,y-2\,0]$	$-\left[\frac{3(y-1)}{8(y+1)}\right]^{1/2}$	$(2y+5)\left[\frac{3(y-1)}{8(y+1)(4y^2+12y-1)}\right]^{1/2}$	$-\left[\frac{(y+2)(y+4)}{(4y^2+12y-1)}\right]^{1/2}$
$[y+1\,y\,1]$	$\left[\frac{(y+2)(y+4)}{2(y+1)(y+3)}\right]^{1/2}$	$(2y-1)\left[\frac{(y+2)(y+4)}{2(y+1)(y+3)(4y^2+12y-1)}\right]^{1/2}$	$-\left[\frac{3(y-1)}{(y+3)(4y^2+12y-1)}\right]^{1/2}$
Σ_1	n	$-[4y^2+12y-1]^{1/2}$	0

TABLE A7.3. $U([yy1][1^3][yy1][1];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$
$[yy0]$	$-\left[\dfrac{y+2}{8y}\right]^{1/2}$	$\dfrac{-(2y-7)[(y+2)]^{1/2}}{[8y(4y^2+12y-1)]^{1/2}}$	$-\left[\dfrac{3(y-1)(y+1)(y+4)}{y(4y^2+12y-1)}\right]^{1/2}$
$[y\ y-1\ 1]$	$-\left[\dfrac{(y-1)(y+1)}{2y(y+2)}\right]^{1/2}$	$\dfrac{(2y+7)[(y-1)(y+1)]^{1/2}}{[2y(y+2)(4y^2+12y-1)]^{1/2}}$	$\left[\dfrac{3(y+4)}{y(4y^2+12y-1)}\right]^{1/2}$
$[y+1\ y+1\ 2]$	$\left[\dfrac{3(y+4)}{8(y+2)}\right]^{1/2}$	$\dfrac{(2y+1)[3(y+4)]^{1/2}}{[8(y+2)(4y^2+12y-1)]^{1/2}}$	$-\left[\dfrac{(y-1)(y+1)}{4y^2+12y-1}\right]^{1/2}$
Σ_1	n	$-[4y^2+12y-1]^{1/2}$	0

TABLE A7.4. $U([y][1^3][y][1];\ [f^{(12)}][f^{(23)}])$

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]
$[y-1\ 00]$	$\left[\dfrac{y}{4(y+3)}\right]^{1/2}$	$-\left[\dfrac{3(y+4)}{4(y+3)}\right]^{1/2}$
$[y+1\ 11]$	$\left[\dfrac{3(y+4)}{4(y+3)}\right]^{1/2}$	$\left[\dfrac{y}{4(y+3)}\right]^{1/2}$
Σ_1	n	$-[3y(y+4)]^{1/2}$

TABLE A7.5. $U([y11][1^3][y11][1];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$
$[y10]$	$-\left[\dfrac{(y+2)}{4(y+1)}\right]^{1/2}$	$-\dfrac{(y-6)}{[4(y+1)(3y+2)]^{1/2}}$	$\left[\dfrac{2(y-1)(y+4)}{(y+1)(3y+2)}\right]^{1/2}$
$[y+1\ 22]$	$\left[\dfrac{(y+4)}{2(y+3)}\right]^{1/2}$	$\left[\dfrac{(y+2)(y+4)}{2(y+3)(3y+2)}\right]^{1/2}$	$\left[\dfrac{(y-1)(y+2)}{(y+3)(3y+2)}\right]^{1/2}$
$[y-1\ 11]$	$\left[\dfrac{(y-1)(y+2)}{4(y+1)(y+3)}\right]^{1/2}$	$-\dfrac{(3y+10)[y-1]^{1/2}}{[4(y+1)(y+3)(3y+2)]^{1/2}}$	$\left[\dfrac{8(y+4)}{(y+1)(y+3)(3y+2)}\right]^{1/2}$
Σ_1	n	$-[(y+2)(3y+2)]^{1/2}$	0

