Equilibria of high pressure elliptic flux-conserving tokamak

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An analytical calculation is carried out to determine the plasma current and poloidal beta for a toroidal plasma with an elliptic cross section under the constraint of flux conservation. It is shown that the pressure buildup during heating occurs in two stages: An outward shift in the magnetic axis and an elongation in the flux surfaces. The total plasma current increases with the pressure variable more rapidly for an elliptic cross section than it does for a circular cross section. This current rise produces a rather slow rise of the poloidal beta with pressure, particularly for large elongations. The ensuing class of flux conserving equilibria is characterized by large beta but modest poloidal beta.

INTRODUCTION

Considerable interest has recently developed in the concept of flux conservation[1,2] in tokamaks as a means of enhancing the beta (ratio of plasma to magnetic field pressures) of these devices. Since the power density of a magnetically confined fusion reactor depends on the square of beta, and enhancement in this quantity could have a major impact on the size and/or cost of the reactor.

High beta equilibria with no limitation on the poloidal beta have been shown[2,3] to exist in axisymmetric devices with circular cross section. Moreover, computer simulations[4] have indicated that magnetohydrodynamic equilibria can exist in tokamaks with D-shaped cross section with β values of up to 30%. Numerical analysis indicates that keeping the poloidal beta close to unity while maximizing beta is desirable[5] for ideal magnetohydrodynamic stability.

In this paper we investigate the equilibrium properties of a flux-conserving tokamak with an elliptic cross section whose inverse ellipticity is given by $\kappa = a^2/b^2$ with a and b being the semi-minor and major axes, respectively. We will examine the effects of eilliptical elongation on the pressure driven toroidal current rise described in Ref. 2 by keeping the safety factor q and the minor plasma radius a the same as in the circular cross section case. We will show that the plasma pressure buildup results in an outward shift of the magnetic axis and an elongation of the magnetic flux surfaces. The plasma current rises nonlinearly with pressure even faster than in the circular case[2] such that the poloidal beta grows slower than linearly, a desirable effect for plasma stability.

ANALYSIS

We begin with the well-known relations between the poloidal beta, β_I, the diamagnetic parameter, μ_I, and the inductance parameter per unit length, l_i, for a toroidal plasma with arbitrary cross section in equilibrium given by[6]

$$\beta_I - \tfrac{1}{3}\mu_I + \tfrac{1}{3} l_i = \tfrac{2}{3}(s_1 + s_2) , \tag{1}$$

$$\beta_I + \mu_I + l_i = 2 s_2 , \tag{2}$$

where

$$\beta_I \equiv 2 \int p\, dV (2\pi R_c I^2)^{-1} , \tag{3a}$$

$$\mu_I \equiv 2 \int \frac{dV}{8\pi} (B_{\varphi 0}^2 - B_\varphi^2)(2\pi R_c I^2)^{-1} , \tag{3b}$$

$$l_i \equiv 2 \int \frac{dV}{8\pi} B_p^2 (2\pi R_c I^2)^{-1} , \tag{3c}$$

and

$$s_1 = \frac{I}{2\pi R I^2} \oint \left[\frac{B_\tau^2 - B_n^2}{8\pi} \rho \hat{n} \cdot \hat{e}_\rho - \frac{B_\tau B_n}{4\pi} \rho \hat{\tau} \cdot \hat{e}_\rho \right] dS_n , \tag{4}$$

$$s_2 = \frac{1}{2\pi I^2} \oint \left[\frac{B_\tau^2 - B_n^2}{8\pi} \hat{n} \cdot \hat{e}_x - \frac{B_n B_\tau}{4\pi} \hat{\tau} \cdot \hat{e}_x \right] dS_n . \tag{5}$$

In Eqs. (1)–(5) B_φ is the toroidal magnetic field inside the volume considered, and B_n and B_τ are the normal and tangential components, respectively, of the poloidal field on the surface (see Fig. 1). The total plasma current $I(\psi)$ inside the flux surface ψ is given by

$$I(\psi) = V'(\psi) \langle B_p^2 \rangle_\psi / 8\pi^2 = \int_\psi \frac{B_p}{4\pi} dl , \tag{6}$$

