Compressible Low Reynolds Number Flow around a Sphere

D. R. KASSOY,* T. C. ADAMSON, JR., AND A. F. MESSITER
Department of Aerospace Engineering, University of Michigan, Ann Arbor, Michigan
(Received 18 October 1965)

The low Reynolds number flow past a sphere is studied for cases involving significant variation in density and temperature. An order of magnitude analysis of the governing equations for a continuum flow indicates the relative importance of compressibility, variable transport properties, and viscous dissipation. The order of magnitude of the nondimensional impressed temperature difference τ is shown to be useful as a guide for distinguishing problems with different governing physical characteristics. The cases $\tau = O(1)$ and $\tau = O(\mathrm{Re})$, where Re is the Reynolds number, are analyzed in detail. Inner and outer asymptotic expansions are derived for each of the flow variables in terms of the parameters Re τ and Mach number M_∞. For $\tau = O(\mathrm{Re})$ the solutions are developed up to and including the first effect of nonzero M_∞. For $\tau = O(1)$ the velocity expansions are calculated to zeroth order and the temperature expansions to order Re. Drag and heat transfer coefficients are calculated for each case.

I. INTRODUCTION

THE solution for the flow of an incompressible gas at low Reynolds number around a sphere was developed by Stokes in 1851.[1] It was predicated on the assumption that the inertia forces in the flow field are negligible in comparison with the viscous and pressure forces. Subsequently, it was found that under such an assumption the analogous problem for the cylinder cannot be solved[1] (Stokes paradox). Furthermore, a second approximation for the three-dimensional case which would match with the uniform stream at infinity cannot be obtained[2] (Whitehead paradox). The difficulty, as shown by Oseen,[3] arises because the Stokes approximation is not correct far from the body. At large distances, the inertia forces can no longer be considered small compared to the viscous forces. Oseen suggested the use of a linearized inertia term far from the body. The resulting "Oseen" equation is found to provide a uniformly valid representation for both plane and three-dimensional flows. However, it has not been possible to obtain a uniformly valid second approximation by using the full Oseen result as the first approximation.[4]

The meaning of the Stokes and Oseen approximations has been explained most clearly in recent years through the work of Kaplun, Lagerstrom, and Cole,[5-7] who have developed systematic procedures for obtaining asymptotic expansions of solutions to differential equations containing a small parameter. Their method, now sometimes called the method of matched asymptotic expansions, has been applied in studies of flow at low Reynolds number, as a means of finding uniformly valid asymptotic expansions for the limit $\mathrm{Re} \to 0$.

Kaplun and Lagerstrom[5,6] have studied both the plane and three-dimensional cases and have shown how to derive the first corrections to the classical solutions for the cylinder and sphere. Additional details were developed by Proudman and Pearson.[8] These and other applications are also discussed by Van Dyke.[4]

More recently there has been interest in calculating the drag and heat transfer for a spherical particle in a variable density flow at low Reynolds number. Acrivos and Taylor[9] calculated the average Nusselt number when a small impressed temperature difference is assumed between the particle and free stream. The flow was considered incompressible, with a Stokes velocity profile, to the order for which the calculation was carried out (although the resulting temperature gradients really will produce a small density variation, and a subsequent effect on the velocity profile). The Nusselt number was derived to $O(\mathrm{Re}^3 \ln \mathrm{Re})$. In a later study, Chang[10] considered the effect of a dimensionless impressed temperature difference ϵ, small compared to one, but large compared to Re. He correctly allowed the density to vary and obtained the leading terms of a two parameter (ϵ, Re) expansion for the drag coefficient valid to $O(\epsilon)$.

* Present address: Department of Aerospace and Mechanical Engineering Sciences, University of California, San Diego, La Jolla, California.

[1] G. G. Stokes, Trans. Camb. Phil. Soc. 9, Pt. II, 8 (1851).
[2] A. N. Whitehead, Quart. J. Math. 23, 143 (1889).
[3] C. W. Oseen, Arkiv. Mat. Astron. Fysik 6 (1910).
[4] M. Van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic Press Inc., New York, 1964).
[5] S. Kaplun and P. A. Lagerstrom, J. Math. Mech. 6, 585 (1957).
[6] S. Kaplun, J. Math. Mech. 6, 595 (1957).
[7] P. A. Lagerstrom and J. D. Cole, J. Ratl. Mech. Anal. 4, 817 (1955).
[8] I. Proudman and J. R. A. Pearson, J. Fluid Mech. 2, 237 (1957).
[9] A. Acrivos and T. D. Taylor, Phys. Fluids 5, 387 (1962).
[10] I.-D. Chang, Stanford University Aeronautical Engineering Department Report 210 (1964).

Actually a total of three nondimensional parameters are of special importance in formulating a study of viscous compressible flow past a sphere. These parameters are the Reynolds number Re, the Mach number M_∞, and a nondimensional impressed temperature difference τ, defined below. The basic simplification of the governing equations comes from the assumption Re $\ll 1$. Since the Navier–Stokes equations are considered to be the exact equations, the continuum condition $(M_\infty/\mathrm{Re}) \ll 1$ will also be imposed. The Prandtl number Pr is assumed of order one and the specific heats are considered constant. Finally, the temperature difference τ will be positive or zero, with value at most of order one.