TABLE A7.6. $U([y\ y-1\ y-1][1^3][y\ y-1\ y-1][1];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$
$[y-1\ y-1\ y-1]$	$\left[\dfrac{(y+2)}{12(y+3)}\right]^{1/2}$	$-\dfrac{(y+14)}{[12(y+3)(3y+2)]^{1/2}}$	$\left[\dfrac{8(y-1)(y+4)}{3(y+3)(3y+2)}\right]^{1/2}$
$[y+1\ y\ y]$	$\left[\dfrac{(y+2)(y+4)}{4y(y+3)}\right]^{1/2}$	$\dfrac{(3y-2)[y+4]^{1/2}}{[4y(y+3)(3y+2)]^{1/2}}$	$\left[\dfrac{8(y-1)}{y(y+3)(3y+2)}\right]^{1/2}$
$[y\ y-1\ y-2]$	$-\left[\dfrac{2(y-1)}{3y}\right]^{1/2}$	$\left[\dfrac{2(y-1)(y+2)}{3y(y+3)}\right]^{1/2}$	$\left[\dfrac{(y+2)(y+4)}{3y(3y+2)}\right]^{1/2}$
Σ_1	n	$-[(y+2)(3y+2)]^{1/2}$	0

TABLE A7.7. $U([y10][1^3][y10][1];\ [f^{(12)}][f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$
$[y00]$	$-\left[\frac{y+1}{12y}\right]^{\frac{1}{2}}$	$\frac{-(y-11)}{[12y(3y+7)]^{\frac{1}{2}}}$	$-\left[\frac{8(y-1)(y+4)}{3y(3y+7)}\right]^{\frac{1}{2}}$
$[y-1\ 10]$	$\left[\frac{(y-1)(y+1)}{4y(y+3)}\right]^{\frac{1}{2}}$	$\frac{-3(y+11)[y-1]^{\frac{1}{2}}}{[4y(y+3)(3y+7)]^{\frac{1}{2}}}$	$-\left[\frac{8(y+4)}{y(y+3)(3y+7)}\right]^{\frac{1}{2}}$
$[y+1\ 21]$	$\left[\frac{2(y+4)}{3(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{2(y+1)(y+4)}{3(y+3)(3y+7)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-1)(y+1)}{3(y+3)(3y+7)}\right]^{\frac{1}{2}}$
Σ_1	n	$-[(y+1)(3y+7)]^{\frac{1}{2}}$	0

TABLE A8.1. $U([yy0][1^2][yy0][1^2];\ [f^{(12)}][f^{(23)}])$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]	[22]
$[y+1\ y+1\ 0]$	$\left[\frac{(y+3)(y+4)}{6(y+1)(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{y(y+3)}{2(y+1)(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{y}{3(y+2)}\right]^{\frac{1}{2}}$
$[y-1\ y-1\ 0]$	$\left[\frac{y(y+1)}{6(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+1)(y+4)}{2(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+4)}{3(y+2)}\right]^{\frac{1}{2}}$
$[y+1\ y1]$	$-\left[\frac{2y(y+4)}{3(y+1)(y+3)}\right]^{\frac{1}{2}}$	$-\left[\frac{2}{(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\frac{1}{\sqrt{3}}$
Σ_2	n_+	$\frac{1}{3}(n+2)[3y(y+4)]^{\frac{1}{2}}$	$\frac{1}{3}[2y(y+1)(y+3)(y+4)]^{\frac{1}{2}}$

