Stability of a current-carrying fluid with a free surface in a transverse magnetic field

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The stability of a viscous, incompressible, electrically conducting fluid with a free surface in a transverse magnetic field is investigated. A horizontal layer of fluid is set in motion by the combined action of the horizontal electric current and a vertically imposed magnetic field. An eigenvalue problem is formulated for infinitesimal disturbances and is solved by the regular perturbation method. It is found that if the electromagnetic effects are sufficient to overcome the stabilizing effect of gravity, the flow is unstable. The instability is due to the stratification of electric conductivity and to the longitudinal Lorentz force. Topological features of the $(c_1)_i$ curves for various cases are depicted. The roles played by the electric current as well as the magnetic field on the stability of the flow system are discussed briefly.

I. INTRODUCTION

In previous papers by Yih[1,2] and by Benjamin,[3] the flow of a layer of viscous fluid down an inclined plane was found to be unstable when the Reynolds number is sufficiently large. The instability is due to stratification in specific weights ρg when there is a longitudinal component of gravity.

Since an electromagnetic field with an attendant electric current gives rise to a body force, there are flows in which this body force, acting in the direction of flow, will cause instability in the same way as the longitudinal component of gravity in the problems studied by Yih and Benjamin. In this paper, the instability of one such flow is investigated. The unstable mode is due to the stratification in electric conductivity in the presence of the longitudinal Lorentz force.

To simplify the analysis, we shall consider the case of single horizontal layer of conducting fluid with a free surface. The air above the fluid is assumed nonconductive, so that the free surface is a surface of discontinuity in current density. The fluid is assumed viscous, incompressible, and its conductivity finite. A current of density j_0 is directed in the horizontal direction, in the direction of decreasing Z (Fig. 1), and a magnetic field of strength H_0 is superimposed in the direction of gravity along which Y is measured. The combined action of j_0 and H_0 caused the fluid to flow in the X direction. (Because in this case the gravitational acceleration is always present, the transverse component of the body force is never zero. Hence, the results given here correspond to the instability studied by Yih and Benjamin when their flow is not vertical.) The fundamental equations and the primary flow are described first. The investigation of its stability then follows. Finally, the results obtained are discussed in the concluding section.

II. FUNDAMENTAL EQUATIONS AND THE PRIMARY FLOW

In the current study, the motion generated by the combined action of the horizontal electric current and the vertical magnetic field is two dimensional (see Fig. 1). The velocity components in the directions of increasing X and Y are denoted by u and v, and the components of the magnetic field strength are denoted by H_x and H_y. The equations of motion for two-dimensional flow are[4,5]

$$\rho\frac{Du}{Dt} = -\frac{\partial}{\partial X}\left(p + \frac{\mu}{8\pi}|H|^2\right) + \rho\nu\Delta u + \frac{\mu}{4\pi}\left(H_x\frac{\partial H_x}{\partial X} + H_y\frac{\partial H_x}{\partial Y}\right), \tag{1}$$

$$\rho\frac{Dv}{Dt} = -\frac{\partial}{\partial Y}\left(p + \frac{\mu}{8\pi}|H|^2\right) + \rho g + \rho\nu\Delta v + \frac{\mu}{4\pi}\left(H_x\frac{\partial H_y}{\partial X} + H_y\frac{\partial H_y}{\partial Y}\right), \tag{2}$$

in which ρ is the density, t is the time, p is the pressure, g is the gravitational acceleration, $\rho\nu$ is the viscosity, μ is the magnetic permeability, and

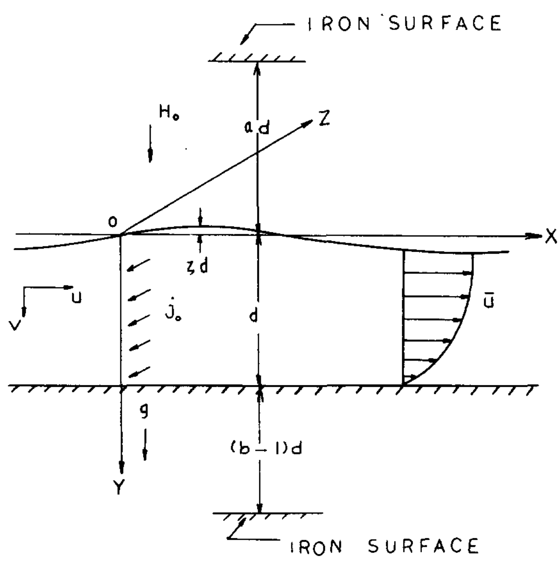

FIG. 1. Definition sketch.