In terms of the limit process, the behavior of the parameters Re, M_∞, and τ therefore is formulated as follows:

$$\mathrm{Re} = (u_\infty a/\nu_\infty) \to 0, \qquad (M_\infty/\mathrm{Re}) \to 0, \qquad \text{(1a, b)}$$

$$\text{(1)} \quad \tau = [(T_w - T_\infty)/T_\infty] \to 0, \quad \text{(2)} \quad \tau = \text{const.} \quad \text{(1c)}$$

Here a = sphere radius, ν = kinematic viscosity, T_w = temperature of the sphere, and the subscript "∞" refers to the free-stream conditions.

It is necessary to discuss the relative sizes of the three parameters, i.e., the rates at which these parameters tend to zero. A consequence of (1a, b) is that $M_\infty^2 \ll \mathrm{Re}$; in order to study the largest possible Mach number effect, it will be assumed that

$$\mathrm{Re}^2 \ln \mathrm{Re} \ll (M_\infty^2/\mathrm{Re}) \ll \mathrm{Re}.$$

The parameter τ is ordered with respect to M_∞ and Re by considering the relative sizes of terms in the governing differential equations. It is found that several different cases might be investigated, depending on the relative importance of different physical effects. The following conclusions are reached: (1) Thermal conduction is the dominant mechanism of heat transfer when $\tau \gg M_\infty^2$ but at most of order one. If $\tau = O(M_\infty^2)$ or less, then the temperature variation is determined by a balance between viscous dissipation and thermal conduction. (2) When $\tau \gg M_\infty^2/\mathrm{Re}$, but at most of order one, the density variation is determined primarily by the conduction controlled temperature gradient. The Mach number effect on the density also appears when $\tau = O(M_\infty^2/\mathrm{Re})$. If $\tau \ll M_\infty^2/\mathrm{Re}$, the density gradient is primarily determined by the Mach number effect; that is, the leading term in the density variation is related to the pressure rather than to the temperature. (3) The velocity is found to be strongly influenced by τ, through the effects of variable density and viscosity, when $\tau = O(1)$. For Re $\ll \tau \ll 1$, the leading term in the velocity profile is given by the classical Stokes solution[1]; the first perturbation is $O(\tau)$. When $\tau = O(\mathrm{Re})$ or less the basic solution remains that of Stokes, and the perturbation is $O(\mathrm{Re})$, due to the inertia terms in the momentum equation.

Actual calculation of solutions to the equations is necessary only for the distinguished cases $\tau = O(1)$, $O(\mathrm{Re})$, $O(M_\infty^2/\mathrm{Re})$, and $O(M_\infty^2)$. Each of the intermediate ranges of τ may be studied as an extension of one of the distinguished cases, because no additional physical features need to be represented in the leading terms. Since it is of interest to determine the effect of the largest values of τ, only the first two cases are analyzed in detail. For $\tau = O(\mathrm{Re})$, the expansions are carried out far enough to include the first Mach number effect. To this order, logarithmic terms need not be calculated. In the case $\tau = O(1)$, only the zeroth-order term is calculated for the velocity, because of the complexity of the equations. For the temperature expansion both the zeroth- and first-order terms are derived.

The limit process for the case $\tau = O(\mathrm{Re})$ is defined by Re $\to 0$, $M_\infty \to 0$, $\tau \to 0$, such that $(M_\infty/\mathrm{Re}) \to 0$ and τ/Re is fixed. An assumed form of the asymptotic expansion for each flow variable is substituted in the governing equations. The equations for the perturbation quantities are found by grouping terms of the same magnitude. At each step the orders of magnitude of the perturbations can be deduced from the equations or from a boundary condition or matching condition. The solutions that follow show that to $O(\mathrm{Re})$ the impressed temperature difference increases the drag coefficient and decreases the average Nusselt number. A Mach number term appears in both the velocity and temperature expansions. However, to this order in Mach number only the Nusselt number is affected. A similar effect does not appear in the drag coefficient because the corresponding velocity perturbation is symmetric in the θ variable.

When $\tau = O(1)$ the limit process is defined by Re $\to 0$ and $(M_\infty/\mathrm{Re}) \to 0$, with τ fixed. The development of the equations for various terms in the expansions is carried out in the manner described for the previous case. The results show that the drag coefficient increases almost linearly with τ. At $\tau \approx 1$ the drag has increased about 70% over the Stokes value. The Nusselt number is observed to decrease with increasing τ.

Many of the details of the calculations which are not included in the following sections may be found in Ref. 11.

II. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

It is assumed that the flow is steady, the gas is perfect, the bulk viscosity is zero, and the Prandtl number and specific heats are constant. The coefficient of viscosity and the thermal conductivity are taken to be linear functions of temperature.

The governing equations may be made nondimensional in various ways (e.g., Ref. 12). Here, variables r and s will be defined, respectively, as the distance from the center of the sphere made nondimensional either with the sphere radius a or with the viscous length ν_∞/u_∞. Also of special importance is the choice of a reference pressure. Here p will denote the difference between the local static pressure and the free-stream value, divided by the viscous pressure $\mu_\infty u_\infty/a$, and P will denote the pressure difference divided by $\rho_\infty u_\infty^2$. Then $p = \mathrm{Re}\, P$. Throughout the following discussions the quantities ρ, T, $\mathbf{v}$, μ, and λ will denote, respectively, the nondimensional density, temperature, velocity, coefficient of viscosity, and thermal conductivity, each having been referred to the free-stream value.