TABLE A8.2. $U([yy0][2^3][yy0][2];\ [f^{(12)}][f^{(23)}])$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]	[422]
$[y\ y-2\ 0]$	$\left[\frac{3y(y-1)}{10(y+1)(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-1)(y+4)}{2(y+1)(y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+4)(y+5)}{5(y+1)(y+2)}\right]^{\frac{1}{2}}$
$[y+1\ y1]$	$-\left[\frac{2y(y+4)}{5(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{6}{(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{3(y-1)(y+5)}{5(y+1)(y+3)}\right]^{\frac{1}{2}}$
$[y+2\ y+2\ 2]$	$\left[\frac{3(y+4)(y+5)}{10(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{y(y+5)}{2(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\left[\frac{y(y-1)}{5(y+2)(y+3)}\right]^{\frac{1}{2}}$
Σ_2	n_-	$-\frac{1}{3}(n-2)[15y(y+4)]^{\frac{1}{2}}$	$\frac{1}{3}[6y(y-1)(y+4)(y+5)]^{\frac{1}{2}}$

TABLE A8.3. $U([y00][1^2][y00][1^2];\ [f^{(12)}][f^{(23)}])$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]
$[y+1\ 10]$	$\left[\frac{y+4}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{y}{2(y+2)}\right]^{\frac{1}{2}}$
$[y\ 11]$	$-\left[\frac{y}{2(y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y+4)}{2(y+2)}\right]^{\frac{1}{2}}$
Σ_2[a]	n_+	$\frac{2}{3}(n-y)[y(y+4)]^{\frac{1}{2}}$

[a] With $[y00] \to [yyy]$, for $[f^{(23)}] = [211]$; $\Sigma_2 \to \frac{2}{3}(n+y+4)[y(y+4)]^{\frac{1}{2}}$.

TABLE A8.4. $U([y00][2^2][y00][2];\ [f^{(12)}][f^{(23)}])$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211]	[422]
$[y-2\ 00]$	$\left[\dfrac{y(y-1)}{10(y+2)(y+3)}\right]^{\frac{1}{2}}$	$-\left[\dfrac{(y-1)(y+4)}{2(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{2(y+4)(y+5)}{5(y+2)(y+3)}\right]^{\frac{1}{2}}$
$[y11]$	$\left[\dfrac{3y}{10(y+2)}\right]^{\frac{1}{2}}$	$\dfrac{-(y+8)}{[6(y+2)(y+4)]^{\frac{1}{2}}}$	$-\left[\dfrac{8(y-1)(y+5)}{15(y+2)(y+4)}\right]^{\frac{1}{2}}$
$[y+2\ 22]$	$\left[\dfrac{3(y+5)}{5(y+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{y(y+5)}{3(y+3)(y+4)}\right]^{\frac{1}{2}}$	$\left[\dfrac{y(y-1)}{15(y+3)(y+4)}\right]^{\frac{1}{2}}$
Σ_2	n_-	$-_{8}(3n+y-4)[5y(y+4)]$	$[y(y-1)(y+4)(y+5)]^{\frac{1}{2}}$

TABLE A8.5. $U([y\ y-1\ 0][1^2][y\ y-1\ 0][1^2];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.[a]

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[22]	[211] $\rho=1$	[211] $\rho=2$
$[y+1\ y\ 0]$	$\left[\dfrac{(y+4)}{6y}\right]^{\frac{1}{2}}$	$\left[\dfrac{(y-1)(y+1)}{3y(y+2)}\right]^{\frac{1}{2}}$	$-\dfrac{(2y-1)[y+4]^{\frac{1}{2}}}{[2y(4y^2+12y-1)]^{\frac{1}{2}}}$	$\left[\dfrac{3(y-1)(y+1)}{y(y+2)(4y^2+12y-1)}\right]^{\frac{1}{2}}$
$[y-1\ y-2\ 0]$	$\left[\dfrac{(y-1)}{6(y+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(y+2)(y+4)}{3(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\dfrac{(2y+7)[y-1]^{\frac{1}{2}}}{[2(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\left[\dfrac{3(y+2)(y+4)}{(y+1)(y+3)(4y^2+12y-1)}\right]^{\frac{1}{2}}$
$[yy1]$	$-\dfrac{1}{\sqrt{6}}$	$\left[\dfrac{(y-1)(y+4)}{12(y+1)(y+2)}\right]^{\frac{1}{2}}$	$\dfrac{-5}{[2(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\dfrac{(2y+3)[3(y-1)(y+4)]^{\frac{1}{2}}}{2[(y+1)(y+2)(4y^2+12y-1)]^{\frac{1}{2}}}$
$[y+1\ y-1\ 1]$	$-\left[\dfrac{(y-1)(y+4)}{2y(y+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(y+1)(y+2)}{4y(y+3)}\right]^{\frac{1}{2}}$	$-\left[\dfrac{3(y-1)(y+4)}{2y(y+3)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\dfrac{(2y+3)[(y+1)(y+2)]^{\frac{1}{2}}}{2[y(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$
Σ_2	n_+	$\frac{1}{2}[2(y-1)(y+1)(y+2)(y+4)]^{\frac{1}{2}}$	$\dfrac{\sqrt{3}\left\{(y-1)(y+1)(2y+7) + x(4y^2+12y-1)\right\}}{2[(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\frac{3}{2}\left[\dfrac{2(y-1)(y+1)(y+2)(y+4)}{(4y^2+12y-1)}\right]^{\frac{1}{2}}$