$$\Delta = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial X} + v\frac{\partial}{\partial Y},$$

$$|H|^2 = H_x^2 + H_y^2.$$

The equations for the magnetic field are

$$\frac{DH_x}{Dt} = H_x\frac{\partial u}{\partial X} + H_y\frac{\partial u}{\partial Y} + \eta\Delta H_x, \quad (3)$$

$$\frac{DH_y}{Dt} = H_x\frac{\partial v}{\partial X} + H_y\frac{\partial v}{\partial Y} + \eta\Delta H_y, \quad (4)$$

in which η is the magnetic diffusivity $(4\pi\mu\sigma)^{-1}$, σ being the electric conductivity of the liquid.

The equation of continuity is

$$(\partial u/\partial X) + (\partial v/\partial Y) = 0. \quad (5)$$

The equation of continuity of magnetic field strength is

$$(\partial H_x/\partial X) + (\partial H_y/\partial Y) = 0, \quad (6)$$

which is consistent with (3) and (4). The relationship between the magnetic field strength H and the current density j is

$$\text{curl } \mathbf{H} = 4\pi\mathbf{j}. \quad (7)$$

The magnetic field induced by the current is, according to (7),

$$\overline{H}_x = 4\pi j_0(Y - \tfrac{1}{2}d), \quad (8)$$

in which d is the depth of the fluid. The constant $\frac{1}{2}d$ has been used to make the magnetic field antisymmetric with respect to the plane $Y = \frac{1}{2}d$, because the magnetic field produced *exclusively* by the current must be antisymmetric with respect to the middle plane of the layer. Thus, the primary field is $(\overline{H}_x, H_0, 0)$.

The primary flow only has the component $\bar{u}$, determined by (1), which assumes the simple form

$$\rho\nu(d^2\bar{u}/dY^2) = -\mu j_0 H_0. \quad (9)$$

Equation (9) can be integrated with the boundary conditions $\bar{u} = 0$, at $Y = d$, and $d\bar{u}/dY = 0$, at $Y = 0$. The result is

$$\bar{u} = (\mu j_0 H_0/2\rho\nu)(d^2 - Y^2). \quad (10)$$

The average value of $\bar{u}$ over d is

$$\bar{u}_a = \mu j_0 H_0 d^2/3\rho\nu. \quad (11)$$

If the dimensionless parameters

$$\overline{H}_1 = \overline{H}_x/4\pi j_0 d, \quad \overline{H}_2 = H_0/4\pi j_0 d, \quad U = \bar{u}/u_a,$$
$$x = X/d, \quad y = Y/d \quad (12)$$

are introduced, then

$$U = \tfrac{3}{2}(1 - y^2), \quad \overline{H}_1 = y - \tfrac{1}{2}, \quad \overline{H}_2 = \gamma, \quad (13)$$

with $\gamma = H_0/4\pi j_0 d$.

To simplify the analysis in the next section, we use, in addition to (12) and (13), the nondimensional quantities

$$(u_1, v_1) = (u, v)/\bar{u}_a, \quad (H_1, H_2) = (H_x, H_y)/4\pi j_0 d,$$
$$P_1 = p/\rho\bar{u}_a^2, \quad \tau = t\bar{u}_a/d, \quad (14)$$

and the nondimensional physical variables

$$F^2 = \bar{u}_a^2/gd, \quad R = \bar{u}_a d/\nu, \quad R_m = \bar{u}_a d/\eta,$$
$$M = 4\pi\mu j_0^2 d^4/\rho\nu^2, \quad (15)$$
$$N = (M\gamma^2\nu/\eta)^{1/2} = \mu H_0 d(\sigma/\rho\nu)^{1/2}.$$

III. THE DIFFERENTIAL SYSTEM GOVERNING STABILITY

As is customary in magnetohydrodynamic problems, an infinitesimal disturbance is applied to the primary flow. The flow will be unstable if the disturbance grows with time, and is stable if it attenuates with time, Lock[6] has shown under certain restrictions, that Squire's theorem[7] held in the problem of pressure flow between parallel planes under a transverse magnetic field, and therefore the motion is more unstable to a purely two-dimensional type of disturbance than to a three-dimensional one. Thus, it is possible to infer that Squire's theorem also holds in the present investigation. We shall henceforth consider only two-dimensional disturbances.