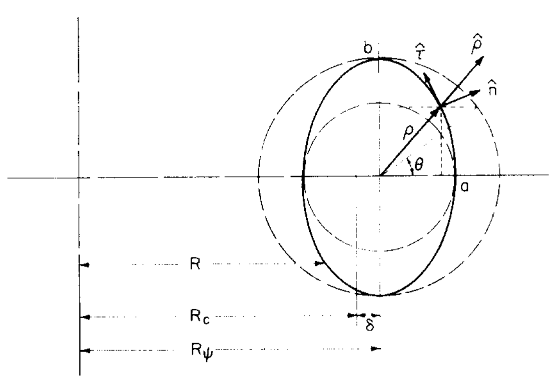

FIG. 1. Elliptic flux-surface geometry.

where B_p is the poloidal magnetic field. Although full determination of the flux function requires the complete solution of the equilibrium problem it is sufficient here to employ representation for ψ corresponding to a Fourier series $\bar{\rho} = \sum a_n(\psi) \cos n\theta$ as follows[7]:

$$\psi = S(\bar{\rho}^2) \tag{7a}$$

$$\bar{\rho}^2 = X^2 + \kappa(\psi) Z^2 + \tau(\psi)[X^3 - 3XZ^2] + \dots \tag{7b}$$

with

$$X = R - R_\psi , \tag{8a}$$

$$R_\psi = R_c + \delta(\psi) , \tag{8b}$$

where $\kappa(\psi) = a^2/b^2$ is the ellipticity parameter introduced earlier, $\tau(\psi)$ is the triangularity parameter, and $\delta(\psi)$ denotes the shift from the geometric center, R_c, of the center of the flux tube R_ψ as illustrated in Fig. 1. It should be noted that the ellipticity parameter takes on a constant value, κ_0, at the plasma boundary and assumes a smaller value at the magnetic axis, i.e., $\kappa(\psi = 0) < \kappa_0$.

The triangularity parameter $\tau(\psi)$ must vanish on the elliptic boundary where we wish to calculate s_1, s_2, and I, but it increases toward the magnetic axis. However, if we focus our attention on a plasma with strong ellipticity at the boundary, the effect of the triangularity becomes less promounced in the inner regions of the plasma and will be neglected in this analysis whose main purpose is to demonstrate that the pressure dependence of the plasma current and the poloidal beta under flux conservation will be enhanced by the effects of ellipticity.

Since the poloidal field, B_p, is related to the flux through $|\nabla\psi| = RB_p$, it can now be written as

$$B_p = \frac{1}{R} \frac{2\dot{S}(X^2 + \kappa^2 Z^2)^{1/2}}{|1 + 2\delta' \dot{S} X - \kappa' \dot{S} Z^2|} , \tag{9}$$

where $\dot{S} = dS/d\bar{\rho}^2$, $\delta' = \partial\delta/2\psi$, and $\kappa' = \partial\kappa/\partial\psi$. To further simplify the calculations we consider the case where the boundary coincides with the flux surfaces so that we can employ the elliptic coordinate system given by[8]

$$X = c \cosh u \cos\theta = a \cos\theta , \tag{10a}$$

$$Z = c \sinh u \sin\theta = b \sin\theta . \tag{10b}$$

In view of these transformations we can write

$$\hat{n} \cdot \hat{e}_p = ab[(a^2 \cos^2\theta + b^2 \sin^2\theta) \times (a^2 \sin^2\theta + b^2 \cos^2\theta)]^{-1/2} , \tag{10c}$$

$$\hat{n} \cdot \hat{e}_x = b \cos\theta (a^2 \sin^2\theta + b^2 \cos^2\theta)^{-1/2} , \tag{10d}$$

$$ds_n = 2\pi R (a^2 \sin^2\theta + b^2 \cos^2\theta)^{1/2} \, d\theta , \tag{10e}$$

so that s_1, and s_2 as given by Eqs. (4) and (5) may now be put in the form

$$s_1 \simeq \frac{(\dot{S}a\tilde{\epsilon})^2}{\sqrt{\kappa} I^2} \oint \frac{d\theta}{2\pi} \frac{[(1-\kappa)\cos^2\theta + \kappa](1 - \tilde{\epsilon}\cos\theta)}{(g\cos^2\theta + d\cos\theta + 1 - g)^2} , \tag{11}$$

$$s_2 \simeq \frac{(\dot{S}a)^2 \tilde{\epsilon}}{\sqrt{\kappa} I^2} \oint \frac{d\theta}{2\pi} \frac{[(1-\kappa)\cos^2\theta + \kappa] \cos\theta (1 - \tilde{\epsilon}\cos\theta)}{(g\cos^2\theta + d\cos\theta + 1 - g)^2} . \tag{12}$$