The nondimensional equations appropriate near the body are expressed in terms of the Stokes variables (i.e., with the sphere radius chosen as the characteristic length):

Continuity,

$$\nabla\cdot\rho\mathbf{v} = 0; \tag{2}$$

Momentum,

$$\begin{aligned}\mathrm{Re}\,\rho\mathbf{v}\cdot\nabla\mathbf{v} = {}& -\nabla p + \mu[\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}) + \nabla^2\mathbf{v}] \\ & + [-\tfrac{2}{3}(\nabla\cdot\mathbf{v})\nabla\mu + \nabla(\mathbf{v}\cdot\nabla\mu) \\ & - (\mathbf{v}\cdot\nabla)\nabla\mu + (\nabla\mu\cdot\nabla)\mathbf{v}];\end{aligned} \tag{3}$$

Energy,

$$\begin{aligned}\mathrm{Re}\,\rho\mathbf{v}\cdot\nabla T = {}& \mathrm{Pr}^{-1}\,\nabla\cdot\lambda\nabla T \\ & + (\gamma - 1)M_\infty^2[\mu\Phi + \mathbf{v}\cdot\nabla p];\end{aligned} \tag{4}$$

State,

$$\rho = (1/T) + (\gamma M_\infty^2/\mathrm{Re})(p/T); \tag{5}$$

Transport properties,

$$\mu = \lambda = T; \tag{6}$$

where the vector differential operator ∇ implies differentiation with respect to the Stokes variables, and

$$\begin{aligned}\Phi = {}& \nabla^2(\mathbf{v}\cdot\mathbf{v}) + 2\nabla\cdot(\boldsymbol{\omega}\times\mathbf{v}) - 2\mathbf{v}\cdot\nabla(\nabla\cdot\mathbf{v}) \\ & + (\boldsymbol{\omega}\cdot\boldsymbol{\omega}) - \tfrac{2}{3}(\nabla\cdot\mathbf{v})^2, \qquad \boldsymbol{\omega} = \nabla\times\mathbf{v}.\end{aligned} \tag{7}$$

The neglect of buoyant force in Eq. (3) implies that the parameter ga/u_∞^2 (reciprocal of the Froude number) is at most of order one, where g = gravitational acceleration (e.g., see Schlichting[13]). Since the flow is axially symmetric, each of the dependent variables is a function of r and θ, where θ is the angle measured from the free-stream direction. The velocity has components $v_r(r, \theta)$ and $v_\theta(r, \theta)$ in the r and θ directions,

$$\mathbf{v} = \mathbf{i}_r v_r + \mathbf{i}_\theta v_\theta.$$

The boundary conditions for the system of equations (2)–(6) are those at the particle surface,

$$\mathbf{v}(1, \theta) = 0, \qquad T(1, \theta) = 1 + \tau. \tag{8}$$

In the limit $\mathrm{Re} \to 0$ (with r, θ fixed), the above equations give an adequate description of the flow only when $r = O(1)$. Thus the asymptotic solutions do not necessarily satisfy the boundary conditions for $r \to \infty$. These conditions are replaced by the requirement that the asymptotic expansions of solutions to Eqs. (2)–(6) must match properly with another set of asymptotic expansions which *are* valid far from the body.

The nondimensional equations appropriate far from the body are expressed in terms of Oseen variables (i.e., with the viscous length chosen as the characteristic length),

$$\nabla\cdot\rho\mathbf{v} = 0, \tag{9}$$

$$\begin{aligned}\rho\mathbf{v}\cdot\nabla\mathbf{v} = {}& -\nabla P + \mu[\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}) + \nabla^2\mathbf{v}] \\ & + [-\tfrac{2}{3}(\nabla\cdot\mathbf{v})\nabla\mu + \nabla(\mathbf{v}\cdot\nabla\mu) \\ & - (\mathbf{v}\cdot\nabla)\nabla\mu + (\nabla\mu\cdot\nabla)\mathbf{v}],\end{aligned} \tag{10}$$

$$\begin{aligned}\rho\mathbf{v}\cdot\nabla T = {}& \mathrm{Pr}^{-1}\,\nabla\cdot\lambda\nabla T \\ & + (\gamma - 1)M_\infty^2[\mu\varphi + \mathbf{v}\cdot\nabla P],\end{aligned} \tag{11}$$

$$\rho = (1/T) + \gamma M_\infty^2(P/T), \tag{12}$$

$$\mu = \lambda = T. \tag{13}$$

The same notation, $\mathbf{v}$, ρ, T, μ, λ, has been used to denote a given nondimensional quantity, whether it is a function of r and θ as in (2)–(6), or as a function of s and θ as in (9)–(13). Also implied above are the relations

$$s = \mathrm{Re}\, r, \qquad p = \mathrm{Re}\, P, \qquad \Phi = \mathrm{Re}\, \varphi. \tag{14}$$

[11] D. R. Kassoy, Ph. D. dissertation, University of Michigan (1965).

[12] P. A. Lagerstrom, in *Theory of Laminar Flows*, edited by F. K. Moore (Princeton University Press, Princeton, New Jersey, 1964).

[13] H. Schlichting, *Boundary Layer Theory*, (McGraw-Hill Book Company, Inc., New York, 1955).

TABLE I. The effect of τ on temperature and density.