Written in nondimensional form, we let

$$(u_1, v_1, P_1, H_1, H_2) = (U + u', v', P + p', \overline{H}_1 + h, \overline{H}_2 + k), \quad (16)$$

where the capital letters are the primary flow, and the lower case letters denote the disturbances. Equations (5) and (6), permit, respectively, the use of a stream function ψ and a magnetic stream function χ, in terms of which

$$(u', v', h, k) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, \frac{\partial\chi}{\partial y}, -\frac{\partial\chi}{\partial x}\right).$$

We shall assume all perturbation quantities together with the free-surface displacement ζ to have an exponential factor

$$(\psi, \chi, p', \zeta) = [\phi(y), q(y), f(y), \zeta_0]\exp[i\alpha(x - c\tau)], \quad (17)$$

in which ζ is the amplitude of the free-surface displacement, α is the dimensionless wavenumber, and c is the complex phase velocity $c_r + ic_i$. In terms of ϕ and q,

$$(u', v', h, k)$$
$$= [(d\phi/dy), -i\alpha\phi, (dq/dy), -i\alpha q]\exp[i\alpha(x - c\tau)]. \quad (18)$$

If now Eqs. (12)–(18) are substituted in Eqs. (1) to (4), the primary flow is separated out, and quadratic terms in the perturbation quantities are neglected, the linearized equations become, with accents indicating differentiation with respect to y

$$i\alpha[(U - c)\phi' - U'\phi] = -i\alpha f + R^{-1}(\phi''' - \alpha^2\phi')$$
$$+ (M/R^2)[\gamma(q'' - \alpha^2 q) - i\alpha q], \quad (19)$$

$$\alpha^2(U - c)\phi = -f' - (i\alpha/R)(\phi'' - \alpha^2\phi)$$
$$- (M/R^2)[(y - \tfrac{1}{2})(q'' - \alpha^2 q) + q'], \quad (20)$$

$$i\alpha[(U-c)q' - \phi - (y-\tfrac{1}{2})\phi'] - \gamma\phi'' + i\alpha q U' = (R_m)^{-1}(q''' - \alpha^2 q'), \tag{21}$$

$$i\alpha(U-c)q - \gamma\phi' - i\alpha(y-\tfrac{1}{2})\phi = (R_m)^{-1}(q'' - \alpha^2 q). \tag{22}$$

It is clear that (21) can be obtained from (22) by differentiation. This is because (3), (4), and (6) are not independent. Elimination of f from (19) and (20) leads to

$$\begin{aligned} &i\alpha[(U-c)(\phi''-\alpha^2\phi) - U''\phi] \\ &\quad - (i\alpha M/R^2)(y-\tfrac{1}{2})(q''-\alpha^2 q) \\ &\quad - (M\gamma/R^2)(q'''-\alpha^2 q') \\ &\quad = R^{-1}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi). \end{aligned} \tag{23}$$

The electromagnetic field in regions outside the fluid layer are governed by the equation of magnetic-field diffusion. The magnetic stream functions of the air and the bottom material are assumed to be

$$(\psi_a, \psi_b) = [q_a(y), q_b(y)]\exp[i\alpha(x - c\tau)].$$

Thus, above the free surface, the magnetic diffusion equation corresponding to (22) is

$$q''_a - \alpha^2 q_a = 0, \tag{24}$$

because the gas above is assumed to be nonconductive of electricity. Similarly, below the bottom of the liquid layer

$$q''_b - \alpha^2 q_b = 0. \tag{25}$$

To solve (22), (23), (24), and (25), ten boundary conditions are needed to completely specify the mathematical problem. The conditions of the continuity of the magnetic flux at the iron surfaces demand that

$$q_a(-a) = 0, \quad q_b(b) = 0. \tag{26}$$

The transitions of the normal component of magnetic induction and the tangential component of magnetic field at the bottom of the fluid layer are continuous, i.e.,

$$\mu q(1) = \mu_b q_b(1), \quad q'(1) = q'_b(1), \tag{27a}$$

where μ_b is the magnetic permeability of the bottom material.