In Eqs. (11) and (12) $\tilde{\epsilon} = a(\psi)/b(\psi)$, and

$$d = 2\dot{S}\delta' a , \tag{13}$$

$$g = \dot{S}\kappa' \kappa^{-1} a^2 , \tag{14}$$

while the total plasma current is given by

$$I(\psi) \simeq \dot{S}a\tilde{\epsilon}\kappa^{-1/2} \oint \frac{d\theta}{2\pi} \frac{[(1-\kappa)\cos^2\theta + \kappa](1 - \tilde{\epsilon}\cos\theta)}{g\cos^2 + d\cos\theta + 1 - g} . \tag{15}$$

Note that the parameter $|d|$ represents the effect of the horizontal displacement of R_ψ (relative to the geometric center R_c) while the elongation parameter g reflects the effect of the vertical deformation during the creation of a high pressure plasma under the constraint of flux conservation. We shall examine these two parameters in detail later, but for the purpose of carrying out the integrations in Eqs. (11), (12), and (15) we first let $t = \tan(\theta/2)$ so that

$$s_1 = \frac{(\dot{S}a\tilde{\epsilon})^2}{\sqrt{\kappa} I^2} \frac{2}{\pi} \int_0^\infty dt \frac{\eta_1 t^6 + \eta_2 t^4 + \eta_3 t^2 + \eta_4}{\{(1-d)t^4 + 2(1-2g)t^2 + 1 - d\}^2} , \tag{16}$$

$$s_2 = \frac{(\dot{S}a)^2 \tilde{\epsilon}}{\sqrt{\kappa} I^2} \frac{2}{\pi} \int_0^\infty dt \frac{\xi_1 t^8 + \xi_2 t^6 + \xi_3 t^4 + \xi_4 t^2 + \xi_5}{(1+t^2)[(1-d)t^4 + 2(1-2g)t^2 + 1 - d]^2} , \tag{17}$$

$$I(\psi) = \dot{S}a\tilde{\epsilon}\kappa^{-1/2} \frac{2}{\pi} \times \int_0^\infty dt \frac{\eta_1 t^6 + \eta_2 t^4 + \eta_3 t^2 + \eta_4}{(1+t^2)^2[(1-d)t^4 + 2(1-2g)t^2 + 1 - d]} , \tag{18}$$

where

$$\eta_1 = 1 + \tilde{\epsilon}, \quad \eta_2 = -1 - 3\tilde{\epsilon} + 4\kappa(1 + \tilde{\epsilon}), \quad \eta_3 = -1 + 3\tilde{\epsilon} + 4\kappa(1 - \tilde{\epsilon}), \quad \eta_4 = 1 - \tilde{\epsilon}, \tag{19a}$$

and

$$\xi_1 = -1 - \tilde{\epsilon}, \quad \xi_2 = 2[\tilde{\epsilon} + (1 - 2\kappa)(1 + \tilde{\epsilon})], \quad \xi_3 = 2\tilde{\epsilon}(4\kappa - 3\tilde{\epsilon}), \quad \xi_4 = 2[\tilde{\epsilon} - (1 - 2\kappa)(1 + \tilde{\epsilon})], \text{ and } \xi_5 = 1 - \tilde{\epsilon}. \tag{19b}$$

These integrals have nontrivial solutions only when the inequality $|1 - 2g| \geq (1 - d^2)^{1/2}$ is satisfied. Since, as pointed out earlier, κ has a smaller value near the magnetic axis than it does at the boundary, it may be noted from (14) that the parameter g will also be smaller in the same region. From (14) and (7) it can be seen that the range of g is between zero and 1/2 so that the domain for $|d|$ and g is below the curve $2g = 1 - (1 - d^2)^{1/2}$ and above $g = 0$ shown in Fig. 2. The curve in question yields some interesting and important information regarding the plasma behavior during the process of

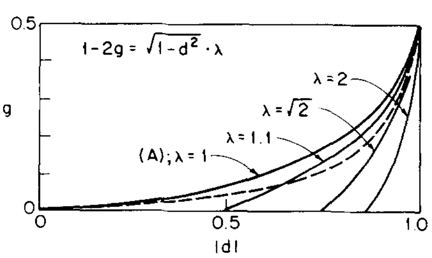

FIG. 2. Variation of the magnetic axis shift with the elongation parameter.