(a) $\tau = O(1)$	$\Delta T = O(\tau)$	$\Delta\rho = O(\tau)$	$\Delta\mathbf{v} = O(\tau)$
(b) $\mathrm{Re} \ll \tau \ll 1$	$\Delta T = O(\tau)$	$\Delta\rho = O(\tau)$	$\Delta\mathbf{v} = O(\tau)$
(c) $\tau = O(\mathrm{Re})$	$\Delta T = O(\tau)$	$\Delta\rho = O(\tau)$	$\Delta\mathbf{v} = O(\mathrm{Re})$
(d) $(M_\infty^2/\mathrm{Re}) \ll \tau \ll \mathrm{Re}$	$\Delta T = O(\tau)$	$\Delta\rho = O(\tau)$	$\Delta\mathbf{v} = O(\mathrm{Re})$
(e) $\tau = O(M_\infty^2/\mathrm{Re})$	$\Delta T = O(\tau)$	$\Delta\rho = O(M_\infty^2/\mathrm{Re})$	$\Delta\mathbf{v} = O(\mathrm{Re})$
(f) $M_\infty^2 \ll \tau \ll M_\infty^2/\mathrm{Re}$	$\Delta T = O(\tau)$	$\Delta\rho = O(M_\infty^2/\mathrm{Re})$	$\Delta\mathbf{v} = O(\mathrm{Re})$
(g) $\tau = O(M_\infty^2)$	$\Delta T = O(M_\infty^2)$	$\Delta\rho = O(M_\infty^2/\mathrm{Re})$	$\Delta\mathbf{v} = O(\mathrm{Re})$
(h) $\tau \ll M_\infty^2$	$\Delta T = O(M_\infty^2)$	$\Delta\rho = O(M_\infty^2/\mathrm{Re})$	$\Delta\mathbf{v} = O(\mathrm{Re})$

In Eqs. (9)–(13) the operator ∇ implies differentiation with respect to the Oseen variables.

The boundary conditions for $s \to \infty$ are

$$\mathbf{v}(\infty, \theta) = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta, \qquad T(\infty, \theta) = 1, \quad P(\infty, \theta) = 0. \tag{15}$$

These will be supplemented by conditions for matching with solutions valid when $r = O(1)$.

III. PHYSICAL CONSIDERATIONS

The sequence in which various physical features become apparent in the asymptotic expansions will depend on the relative sizes of the parameters Re, M_∞, and τ. To understand the significance of different choices it is necessary to study the orders of the terms in the equations. Equations (2)–(6) are studied, since it is near the body that the greatest changes occur in the flow variables.

Since the nondimensional wall temperature is $1 + \tau$, then for $r = O(1)$ changes in the nondimensional temperature T are measured by $\Delta T = T - (1 + \tau)$. In the energy equation (4) the convection, conduction, and dissipation terms are, respectively, $O(\mathrm{Re}\,\Delta T)$, $O(\Delta T)$, $O(M_\infty^2)$. For $\Delta T \gg M_\infty^2$, the conduction term dominates the equation and the temperature variation ΔT is of the same order as the impressed temperature difference τ; this result also implies $\tau \gg M_\infty^2$. If $\tau = O(M_\infty^2)$ the dissipation and conduction terms are of the same order, and $\Delta T = O(\tau) = O(M_\infty^2)$. If $\tau \ll M_\infty^2$, temperature differences due to viscous heating are greater than $O(\tau)$, and the approximate equation still describes a balance between dissipation and thermal conduction.

The equation of state (5) shows that density changes are primarily related to temperature changes if $\tau \gg M_\infty^2/\mathrm{Re}$ and are determined predominantly by pressure changes if $\tau \ll M_\infty^2/\mathrm{Re}$. Both effects enter if τ and M_∞^2/Re are of the same order. From the continuity equation it is seen that changes in velocity are at least as large, in order of magnitude, as changes in density. The momentum equation shows that for $\tau \gg \mathrm{Re}$ the variation $O(\tau)$ in viscosity has greater influence than the inertia terms, which are $O(\mathrm{Re})$. It follows that for $\tau \gg \mathrm{Re}$ the corrections $\Delta\mathbf{v}$ to the Stokes solution for velocity are $O(\tau)$, due to variable density and viscosity. But if $\tau \ll \mathrm{Re}$ the corrections are $O(\mathrm{Re})$, due to the effect of inertia forces.

The results are summarized in Table I. For $\tau \to 0$ increasingly fast (i.e., passing from top to bottom in the first column of Table I), it is seen from the equations that some new physical feature becomes important at each of cases a, c, e, and g. These might be called distinguished cases. Each of the intermediate cases b, d, f, h depends at most on the same physical processes as appear in the immediately preceding distinguished case. This implies that an approximate solution for a particular distinguished case can be extended to include the succeeding intermediate case (or, for that matter, to include the immediately preceding case). In this sense it is only necessary to study the distinguished cases.

Since it is of interest to examine the effect of large values of τ, the present analysis is limited to the cases $\tau = O(1)$ and $\tau = O(\mathrm{Re})$.