At the free surface, the condition of the continuity of the tangential component of the magnetic field yields

$$q'(0) + \zeta_0 = q'_a(0). \tag{27b}$$

The kinematic boundary condition at the free surface is

$$(\partial\zeta/\partial\tau) + U(\partial\zeta/\partial x) = v, \tag{27c}$$

which, by virtue of (17) and (18), can be put in the form

$$\zeta_0 = \phi(0)/c', \quad c' = c - U(0) = c - \tfrac{3}{2}. \tag{27d}$$

Thus, the boundary conditions of the magnetic field at the free surface are

$$\mu q(0) = \mu_a q_a(0), \qquad q'(0) + [\phi(0)/c'] = q'_a(0). \tag{28}$$

At the bottom of the fluid, the nonslip conditions imply

$$\phi(1) = 0, \qquad \phi'(1) = 0. \tag{29}$$

The conditions corresponding to zero shear stress and zero normal stress at the free surface are, respectively,[5]

$$\begin{aligned} &\phi''(0) + [\alpha^2 - (3/c')]\phi(0) = 0, \\ &\alpha(RF^{-2} + \alpha^2 SR)[\phi(0)/c'] + \alpha(Rc' + 3\alpha i)\phi'(0) \\ &\quad - i\phi'''(0) - (\alpha M/R)\{q(0) - [\phi(0)/2c']\} \\ &\quad - (iM\gamma/R)[q''(0) - \alpha^2 q(0)] = 0, \end{aligned} \tag{30}$$

where $S = T/\rho\bar{u}_a^2 d$, and T denotes the surface tension.

The differential equations (22), (23), (24), (25), and the boundary conditions (26), (27a), (28), (29), and (30) constitute the differential system governing the stability problem.

IV. THE SOLUTION OF THE STABILITY PROBLEM

Since one expects long waves ($\alpha \ll 1$) to be unstable, the method of regular perturbation of Yih[2] will be used to solve the stability problem. We assume

$$\begin{aligned} &\phi = \sum_{i=0}^{\infty} \alpha^i \phi_i, \qquad q = \sum_{i=0}^{\infty} \alpha^i q_i, \qquad c = \sum_{i=0}^{\infty} \alpha^i c_i. \\ &q_a = \sum_{i=0}^{\infty} \alpha^i q_{a,i} \qquad q_b = \sum_{i=0}^{\infty} \alpha^i q_{b,i} \end{aligned} \tag{31}$$

If (31) are substituted into the governing differential system and the various powers in α are separated out, we have the governing equations and the boundary conditions for each order of approximation.

A. The zero-order approximation

The equations corresponding to α^0 are

$$\begin{aligned} &\phi_0'''' + (M\gamma/R)q_0''' = 0, \qquad q''_0 + \gamma R_m \phi'_0 = 0, \\ &\qquad q''_{a,0} = 0, \qquad\qquad q''_{b,0} = 0. \end{aligned} \tag{32}$$

The first two equations of (32) can be combined to produce

$$\phi_0'''' - N^2\phi''_0 = 0, \qquad \text{with } N^2 = M\gamma^2 R_m/R = M\gamma^2 \nu/\eta.$$

The solutions of the zero-order approximation are

$$\begin{aligned} &\phi_0 = 1 + By + C\cosh Ny + D\sinh Ny, \\ &q_0 = -\gamma R_m\left(\frac{By^2}{2} + \frac{C}{N}\sinh Ny + \frac{D}{N}\cosh Ny\right) \\ &\qquad + Ey + G, \\ &q_{a,0} = A_a + B_a y, \qquad q_{b,0} = A_b + B_b y. \end{aligned} \tag{33}$$

The boundary conditions (29) and (30) for terms corresponding to α^0 are

$$\begin{aligned} &\phi_0(1) = 0, \quad \phi'_0(1) = 0, \quad \phi''_0(0) - (3/c'_0)\phi_0(0) = 0, \\ &\phi'''_0(0) + (M\gamma/R)q''_0(0) = 0. \end{aligned} \tag{34}$$

From (34), we obtain

$$B = 0, \quad C = -\cosh N, \quad D = \sinh N,$$
$$c_0' = 3(\cosh N - 1)/N^2 \cosh N. \tag{35a}$$

Other coefficients E, G, A_a, B_a, A_b, and B_b can be determined by the magnetic boundary conditions (26), (27a) (28), and are given in the appendix. The eigenvalue $c_0(= c_0' + \frac{3}{2})$ is given by

$$c_0 = [3(\cosh N - 1)/N^2 \cosh N] + \tfrac{3}{2}. \tag{35b}$$

Since c_0 is real, no instability is manifested at this stage.