pressure buildup. Two different phenomena take place: a shift of the magnetic axis and an elongation of the flux surface. In the early stages of heating the dominant effect is the outward shift of the plasma since the elongation occurs slowly. However, as the plasma pressure increases, the secondary stage sets in at a turning point defined by $dg/d|d|=1$ for which $|d|$ has a value of 0.8. At this point the effect of the magnetic axis shift saturates and the effect of elongation becomes dominant. This observation is in agreement with the conclusions of the computer studies by Dory and Peng.[3] [Present day low beta tokamak experiments seem to indicate that $|d|$ is of the order of ϵ (the ratio of the minor radius to the major radius at the boundary) while the elongation effect is so weak that g is of the third order in ϵ.]

These considerations suggest replacing the inequality connecting $|d|$ and g by the relation $1-2g=\lambda(1-d^2)$ where λ is a constant equal to or larger than unity. We have sketched this equation in Fig. 2 for $\lambda=1.1$, $\sqrt{2}$, and 2. We observe that for large values of λ the elongation effect rises very sharply but sets in at relatively large shifts $|d|$. It appears therefore that the most probable connection between $|d|$ and g is likely to be given by $1-2g=[1+\hat{\beta}(d)](1-d^2)^{1/2}$ where $\hat{\beta}(d)$ may be of order of β^2. The corresponding curve in Fig. 2 is the one shown by the broken line. It follows curve (A) closely at low pressure because of the small value of β in this region. However, as the plasma pressure increases, β also increases and the curve departs from (A) and asymptotically approaches the curve associated with large $\hat{\beta}$. The transition from one curve to the other does not have a serious effect on our analysis so that to zero order approximation in β we shall take curve (A), namely, $1-2g=(1-d^2)^{1/2}$, as the link between the parameters $|d|$ and g. The ensuing equilibrium model contains the essence of the pressure driven noncircular distortions yet enables an analytic investigation of flux conserving effects.

With this we can evaluate the integrals in s_1, s_2, and the plasma current to obtain

$$s_1=\frac{(\dot{S}\tilde{\epsilon}a)^2}{\sqrt{\kappa}I^2}\,\frac{1}{(1-d)^2}\left(\frac{5}{16}\frac{\eta_1}{\sqrt{\alpha}}+\frac{1}{16}\frac{\eta_2}{\sqrt{\alpha^3}}+\frac{1}{16}\frac{\eta_3}{\sqrt{\alpha^5}}+\frac{5}{16}\frac{\eta_4}{\sqrt{\alpha^7}}\right), \tag{20}$$

$$s_2=\frac{(\dot{S}a)^2\tilde{\epsilon}}{\sqrt{\kappa}\,I^2}\,\frac{1}{(1-d)^2}\left[\frac{\xi_1-\xi_2+\xi_3-\xi_4+\xi_5}{(1-\alpha)^4}\right.$$
$$\times\left(1-\frac{5}{16\sqrt{\alpha}}+\frac{1-4\alpha}{16\sqrt{\alpha^3}}-\frac{1-4\alpha+6\alpha^2}{16\sqrt{\alpha^5}}-\frac{5\alpha^4}{16\sqrt{\alpha^7}}\right)$$
$$\left.+\frac{5\xi_1}{16\sqrt{\alpha}}+\frac{-\xi_1+\xi_2}{16\sqrt{\alpha^3}}+\frac{\xi_1-\xi_2+\xi_3}{16\sqrt{\alpha^5}}+\frac{5\xi_5}{16\sqrt{\alpha^7}}\right], \tag{21}$$

and

$$(\psi)=\dot{S}a\tilde{\epsilon}\kappa^{-1/2}\,\frac{1}{2(1-d)}$$
$$\times\left[\frac{(1-3\alpha)\eta_1+2\alpha\eta_2-(1+\alpha)\eta_3+2\eta_4}{(1-\alpha)^3}\left(1-\frac{1}{\sqrt{\alpha}}\right)\right.$$
$$+\frac{2\alpha\eta_1-(1+\alpha)\eta_2+2\eta_3-(3-\alpha)\eta_4}{(1-\alpha)^3}$$
$$\left.\times(-1+\sqrt{\alpha})+\frac{\eta_1}{\sqrt{\alpha}}+\frac{\eta_3}{\sqrt{\alpha^3}}\right], \tag{22}$$