[a] $U([yy1][1^2][yy1][1^2];\ [f^{(12)}][f^{(23)}]\rho) = (-1)^{\rho+1}U([yy-1\ 0][1^2][y\ y-1\ 0][1^2];\ [f^{(12)*}][f^{(23)}]\rho)$. The Σ_2 values are the same as the above except for [211] $\rho=1$ for which $\Sigma_2 = \{\sqrt{3}[(y+1)(y+2)(2y+3) + x(4y^2+12y-1)]\}/\{2[(4y^2+12y-1)]\}^{\frac{1}{2}}$.

TABLE A8.6. $U([y\ y-1\ 0][2^3][y\ y-1\ 0][2];\ [f^{(12)}], [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$	[422] $\rho = 1$	[422] $\rho = 2$
$[y-1\ y-2\ 0]$	$-\left[\frac{y-1}{10(y+3)}\right]^{\frac{1}{2}}$	$\frac{(2y+11)[y-1]^{\frac{1}{2}}}{[6(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(4y-5)[(y+2)(y+4)]^{\frac{1}{2}}}{3[(y+1)(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(3y+22)[y+4]^{\frac{1}{2}}}{3[5(y+3)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{2(y-2)(y+2)(y+4)(y+5)}{3(y+1)(y+3)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[yy1]$	$\frac{1}{\sqrt{10}}$	$\frac{-3\sqrt{3}}{[2(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{-2(y+11)[(y-1)(y+2)]^{\frac{1}{2}}}{2[(y+1)(y+4)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(3y+22)[y-1]^{\frac{1}{2}}}{2[5(y+4)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{3(y-1)(y-2)(y+2)(y+5)}{2(y+1)(y+4)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[y\ y-3\ 0]$	$\left[\frac{(y-1)(y-2)}{5y(y+1)}\right]^{\frac{1}{2}}$	$\frac{-(2y+5)[(y-1)(y-2)]^{\frac{1}{2}}}{[3y(y+1)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{2}{3}\left[\frac{2(y-2)(y+2)(y+4)}{y(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{-2(3y+7)[(y-2)(y+4)]^{\frac{1}{2}}}{3[10y(y+1)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{(y+2)(y+4)(y+5)}{3y(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[y+1\ y-1\ 1]$	$-\left[\frac{3(y-1)(y+4)}{10y(y+3)}\right]^{\frac{1}{2}}$	$5\left[\frac{(y-1)(y+4)}{2y(y+3)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{-(2y^2+11y+32)[y+1]^{\frac{1}{2}}}{2[3y(y+2)(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{-(9y^2+17y-56)}{2[15y(y+3)(3y^2+9y-2)]^{\frac{1}{2}}}$	$\frac{-(y+4)[(y+1)(y-2)(y+5)]^{\frac{1}{2}}}{[2y(y+2)(y+3)(3y^2+9y-2)]^{\frac{1}{2}}}$
$[y+2\ y+1\ 2]$	$\left[\frac{3(y+5)}{10(y+1)}\right]^{\frac{1}{2}}$	$\frac{(2y-1)[y+5]^{\frac{1}{2}}}{[2(y+1)(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\left[\frac{3(y-1)(y+5)}{(y+2)(y+4)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{-(y-1)[3(y-1)(y+5)]^{\frac{1}{2}}}{[5(y+1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{2(y-1)(y-2)}{(y+2)(y+4)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
Σ_2	n_-	$\frac{-3\sqrt{5}\left\{\begin{array}{l}(y-1)(y+3)(2y-1)\\ \quad +x(4y^2+12y-1)\end{array}\right\}}{2[3(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\left[\frac{5(y-1)(y+1)(y+2)(y+4)}{2(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{1}{2}[2(y-1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}$	0