B. The first-order approximation

The governing equations obtained from the coefficients of α^1 are

$$\phi_1'''' + (M\gamma/R)q_1''' = iR[(U - c_0)\phi_0'' - U''\phi_0 - (M/R^2)(y - \tfrac{1}{2})q_0''],$$
$$q_1'' + \gamma R_m \phi_1' = iR_m[(U - c_0)q_0 - (y - \tfrac{1}{2})\phi_0], \tag{36}$$
$$q_{a,1}'' = 0, \qquad \text{and } q_{b,1}'' = 0.$$

Eliminating q_1 from the first two equations of (36) and using the zero-order solutions, we have

$$\phi_1'''' - N^2\phi_1'' = i[I_1 y^2 + I_2 y + I_3 + \cosh Ny(I_4 y^2 + I_6 y + I_8) + \sinh Ny(I_5 y^2 + I_7 y + I_9)], \tag{37}$$

where $I_1, I_2, \ldots, I_8$, and I_9 are listed in the appendix.

The solution of (37) can be written as

$$\phi_1 = \bar{B}_1 y + \bar{C}_1 \cosh Ny + \bar{D}_1 \sinh Ny + i[y^2(J_1 y^2 + J_2 y + J_3) + \cosh Ny(J_4 y^3 + J_6 y^2 + J_8 y) + \sinh Ny(J_5 y^3 + J_7 y^2 + J_9 y)], \tag{38}$$

in which $J_1, J_2, \ldots, J_8$, and J_9 are given in the appendix. For the first-order approximation, the boundary conditions corresponding to (29) and (30) are

$$\phi_1(1) = 0, \quad \phi_1'(1) = 0,$$
$$\phi_1''(0) - (3/c_0')[\phi_1(0) - (c_1'/c_0')\phi_0(0)] = 0,$$
$$\phi_1'''(0) + (M\gamma/R)q_1''(0) = -i\{(RF^{-2} + SR\alpha^2)[\phi_0(0)/c_0'] + Rc_0'\phi_0'(0) - (M/R)[q_0(0) - (1/2c_0')\phi_0(0)]\}. \tag{39}$$

The constants $\bar{B}_1, \bar{C}_1, \bar{D}_1$ and the eigenvalue c_1' can be determined by the boundary conditions (39). Separating $c_1'(= c_1)$ into real and imaginary parts

$$c_1' = (c_1)_r' + i(c_1)_i',$$

and using (39), we obtain, after some lengthy mathematical manipulations,

$$(c_1)_r = (c_1')_r = 0,$$
$$(c_1)_i = (c_1')_i = \frac{3}{N^2 \cosh N}\left[b_4\left(1 - \frac{\tanh N}{N}\right) + b_3 - b_1 + b_2\frac{\tanh N}{N}\right], \tag{40}$$

in which b_1, b_2, b_3, and b_4 are also given in the appendix.

V. RESULTS AND DISCUSSION

The eigenvalues of the zeroth-order approximation and first-order approximation are given, respectively, by (35b) and (40). The flow system is stable, unstable, or neutrally stable according to whether $(c_1)_i$ is negative, positive, or zero. From (40), and the definition of b_4, we can verify that the coefficients associated with RF^{-2} and SR are all negative; this means that the effects of gravity and surface tension are always stabilizing, as expected. For long waves ($\alpha \ll 1$), neglecting the effect of surface tension, from (40) we know that $(c_1)_i$ will depend implicitly on all the physical variables as well as the geometrical parameters, or

$$(c_1)_i = (c_1')_i = K(R, R_m, F, M \text{ or } N, \gamma, m_a, m_b, a, b), \tag{41}$$

where $m_a = \mu_a/\mu$ and $m_b = \mu_b/\mu$, our task is to evaluate K numerically.

In order to examine the stability or instability of the flow system, numerical calculations were carried out by the IBM 360 computer. Contour lines corresponding to different values of $(c_1)_i$ are shown in the graphs. All curves were constructed by the method of linear interpolation. Because of the limitation on available computing time, the results obtained are restricted to fairly small physical variables only. Of course, they can be extended to any finite physical variables.

For the meaning of the geometric parameters a and b, we recall that ad is the depth of the air above and $(b - 1)d$ is the depth of the bottom plate. We have taken $a = b = 3$. A calculation based on $a = b = 15$ displayed very little difference in the results. It is reasonable to assume $m_a = m_b = 1$ in the forthcoming calculations.

Since our chief purpose is to investigate the interactions between the electromagnetic effect and the gravitational field, we choose mercury to illustrate our analysis. For mercury at room temperature $\rho = 13.6$ g/cm^3, $\nu = 1.12 \times 10^{-3}$ cm^2/sec, $\eta = 8000$ cm^2/sec.