IV. CONSTRUCTION OF THE SOLUTIONS

The solutions of Eqs. (2)–(6) and (9)–(13), subject to boundary conditions (8) and (15), are assumed to possess asymptotic expansions of the following form:

Inner Outer

Density,

$$\rho \sim \sum_{n=0} \alpha_n \rho_n(r, \theta), \quad \rho \sim \sum_{n=0} a_n R_n(s, \theta);$$

Velocity,

$$\mathbf{v} \sim \sum_{n=0} \beta_n \mathbf{v}_n(r, \theta), \quad \mathbf{v} \sim \sum_{n=0} b_n \mathbf{V}_n(s, \theta);$$

Pressure,

$$p \sim \sum_{n=0} \delta_n p_n(r, \theta), \quad P = \frac{p}{\mathrm{Re}} \sim \sum_{n=0} d_n P_n(s, \theta);$$

Temperature,

$$T \sim \sum_{n=0} \epsilon_n t_n(r, \theta), \quad T \sim \sum_{n=0} e_n T_n(s, \theta); \tag{16}$$

where $\alpha_n, \cdots, \epsilon_n$ and $a_n, \cdots, e_n$ are, in general, functions of M_∞, Re, and τ, and

$$\alpha_0 = \beta_0 = \delta_0 = \epsilon_0 = a_0 = b_0 = d_0 = e_0 = 1.$$

The "outer" variable s is related to the "inner" variable r by $s = \mathrm{Re}\, r$. Components of $\mathbf{V}_n$ will be denoted by $V_{ns}(s, \theta)$, $V_{n\theta}(s, \theta)$.

By definition each inner expansion, of the form

$$q \sim \sum_{n=0} \varphi_n(\mathrm{Re}, M_\infty, \tau) q_n(r, \theta) \tag{17}$$

is an asymptotic expansion if, for each n,

$$\lim \frac{\varphi_{n+1}}{\varphi_n} = 0, \qquad \lim \frac{q - \sum_{k=0}^{n} \varphi_k q_k}{\varphi_n} = 0, \tag{18}$$

for $\mathrm{Re} \to 0$, $(M_\infty/\mathrm{Re}) \to 0$, and (in the cases chosen) either τ fixed or τ/Re fixed. Similar properties hold for the outer expansions

$$q \sim \sum_{n=0} \Phi_n(\mathrm{Re}, M_\infty, \tau) Q_n(s, \theta). \tag{19}$$

The differential equations for the perturbation quantities in Eqs. (16) are found by substituting in the inner and outer governing equations and grouping terms of the same order of magnitude. It is then required that the equations be satisfied to each order of magnitude, in the limit for $\mathrm{Re} \to 0$, $(M_\infty/\mathrm{Re}) \to 0$ and either τ/Re fixed or τ fixed. The boundary conditions to be satisfied by terms in the Eqs. (16) are found by substituting in Eqs. (8) and (15), again requiring that the equations be satisfied to each order of magnitude. The matching condition for each flow variable q states that to each order in Re, M_∞, τ there exists some overlap domain for $s \to 0$ sufficiently slowly, and $r \to \infty$, such that both inner and outer expansions are valid (see Ref. 5). That is, to each order in the small parameters,

$$q_0 + \varphi_1 q_1 + \varphi_2 q_2 + \cdots \sim Q_0 + \Phi_1 Q_1 + \Phi_2 Q_2 + \cdots, \tag{20}$$

for $\mathrm{Re} \to 0$, $(M_\infty/\mathrm{Re}) \to 0$, either τ fixed or τ/Re fixed, $r \to \infty$, and $s \to 0$ (sufficiently slowly).

V. SOLUTIONS FOR $\tau = O(\mathrm{Re})$

For $\tau = O(\mathrm{Re})$ the zeroth- and first-order terms can be found by appropriately combining results from Refs. 1, 3, 5, 8, 9, and 10, or can be rederived using a single consistent approximation scheme, as discussed in the preceding section. An outline of the derivation of these terms is given below. Terms beyond this group represent new results and are discussed in greater detail.

The leading terms in the outer expansions describe the free stream,

$$T_0 = R_0 = 1, \ \mathbf{V}_0 = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta, \ P_0 = 0. \tag{21}$$

Since temperature variations are small, the leading terms of the inner expansion are the same as the Stokes solution,

$$t_0 = \rho_0 = 1,$$

$$\mathbf{v}_0 = \mathbf{i}_r\left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\cos\theta + \mathbf{i}_\theta\left(-1 + \frac{3}{4r} + \frac{1}{4r^3}\right)\sin\theta, \tag{22}$$

$$p_0 = -\tfrac{3}{2}(\cos\theta/r^2).$$

The solutions for t_1 and ρ_1 now also can be obtained; the boundary condition suggests the choice $\epsilon_1 = \tau$, or

$$\epsilon_1 = \mathrm{Re}\, \tau/\mathrm{Re}. \tag{23a}$$

The temperature distribution is determined entirely by conduction. That is, t_1 satisfies Laplace's equation

$$\nabla^2 t_1 = 0 \tag{23b}$$

subject to the boundary condition $t_1(1, \theta) = 0$ and the matching condition, for $r \to \infty$,

$$1 + \tau t_1 + \cdots \sim 1 + \cdots.$$

The solution, given by Acrivos and Taylor,[9] is seen to be

$$t_1 = r^{-1}. \tag{24}$$

From the equation of state, it is found also that

$$\alpha_1 = \mathrm{Re}\, \tau/\mathrm{Re}, \qquad \rho_1 = -t_1 = -r^{-1}. \tag{25}$$

The orders of magnitude for the first-order outer solutions $\mathbf{V}_1$ and T_1 are determined from the matching conditions. For $r \to \infty$, Eqs. (21), (22), and (24) show that $\mathbf{v}_0 - \mathbf{V}_0 = O(\mathrm{Re}/s)$ and $t_1 = O(\mathrm{Re}/s)$. Hence, the matching suggests $b_1 = O(\mathrm{Re})$ and $e_1 = O(\mathrm{Re}^2)$. The equation of state and the momentum equation require, respectively, $a_1 = O(\mathrm{Re}^2)$ and $d_1 = O(\mathrm{Re})$. It is convenient to include certain constant factors, as follows:

$$b_1 = d_1 = \mathrm{Re}, \qquad a_1 = e_1 = \mathrm{Re}^2\,(\tau/\mathrm{Re})\,\mathrm{Pr}. \tag{26}$$