where

$$\alpha=\frac{1-2g}{1-d}=\left(\frac{1+d}{1-d}\right)^{1/2}. \tag{23}$$

It might be noted in Eqs. (21) and (22) that the terms containing $(1-\alpha)^4$ and $(1-\alpha)^3$ in the denominator come from the substitution for $\cos\theta$ in the original integrals. These terms cancel out in the low pressure limit, i.e., $\alpha\rightarrow 1$, and do not contribute significantly in the high pressure limit, $\alpha\rightarrow 0$, compared with the rest of the terms. We therefore find that the total plasma current approximately reduces to

$$I(\psi)=\dot{S}a\tilde{\epsilon}\kappa^{-1/2}\,\frac{1}{2(1-d)}$$
$$\times\left[(1+\tilde{\epsilon})\left(\frac{1-d}{1+d}\right)^{1/4}+(1-\tilde{\epsilon})\left(\frac{1-d}{1+d}\right)^{3/4}\right] \tag{24}$$

which increases with ellipticity by the factor $\kappa^{-1/2}$ as it does in the low beta case. The second term in Eq. (24) reveals the effect of the elongation parameter g, and it shows that this effect results in a larger plasma current by a factor $[(1-d)/(1-2g)]^{1/2}>1$ compared with the circular case of Ref. 2.

In the high pressure limit, the terms containing $\alpha^{-5/2}$ and $\alpha^{-7/2}$ in Eqs. (20) and (21) will dominate. Taking these leading terms and substituting these results in Eqs. (1) and (2) we obtain the following relations between β_I, μ_I, and l_i, at the boundary,

$$\beta_I+\frac{l_i}{2}=\frac{\sqrt{\kappa}}{4}\,\frac{\left(-\frac{1}{2}-\frac{9}{2}\epsilon+10\kappa-2\kappa\epsilon\right)\left(\frac{1-d}{1+d}\right)^{5/4}+5\left(-\frac{1}{2}-\frac{\epsilon}{2}+\frac{1}{\epsilon}\right)\left(\frac{1-d}{1+d}\right)^{7/4}}{\left[(1+\epsilon)\left(\frac{1-d}{1+d}\right)^{1/4}+(1-\epsilon)\left(\frac{1-d}{1+d}\right)^{3/4}\right]^2} \tag{25}$$

$$\mu_I+\frac{l_i}{2}=\frac{\sqrt{\kappa}}{4}\,\frac{\left(\frac{1}{2}-\frac{15}{2}\epsilon+6\kappa+2\kappa\epsilon\right)\left(\frac{1-d}{1+d}\right)^{5/4}+5\left(-\frac{3}{2}+\frac{\epsilon}{2}+\frac{1}{\epsilon}\right)\left(\frac{1-d}{1+d}\right)^{7/4}}{\left[(1+\epsilon)\left(\frac{1-d}{1+d}\right)^{1/4}+(1-\epsilon)\left(\frac{1-d}{1+d}\right)^{3/4}\right]^2}. \tag{26}$$

Keeping only the leading term in Eqs. (25) and (26), we obtain

$$\beta_I - \mu_I \approx \frac{5\sqrt{\kappa}}{4(1-\epsilon)}\left(\frac{1-d}{1+d}\right)^{1/4} \quad . \tag{27}$$

We now introduce the pressure variable which, when normalized to the initial current, I_0, can be defined as

$$\bar{\beta}_I \equiv 2 \int P\, dV (2\pi R_c I_0^2)^{-1} \quad . \tag{28}$$

Then, it can be seen that

$$\frac{\beta_I}{\bar{\beta}_I} \simeq \frac{I_0^2}{I^2} = \frac{4\dot{S}_0(1-d)^2}{\dot{S}\left[(1+\epsilon)\left(\frac{1-d}{1+d}\right)^{1/4} + (1-\epsilon)\left(\frac{1-d}{1+d}\right)^{3/4}\right]^2} \quad . \tag{29}$$

To determine $\dot{S}$ we recall the relation for the safety factor, namely,

$$\frac{q}{F} = \frac{V'\langle R^{-2}\rangle}{4\pi^2} = (2\dot{S}\sqrt{\kappa}R_\psi)(1 - \tfrac{1}{2}(\epsilon d) - \tfrac{1}{2}g) \quad . \tag{30}$$