Figure 2 corresponds to the case of a fixed value M. In this case, R_m, N, and γ are all proportional to R. From the definition of the various quantities, we find

$$M = 4\pi\mu j_0^2 d^4/\rho\nu^2, \quad R = (H_0 d/6\nu)(M\mu/\pi\rho)^{1/2},$$

and

$$F = (H_0/d^{1/2})(M\mu/\pi\rho g)^{1/2}.$$

To investigate the influence of the magnetic field on the stability problem, we vary the field strength H_0 only. It is apparent that as H_0 increases, R and F both increase. From Fig. 2, we see that if one moves along a line of constant slope F/R, in the direction of increasing R and F, one will move from the stable region to the unstable region. Thus, the destabilizing effect of the electromagnetic field is clearly seen. This effect results from the fact that as H_0 increases, the electromagnetic force acting on the fluid also increases, much as a longitudinal gravita-

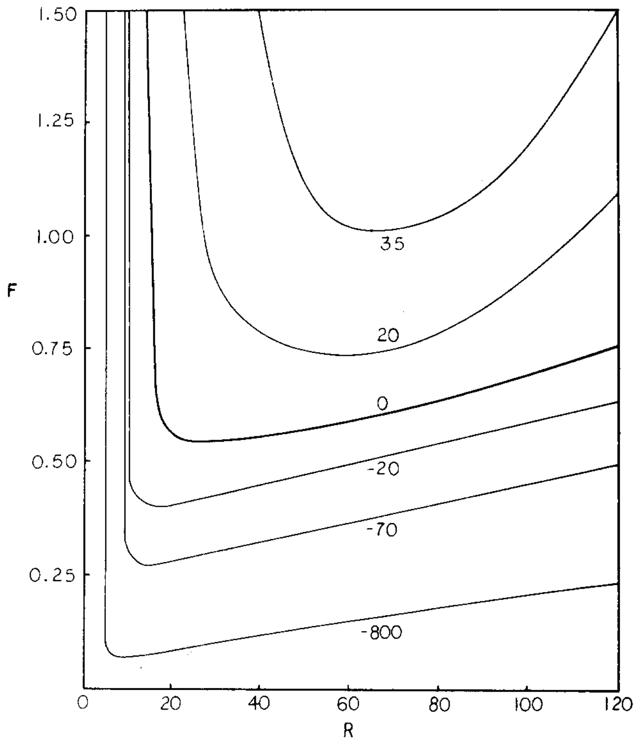

FIG. 2. Curves of constant $(c_1)_i$ for $M = 0.015$.

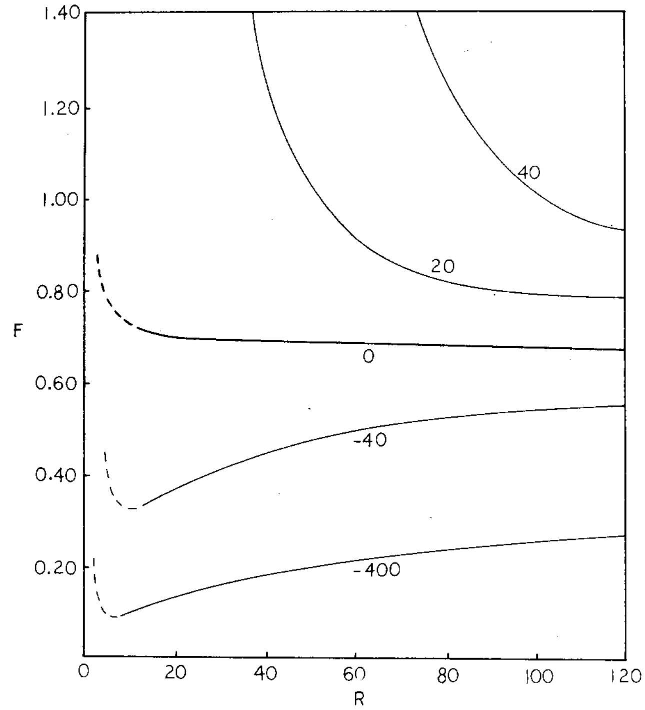

FIG. 3. Curves of constant $(c_1)_i$ for $N = 0.15$.

tional force. If the motivating effect of the electromagnetic field is sufficient to overcome the stabilizing effects of gravity and magnetic diffusion, as well as any other stabilizing effect the magnetic field may have, the flow is unstable.

For constant M, along a line of constant R, we see that as F increases, the flow becomes increasingly unstable (Fig. 2). (Note that as F increases while R and M are kept constant, d decreases and H_0 and j_0 both increase.)