The first-order outer solutions satisfy the differential equations

$$\nabla\cdot\mathbf{V}_1 = 0, \qquad \mathbf{V}_0\cdot\nabla\mathbf{V}_1 = -\nabla P_1 + \nabla^2\mathbf{V}_1,$$
$$\mathbf{V}_0\cdot\nabla T_1 = \mathrm{Pr}^{-1}\,\nabla^2 T_1, \qquad R_1 = -T_1, \tag{27}$$

subject to boundary conditions

$$T_1(\infty, \theta) = P_1(\infty, \theta) = \mathbf{V}_1(\infty, \theta) = 0 \tag{28}$$

and matching conditions, for some overlap domain where $r \to \infty$ and $s \to 0$,

$$1 + \mathrm{Re}^2\frac{\tau}{\mathrm{Re}}\,\mathrm{Pr}\,T_1 + \cdots \sim 1 + \mathrm{Re}\frac{\tau}{\mathrm{Re}}\,t_1 + \cdots,$$
$$\mathbf{V}_0 + \mathrm{Re}\,\mathbf{V}_1 + \cdots \sim \mathbf{v}_0 + \cdots. \tag{29}$$

The solutions for V_1 and P_1 are the Oseen solutions, and the solution for T_1 was given by Acrivos and Taylor,[9]

$$V_{1_s} = \frac{3}{2s^2}\left[1 - e^{-\frac{1}{2}s(1-\cos\theta)}\right] - \frac{3}{4s}(1 + \cos\theta)e^{-\frac{1}{2}s(1-\cos\theta)},$$
$$V_{1_\theta} = \frac{3}{4s}e^{-\frac{1}{2}s(1-\cos\theta)}\sin\theta, \quad P_1 = -\frac{3}{2s^2}\cos\theta, \tag{30}$$
$$T_1 = \frac{1}{\mathrm{Pr}\,s}e^{-\frac{1}{2}s\,\mathrm{Pr}\,(1-\cos\theta)}, \qquad R_1 = -T_1.$$

For the first-order inner solution, the momentum equation suggests that β_1 and δ_1 should be $O(\mathrm{Re})$, the same order as the inertia terms and the variations in density and viscosity,

$$\beta_1 = \delta_1 = \mathrm{Re}. \tag{31}$$

Therefore, $\mathbf{v}_1$ and p_1 satisfy

$$\nabla\cdot\mathbf{v}_1 = -(\tau/\mathrm{Re})\mathbf{v}_0\cdot\nabla\rho_1,$$
$$\nabla^2\mathbf{v}_1 - \nabla p_1 = \mathbf{v}_0\cdot\nabla\mathbf{v}_0 - \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}_1) - (\tau/\mathrm{Re})[\mu_1\nabla^2\mathbf{v}_0 + \nabla(\mathbf{v}_0\cdot\nabla\mu_1) - (\mathbf{v}_0\cdot\nabla)\nabla\mu_1 + (\nabla\mu_1\cdot\nabla)\mathbf{v}_0], \tag{32}$$

where $\rho_1 = -t_1 = -r^{-1}$ and $\mu_1 = t_1 = r^{-1}$. The momentum equation includes a convective term (first term on the right-hand side), an effect of variable density (second term), and an effect of variable viscosity (third term). The boundary conditions for $r = 1$ and the matching conditions for $r \to \infty$ are

$$\mathbf{v}_1(1, \theta) = 0,$$
$$\mathbf{v}_0 + \mathrm{Re}\,\mathbf{v}_1 + \cdots \sim \mathbf{V}_0 + \mathrm{Re}\,\mathbf{V}_1 + \cdots, \tag{33}$$
$$p_0 + \mathrm{Re}\,p_1 + \cdots \sim \mathrm{Re}\,P_1 + \cdots.$$

The matching conditions imply that $p_1 \to 0$ as $r \to \infty$ and that the leading term in $\mathbf{v}_1$ for $r \to \infty$ must agree with the term of order one in $\mathbf{V}_1$ for $s \to 0$.

The solution of this system of equations, found by separation of variables and variation of parameters, is given by

$$v_{1_r} = \left\{\frac{3}{8} - \left[\frac{9}{16} + \frac{13}{16}\frac{\tau}{\mathrm{Re}}\right]\frac{1}{r} + \frac{15}{8}\frac{\tau}{\mathrm{Re}}\frac{1}{r^2} + \left[\frac{3}{16} - \frac{21}{16}\frac{\tau}{\mathrm{Re}}\right]\frac{1}{r^3} + \frac{1}{4}\frac{\tau}{\mathrm{Re}}\frac{1}{r^4}\right\}\cos\theta - \frac{3}{16}\left\{2 - \frac{3}{r} + \frac{1}{r^2} - \frac{1}{r^3} + \frac{1}{r^4}\right\}\tfrac{1}{2}(3\cos^2\theta - 1), \tag{34}$$