Since $F = RB_\varphi = RB_{\varphi 0}[1 + O(\beta)]$, where $B_{\varphi 0}$ is the toroidal field in vacuum, it is a constant to zero order in β. Moreover, the changes in the ellipticity κ are also of order $O(\beta, \epsilon)$; therefore, it is readily seen that $\dot{S}$ undergoes only additive nonsingular changes of $O(\beta)$ which we neglect. From Eqs. (25) and (29), the parameter $|d|$ is related to the pressure variable as

$$\bar{\beta}_I = \frac{\sqrt{\kappa}}{16(1-d)^2}\left[\left(-\frac{1}{2} - \frac{9\epsilon}{2} + 10\kappa - 2\kappa\epsilon\right)\left(\frac{1-d}{1+d}\right)^{5/4} + 5\left(-\frac{1}{2} - \frac{\epsilon}{2} + \frac{1}{\epsilon}\right)\left(\frac{1-d}{1+d}\right)^{7/4}\right] , \tag{31}$$

where we have ignored l_i. If we only keep the leading last term in Eq. (31), then d can be obtained to yield

$$-d = 1 - \left(\frac{5\sqrt{\kappa}}{16\epsilon\bar{\beta}_I}\right)^{4/7} \tag{32}$$

valid for $\bar{\beta}_I > 16\epsilon/5\sqrt{\kappa}$.

Figure 3 shows the relations between the normalized current, I/ψ_0 and the pressure variable, $\bar{\beta}_I$, for various ellipticities at the plasma boundary calculated numerically from Eqs. (25) and (31). We see that the current increase with pressure is more dramatic for large ellipticities. In addition, we observe that the stronger the ellipticity the quicker the outward shift in the magnetic axis takes place, and hence the more effective the role of elongation is in the increase in the plasma current.

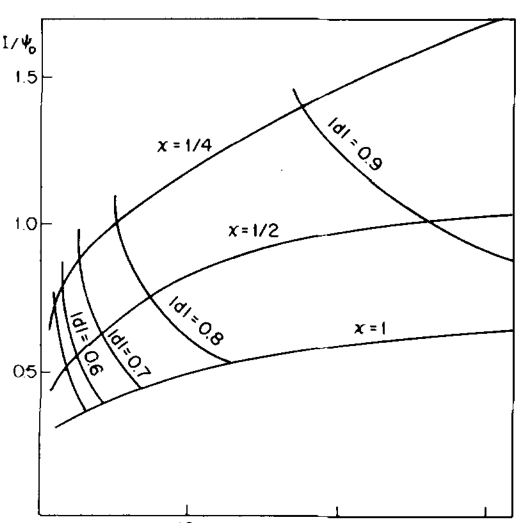

FIG. 3. Variation of the normalized plasma current with the pressure variable for various ellipticities.

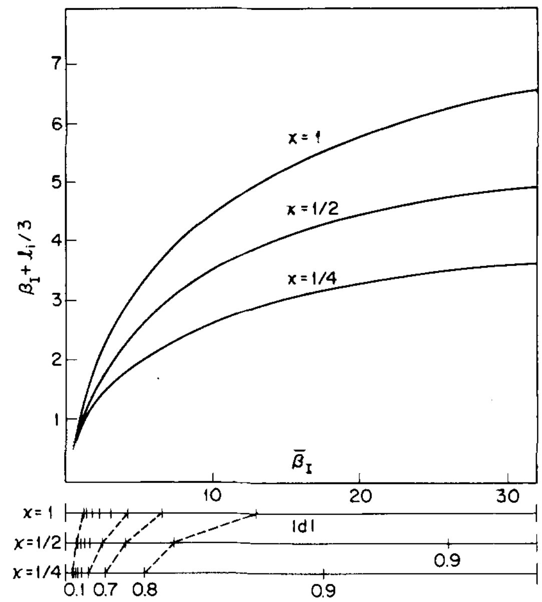

FIG. 4. Variation of the poloidal beta with the pressure variable and the magnetic shift for several ellipticities.

The variation of the poloidal beta with plasma pressure is illustrated in Fig. 4 where we observe that the effect of ellipticity is correspondingly less dramatic than it is in Fig. 3 for the current. At the bottom of Fig. 4 we show the parameter $|d|$ as a function of $\bar{\beta}_I$ for the three values of κ. We see that, in contrast to its effect on the current, the effect of strong ellipticity is to depress the rate of increase of the poloidal beta with the plasma pressure as may be readily deduced from Eq. (3a). Although not as striking as in the case of a circular cross section, the poloidal beta does nevertheless increase with pressure for a plasma with an elliptic cross section.

ACKNOWLEDGMENT

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