TABLE A8.7. $U([yy1][2^3][yy1][2];\ [f^{(12)}], [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$	[422] $\rho = 1$	[422] $\rho = 2$
$[y+1\ y+1\ 1]$	$-\left[\frac{y+4}{10y}\right]^{\frac{1}{2}}$	$-\frac{(2y-5)[y+4]^{\frac{1}{2}}}{[6y(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{-(4y+17)[(y-1)(y+1)]^{\frac{1}{2}}}{3[y(y+2)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(3y-13)[y-1]^{\frac{1}{2}}}{[45y(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{2(y-1)(y+1)(y-2)(y+5)}{3y(y+2)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[y\ y-1\ 0]$	$\frac{1}{\sqrt{10}}$	$\frac{-3\sqrt{3}}{[2(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(2y-5)[(y+1)(y+4)]^{\frac{1}{2}}}{2[(y-1)(y+2)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{(3y-13)[y+4]^{\frac{1}{2}}}{2[5(y-1)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{3(y+1)(y-2)(y+4)(y+5)}{2(y-1)(y+2)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[y+2\ y+2\ 3]$	$\left[\frac{(y+4)(y+5)}{5(y+2)(y+3)}\right]^{\frac{1}{2}}$	$\frac{(2y+1)[(y+4)(y+5)]^{\frac{1}{2}}}{[3(y+2)(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\frac{2}{3}\left[\frac{2(y-1)(y+1)(y+5)}{(y+3)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{-(3y+2)[2(y-1)(y+5)]^{\frac{1}{2}}}{3[5(y+2)(y+3)(3y^2+9y-2)]^{\frac{1}{2}}}$	$-\left[\frac{(y-1)(y+1)(y-2)}{3(y+3)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
$[y+1\ y\ 2]$	$-\left[\frac{3(y-1)(y+4)}{10y(y+3)}\right]^{\frac{1}{2}}$	$5\left[\frac{(y-1)(y+4)}{2y(y+3)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{(2y^2+y+17)[y+2]^{\frac{1}{2}}}{2[3y(y+1)(y+3)(4y^2+12y-1)]^{\frac{1}{2}}}$	$\frac{-(9y^2+37y-26)}{2[15y(y+3)(3y^2+9y-2)]^{\frac{1}{2}}}$	$\frac{-(y-1)[(y-2)(y+2)(y+5)}{[2y(y+1)(y+3)(3y^2+9y-2)^{\frac{1}{2}}}$
$[y\ y-2\ 1]$	$\left[\frac{3(y-2)}{10(y+2)}\right]^{\frac{1}{2}}$	$\frac{-(2y+7)[y-2]^{\frac{1}{2}}}{[2(y+2)(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\left[\frac{3(y-2)(y+4)}{(y-1)(y+1)(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$\frac{-(y+4)[3(y-2)(y+4)]^{\frac{1}{2}}}{[5(y-1)(y+2)(3y^2+9y-2)^{\frac{1}{2}}}$	$\left[\frac{2(y+4)(y+5)}{(y-1)(y+1)(3y^2+9y-2)}\right]^{\frac{1}{2}}$
Σ_2	n_-	$\frac{-\sqrt{15}\left\{\begin{array}{l}(y-1)(2y^2+7y+2)\\ \quad +x(4y^2+12y-1)\end{array}\right\}}{2[(4y^2+12y-1)]^{\frac{1}{2}}}$	$-\left[\frac{5(y-1)(y+1)(y+2)(y+4)}{2(4y^2+12y-1)}\right]^{\frac{1}{2}}$	$-\frac{1}{2}[2(y-1)(y+4)(3y^2+9y-2)]^{\frac{1}{2}}$	0