Similarly, if one moves along a line of constant F, in the direction of increasing R, one moves from the stable region to the unstable (or less stable, if F is small) region and returns to the stable (or more stable for small F) region as the destabilizing effect of the electromagnetic force was suppressed by the combination of the effects of viscosity, diffusivity, gravity, and possibly of the magnetic field itself. (Note that as R increases while F and M are kept constant, H_0 and d increase and j_0 decreases.)

For a given fluid and a case of fixed value of N, we again find that R, M, and γ^{-1} are all proportional to R. From the definitions, we obtain

$$N = (H_0 d/2\nu)(\mu\nu/\pi\rho\eta)^{1/2}, \quad R = (2Nj_0 d^2/3\nu)(\pi\mu\eta/\rho\nu)^{1/2},$$

and

$$F = (2Nj_0 d^{1/2}/3)(\pi\mu\eta/\rho\nu g)^{1/2}.$$

General features of curves of constant $(c_1)_i$ are shown in Fig. 3. To see the effect of the current density j_0, we keep H_0 and d equal to constant and change j_0 only. As j_0 increases, both R and F increase, and from Fig. 3 it is seen that $(c_1)_i$ increases, i.e., j_0 is destabilizing.

In the case of fixed value γ, the dimensionless parameters can be written as

$$\gamma = H_0/4\pi j_0 d, \quad N = (H_0 d/2\nu)(\mu\nu/\pi\rho\eta)^{1/2},$$

and

$$F = (\mu\gamma H_0^2 d^{1/2}/12\pi\rho\nu)(1/g)^{1/2}.$$

From Fig. 4, it can be easily shown that if we keep d constant and increase the field H_0 and the current density j_0, both N and F will increase and F increases as N^2; hence, the flow will become unstable.

Curves of constant $(c_1)_i$ for case of fixed R are exhi-

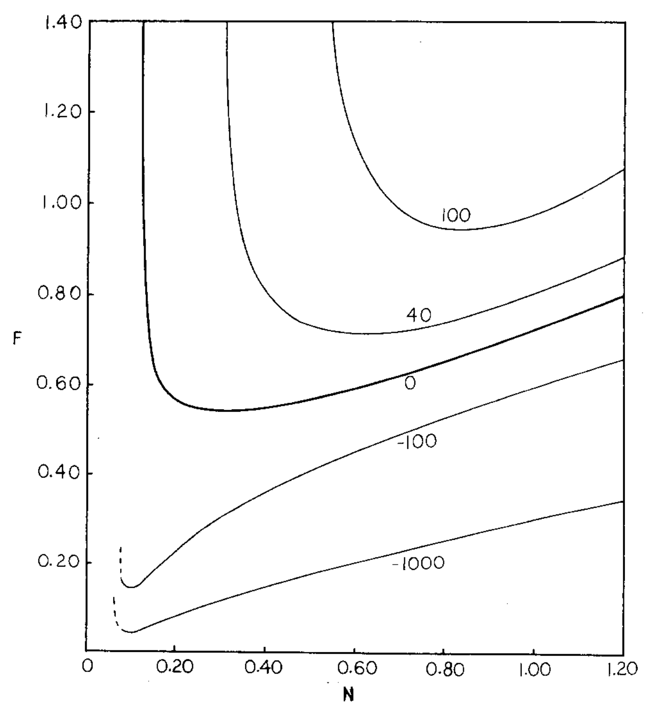

FIG. 4. Curves of constant $(c_1)_i$ for $\gamma = 5000$.

bited in Fig. 5. The dimensionless parameters have the following form

$$R = \mu j_0 H_0 d^3/3\rho\nu^2, \qquad N(3\nu R/4j_0 d^2)(\rho\nu/\pi\mu\eta)^{1/2},$$

and

$$F = (R\nu/d^{3/2})(1/g)^{1/2}.$$

By keeping j_0 equal to a constant, one can verify that decreasing d will cause an increase in H_0, which will result in an increase in N and F, with N increasing as $F^{4/3}$. Thus, as d decreases and H_0 increases at constant R, the flow becomes increasingly unstable (Fig. 5) at small values of N and F, but, in part of the $F-N$ plane at least, increasingly stable at larger values of N and F.