$$v_{1_\theta} = \left\{-\frac{3}{8} + \left[\frac{9}{32} - \frac{3}{32}\frac{\tau}{\mathrm{Re}}\right]\frac{1}{r} + \frac{3}{4}\frac{\tau}{\mathrm{Re}}\frac{1}{r^2} + \left[\frac{3}{32} - \frac{21}{32}\frac{\tau}{\mathrm{Re}}\right]\frac{1}{r^3}\right\}\sin\theta + \left\{\frac{3}{8} - \frac{9}{32r} + \frac{3}{32r^3} - \frac{3}{16r^4}\right\}\sin\theta\cos\theta, \tag{35}$$

$$p_1 = \left\{-\frac{3}{16r^2} + \frac{3}{16r^4} - \frac{1}{16r^6}\right\} + \left\{\left[-\frac{27}{16} - \frac{103}{48}\frac{\tau}{\mathrm{Re}}\right]\frac{1}{r^2} + \frac{11}{4}\frac{\tau}{\mathrm{Re}}\frac{1}{r^3} - \frac{7}{6}\frac{\tau}{\mathrm{Re}}\frac{1}{r^5}\right\}\cos\theta + \left\{\frac{3}{8r^2} - \frac{7}{8r^3} + \frac{3}{4r^4} - \frac{1}{16r^6}\right\}\tfrac{1}{2}(3\cos^2\theta - 1). \tag{36}$$

The terms independent of τ were given by Kaplun and Lagerstrom[5] and by Proudman and Pearson,[8] and the terms proportional to τ/Re can be obtained from Chang's solutions.[10] The results imply that the first-order coefficient will differ from that given by Proudman and Pearson.

The solution for t_2 can now also be obtained. In the energy equation (4) the convection term and the effect of variable conductivity lead to a perturbation in temperature which is of order Re^2. There is no physical reason for suggesting a larger perturbation. Furthermore, if it were assumed that the term $\epsilon_2 t_2$ were larger, such that $\mathrm{Re}^2 \ll \epsilon_2 \ll \mathrm{Re}$, then t_2 would have to satisfy Laplace's equation, and the boundary and matching conditions could be satisfied only if $t_2 \equiv 0$. Hence, $\epsilon_2 = O(\mathrm{Re}^2)$. On the other hand, it is found from the equation of state that the changes in pressure lead to a perturbation $\alpha_2\rho_2$ in density such that $\alpha_2 = M_\infty^2/\mathrm{Re}$, larger than Re^2 according to the assumption following Eq. (1). It is found, however, that the equations defining t_2 do not involve any terms associated with the Mach number and, therefore, can be solved without further study of such terms.

The result $\epsilon_2 = O(\mathrm{Re}^2)$ is consistent with the matching condition, since t_2 must match with a term in T_1 which is $O(1)$ for $s \to 0$. For convenience a constant factor is included in ϵ_2,

$$\epsilon_2 = \mathrm{Re}^2\,\tau/\mathrm{Re}. \tag{37}$$

The differential equation for t_2 is

$$\nabla^2 t_2 = \Pr \mathbf{v}_0\cdot\nabla t_1 - (\tau/\mathrm{Re})\tfrac{1}{2}(\nabla^2 t_1^2), \tag{38}$$

which includes on the right-hand side a convection term and a term showing the effect of variable conductivity. The boundary condition is

$$t_2(1, \theta) = 0 \tag{39}$$

and the matching condition, for $r \to \infty$ and $s \to 0$, is

$$1 + \tau t_1 + \tau \,\mathrm{Re}\, t_2 + \cdots \sim 1 + \tau\,\mathrm{Re}\Pr T_1 + \cdots . \tag{40}$$

The solution for t_2 is found to be

$$t_2 = \left[\tfrac{1}{2}\Pr\left(\frac{1}{r} - 1\right) + \frac{\tau}{\mathrm{Re}}\left(\frac{1}{2r} - \frac{1}{2r^2}\right)\right] + \Pr\left[\frac{1}{2} - \frac{3}{4r} + \frac{3}{8r^2} - \frac{1}{8r^3}\right]\cos\theta. \tag{41}$$

The terms independent of τ were given by Acrivos and Taylor. In the present solution an additional term, proportional to τ/Re, appears because the conductivity is allowed to vary.

At this point the usual procedure would be to study the second-order outer solutions. However, it is found that the next terms of the inner expansions may be obtained without knowledge of additional solutions from the outer equations. These are the first terms involving the Mach number, and their effect on the drag and heat transfer will be of special interest.

For the inner expansion the energy equation shows that the viscous dissipation leads to temperature variation of order $\epsilon_3 = O(M_\infty^2)$. As already noted, the equation of state requires the density variation due to changes in pressure to be of order $\alpha_2 = O(M_\infty^2/\mathrm{Re})$. The continuity equation then shows that the corresponding velocity changes are measured by $\beta_2 = O(M_\infty^2/\mathrm{Re})$. Since a pressure force must appear in the momentum equation, $\delta_2 = O(M_\infty^2/\mathrm{Re})$. The choices will be

$$\epsilon_3 = M_\infty^2, \qquad \alpha_2 = \beta_2 = \delta_2 = M_\infty^2/\mathrm{Re}. \tag{42}$$

Thus it is convenient to study t_3 along with $\mathbf{v}_2$, p_2, and ρ_2.