TABLE A8.8. $U([y11][1^2][y11][1^2];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.[a]

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$	[22]
$[y00]$	$-\frac{1}{3}\left[\frac{y+2}{2y}\right]^{\frac{1}{2}}$	$\frac{(y-6)}{[6y(3y+2)]^{\frac{1}{2}}}$	$-\left[\frac{4(y-1)(y+4)}{3y(3y+2)}\right]^{\frac{1}{2}}$	$\frac{2}{3}\left[\frac{(y-1)}{y}\right]^{\frac{1}{2}}$
$[y-1\ 10]$	$-\left[\frac{(y-1)(y+2)}{6y(y+3)}\right]^{\frac{1}{2}}$	$\frac{-(y+6)[y-1]^{\frac{1}{2}}}{[2y(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{-(y-2)[y+4]^{\frac{1}{2}}}{[y(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{-(y+2)}{[3y(y+3)]^{\frac{1}{2}}}$
$[y+1\ 21]$	$\frac{2}{3}\left[\frac{(y+2)(y+4)}{(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\frac{-2y[y+4]^{\frac{1}{2}}}{[3(y+1)(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{-(y-2)[y-1]^{\frac{1}{2}}}{[6(y+1)(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{1}{3}\left[\frac{(y-1)(y+4)}{2(y+1)(y+3)}\right]^{\frac{1}{2}}$
$[y22]$	$-\left[\frac{(y-1)}{3(y+1)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-1)(y+2)}{(y+1)(3y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+2)(y+4)}{2(y+1)(3y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{(y+2)}{6(y+1)}\right]^{\frac{1}{2}}$
Σ_2	n_+	$\frac{\{2(y+4)+x(3y+2)\}}{2}\left[\frac{3(y+2)}{(3y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{6(y-1)(y+2)(y+4)}{(3y+2)}\right]^{\frac{1}{2}}$	$[2(y-1)(y+2)]^{\frac{1}{2}}$

[a] $U([y y-1\ y-1][1^2][y y-1\ y-1][1^2];\ [f^{(12)}],[f^{(23)}]\rho) = (-1)^{\rho+1}\, U([y11][1^2][y11][1^2];\ [f^{(12)*}],[f^{23}]\rho)$. The Σ_2 values are the same as the above except for [211] $\rho = 1$ for which $\Sigma_2 = \frac{1}{2}[3(y+2)(y-1) + x(3y+2)][3(y+2)/(3y+2)]^{\frac{1}{2}}$.

TABLE A8.9. $U([y11][2^3][y11][2];\ [f^{(12)}],\ [f^{(23)}]\rho_{1,23})$.