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APPENDIX

$$\begin{aligned} E &= D^{-1}(\bar{b}_1 - \bar{b}_2),\\ G &= D^{-1}\{am_a\bar{b}_1 + [1 - m_b(1-b)]\bar{b}_2\},\\ A_a &= (m_a)^{-1}[G - \gamma R_m(D/N)],\\ A_b &= -b(\gamma R_m + E),\\ B_a &= a^{-1}A_a, \qquad \text{and } B_b = -b^{-1}A_b, \end{aligned} \tag{A1}$$

in which

$$m_a = \mu_a/\mu, \quad m_b = \mu_b/\mu, \quad \bar{b}_1 = \gamma R_m(1-b)m_b,$$

$$\bar{b}_2 = \gamma R_m[(D/N) - am_a C] + (am_a CN^2/3),$$

$$D = 1 + am_a - m_b(1-b).$$

$$I_1 = 4.5(N^2/\gamma)E, \; I_2 = 3(N^2/\gamma)G,$$

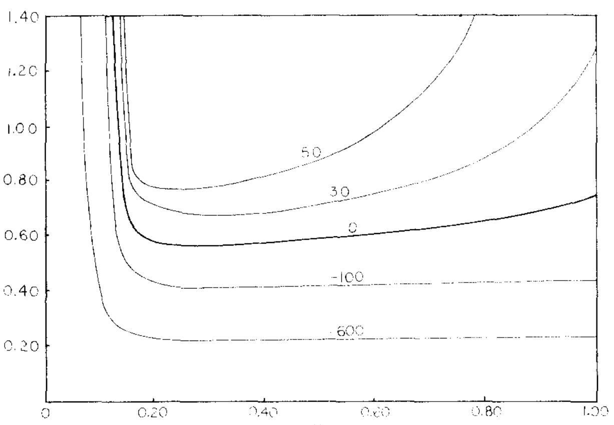

FIG. 5. Curves of constant $(c_1)_i$ for $R = 100$.

$$\begin{aligned} I_3 &= 3R - [(\tfrac{3}{2} - c_0)E - 1](N^2/\gamma),\\ I_4 &= -\tfrac{3}{2}N^2(R + R_m)C, \; I_5 = -\tfrac{3}{2}N^2(R+R_m)D,\\ I_6 &= DN[(2N^2/\gamma) - 3R_m], \; I_7 = CN[(2N^2/\gamma) - 3R_m],\\ I_8 &= C[N^2(R+R_m)(\tfrac{3}{2} - c_0) + 3R + (N^2/\gamma)] - (DN^3/\gamma),\\ I_9 &= D[N^2(R+R_m)(\tfrac{3}{2} - c_0) + 3R + (N^2/\gamma)] - (CN^3/\gamma). \end{aligned} \tag{A2}$$

$$\begin{aligned} J_1 &= -I_1/12N^2, \quad J_2 = -I_2/6N^2,\\ J_3 &= -\frac{1}{N^2}\left(\frac{I_1}{N^2} + \frac{I_3}{2}\right),\\ J_4 &= I_5/6N^3, \quad J_5 = I_4/6N^3,\\ J_6 &= \frac{1}{4N^3}\left(I_7 - \frac{5I_4}{N}\right),\\ J_7 &= \frac{1}{4N^3}\left(I_6 - \frac{5I_5}{N}\right),\\ J_8 &= \frac{1}{2N^3}\left(I_9 - \frac{10I_6}{4N} + \frac{204I_5}{24N^2}\right),\\ J_9 &= \frac{1}{2N^3}\left(I_8 - \frac{10I_7}{4N} + \frac{204I_4}{24N^2}\right). \end{aligned} \tag{A3}$$

$$\begin{aligned} b_1 &= -[J_1 + J_2 + J_3 + (J_4 + J_6 + J_8)\cosh N\\ &\qquad + (J_5 + J_7 + J_9)\sinh N],\\ b_2 &= -\{4J_1 + 3J_2 + 2J_3\\ &\qquad + [(3J_5 + 2J_7 + J_9) + N(J_4 + J_6 + J_8)]\sinh N\\ &\qquad + [(3J_4 + 2J_6 + J_8) + N(J_5 + J_7 + J_9)]\cosh N\}, \end{aligned} \tag{A4}$$

$$\begin{aligned} b_3 &= \tfrac{2}{3}c_0'(J_3 + J_6 + NJ_9),\\ b_4 &= \frac{1}{N^2}\Big\{(6J_2 + 6J_4 + 6NJ_7 + 2N^2J_8)\\ &\qquad + \frac{N^2}{\gamma}\left[\left(\frac{3}{2} - c_0\right)\left(G - \gamma R_m\frac{D}{N}\right) + \frac{1+C}{2}\right]\\ &\qquad + \left(RF^{-2} + \alpha^2 SR + \frac{M}{2R}\right)\frac{1+C}{c_0'}\\ &\qquad + \left(Rc_0' + \frac{1}{\gamma}\right)DN - G\frac{M}{R}\Big\}. \end{aligned}$$

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