$[f^{(12)}]$ \ $[f^{(23)}]$	[0]	[211] $\rho = 1$	[211] $\rho = 2$	[422] $\rho = 1$	[422] $y = 2$
$[y-1\ 10]$	$-\left[\frac{(y-1)(y+2)}{10y(y+3)}\right]^{\frac{1}{2}}$	$\frac{(y+10)[y-1]^{\frac{1}{2}}}{[6y(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{(y-4)[y+4]^{\frac{1}{2}}}{[3y(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{-(2y-11)[2(y+4)]^{\frac{1}{2}}}{[15y(y+3)(3y+1)]^{\frac{1}{2}}}$	$\left[\frac{5(y-2)(y+4)(y+5)}{3y(y+3)(3y+1)}\right]^{\frac{1}{2}}$
$[y-2\ 11]$	$\left[\frac{(y-1)(y-2)}{10y(y+3)}\right]^{\frac{1}{2}}$	$\frac{-(3y+10)[(y-1)(y-2)]^{\frac{1}{2}}}{[6y(y+3)(y+2)(3y+2)]^{\frac{1}{2}}}$	$4\left[\frac{(y-2)(y+4)}{3y(y+3)(y+2)(3y+2)}\right]^{\frac{1}{2}}$	$\frac{-(3y+11)[2(y-2)(y+4)]^{\frac{1}{2}}}{[15y(y+3)(y+2)(3y+1)]^{\frac{1}{2}}}$	$2\left[\frac{5(y+4)(y+5)}{3y(y+3)(y+2)(3y+1)}\right]^{\frac{1}{2}}$
$[y22]$	$\left[\frac{(y-1)}{5(y+1)}\right]^{\frac{1}{2}}$	$\frac{-(y+6)[y-1]^{\frac{1}{2}}}{[3(y+1)(y+2)(3y+2)]^{\frac{1}{2}}}$	$\frac{(y^2+4y+20)}{[6(y+1)(y+4)(y+2)(3y+2)]^{\frac{1}{2}}}$	$\frac{(4y^2+19y+2)}{[15(y+1)(y+2)(y+4)(3y+1)]^{\frac{1}{2}}}$	$y\left[\frac{5(y-2)(y+5)}{6(y+1)(y+2)(y+4)(3y+1)}\right]^{\frac{1}{2}}$
$[y+1\ 21]$	$-\left[\frac{4(y+2)(y+4)}{15(y+1)(y+3)}\right]^{\frac{1}{2}}$	$\frac{-2(y-4)[y+4]^{\frac{1}{2}}}{3[(y+1)(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{(5y+22)[y-1]^{\frac{1}{2}}}{3[2(y+1)(y+3)(3y+2)]^{\frac{1}{2}}}$	$\frac{(2y-11)[y-1]^{\frac{1}{2}}}{3[5(y+1)(y+3)(3y+1)]^{\frac{1}{2}}}$	$-\frac{1}{3}\left[\frac{5(y-1)(y-2)(y+5)}{2(y+1)(y+3)(3y+1)}\right]^{\frac{1}{2}}$
$[y+2\ 33]$	$\left[\frac{(y+5)}{3(y+3)}\right]^{\frac{1}{2}}$	$\frac{1}{3}\left[\frac{5(y+2)(y+5)}{(y+3)(3y+2)}\right]^{\frac{1}{2}}$	$\frac{1}{3}\left[\frac{10(y-1)(y+2)(y+5)}{(y+3)(y+4)(3y+2)}\right]^{\frac{1}{2}}$	$-\frac{1}{3}\left[\frac{(y-1)(y+2)(y+5)}{(y+3)(y+4)(3y+1)}\right]^{\frac{1}{2}}$	$-\frac{1}{3}\left[\frac{2(y-1)(y-2)(y+2)}{(y+3)(y+4)(3y+1)}\right]^{\frac{1}{2}}$
Σ_2[a]	n_-	$-\frac{\left\{\begin{array}{l}(y-1)(y+2)\\ \quad +x(3y+2)\end{array}\right\}}{2}\left[\frac{15(y+2)}{(3y+2)}\right]^{\frac{1}{2}}$	$\left[\frac{10(y-1)(y+2)(y+4)}{3(3y+2)}\right]^{\frac{1}{2}}$	$-\left[\frac{(y-1)(y+2)(y+4)(3y+1)}{3}\right]^{\frac{1}{2}}$	0

[a] If $[y11] \rightarrow [y\ y-1\ y-1]$, the Σ_2 values are the same as above except for [211] $\rho = 1$ for which $\Sigma_2 = -\frac{1}{2}\{2(y-1)y + x(3y+2)\}[15(y+2)/(3y+2)]^{\frac{1}{2}}$.