For the second-order terms in the outer expansion, it can be seen from the differential equations or from the matching conditions that appropriate choices for the orders of magnitude are

$$a_2 = \mathrm{Re}^2\,\tau\Pr, \qquad b_2 = d_2 = \mathrm{Re}^2, \qquad e_2 = \mathrm{Re}^2\,\tau\Pr. \tag{43}$$

But the inner solutions $\mathbf{v}_2$, p_2, ρ_2, and t_3 can only match with terms which will be multiplied by M_∞^2. In the outer equations the dissipation leads to a variation in temperature which is $O(M_\infty^2\,\mathrm{Re})$, of smaller order than either e_2 or ϵ_3. It will be sufficient to replace the matching conditions for t_3, $\mathbf{v}_2$, p_2, and ρ_2 by the requirement that these functions approach zero as $r \to \infty$.

These first terms involving Mach number are found to satisfy the equations

$$\begin{aligned}\nabla^2 t_3 &= -\Pr(\gamma - 1)[\Phi_0 + \mathbf{v}_0\cdot\nabla p_0], \quad \rho_2 = \gamma p_0,\\ \nabla\cdot\mathbf{v}_2 &= -\mathbf{v}_0\cdot\nabla\rho_2,\\ 0 &= -\nabla p_2 + \nabla^2\mathbf{v}_2 + \tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}_2),\end{aligned} \tag{44}$$

subject to boundary conditions,

$$\mathbf{v}_2(1, \theta) = t_3(1, \theta) = 0, \tag{45}$$

and matching conditions for $r \to \infty$,

$$\mathbf{v}_2(r, \theta) \to 0, \qquad p_2(r, \theta) \to 0, \qquad t_3(r, \theta) \to 0. \tag{46}$$

The solutions are

$$\begin{aligned}\frac{t_3}{(\gamma - 1)\Pr} &= \left[\frac{101}{80r} - \frac{3}{2r^2} + \frac{5}{16r^4} - \frac{3}{40r^6}\right] + \left[\frac{1}{2r} - \frac{3}{16r^2} - \frac{3}{2r^3} + \frac{11}{8r^4} - \frac{3}{16r^6}\right]\tfrac{1}{2}(3\cos^2\theta - 1),\\ v_{2r} &= \gamma\left[\frac{1}{2r^2} - \frac{3}{4r^3} + \frac{1}{4r^5}\right] + \left[-\frac{11}{16r^2} + \frac{15}{8r^3} - \frac{27}{16r^4} + \frac{1}{2r^5}\right]\frac{\gamma}{2}(3\cos^2\theta - 1),\\ v_{2\theta} &= \gamma\left[-\frac{3}{2r^2} + \frac{45}{16r^3} - \frac{27}{16r^4} + \frac{3}{8r^5}\right]\sin\theta\cos\theta,\\ p_2 &= \gamma\left[\frac{1}{r^4} - \frac{1}{r^6}\right] + \left[-\frac{35}{8r^3} + \frac{5}{r^4} - \frac{1}{r^6}\right]\frac{\gamma}{2}\cdot(3\cos^2\theta - 1).\end{aligned} \tag{47}$$

The results of the preceding analysis are summarized in the following inner expansions:

$$\begin{aligned}\rho(r, \theta) &= 1 + \mathrm{Re}\,(\tau/\mathrm{Re})\rho_1 + (M_\infty^2/\mathrm{Re})\rho_2 + \mathrm{Re}^2\,(\tau/\mathrm{Re})\rho_3 + O(\alpha_4),\\ \mathbf{v}(r, \theta) &= \mathbf{v}_0 + \mathrm{Re}\,\mathbf{v}_1 + (M_\infty^2/\mathrm{Re})\mathbf{v}_2 + O(\beta_3),\\ p(r, \theta) &= p_0 + \mathrm{Re}\,p_1 + (M_\infty^2/\mathrm{Re})p_2 + O(\delta_3),\\ T(r, \theta) &= 1 + \mathrm{Re}\,(\tau/\mathrm{Re})t_1 + \mathrm{Re}^2\,(\tau/\mathrm{Re})t_2 + M_\infty^2 t_3 + O(\epsilon_4),\\ \mu(r, \theta) &= \lambda(r, \theta) = T(r, \theta).\end{aligned} \tag{48}$$

In general, the expansions have been calculated out to the first Mach number term. The results of Proudman and Pearson[8] and Acrivos and Taylor[9] imply that the logarithm of Reynolds number should appear in terms ordered by β_3, δ_3, and ϵ_4. From the equation of state, it follows that $\alpha_4 = O(M_\infty^2)$, and α_5 will be logarithmic in Re. While no attempt has been made to find these terms, which arise from nonhomogeneous terms in the higher-order energy and momentum equations, it appears that they will differ from the logarithmic terms calculated previously, due to the inclusion of the additional temperature effects.

It is of interest to compare the present expansions for temperature and velocity with those of Acrivos and Taylor[9] and Proudman and Pearson[8] in order to indicate the conditions under which the latter are valid representations.

In the temperature expansions, h_0 and h_1 are solutions given in Ref. 9, $\bar{t}_2$ is the τ/Re term in Eq. (41), and t_3 is given by Eq. (47). In the velocity expansions; $\mathbf{v}_1$ has been split into incompressible ($\mathbf{v}_{10}$) and compressible ($\mathbf{v}_{11}$) parts.

Present solution:

$$T = 1 + \text{Re}\frac{\tau}{\text{Re}}h_0 + \text{Re}^2\frac{\tau}{\text{Re}}\left[P_r h_1 + \frac{\tau}{\text{Re}}\bar{t}_2\right] + M_\infty^2 t_3 + O(\text{Re}^3 \ln \text{Re}), \qquad (49)$$

$$\mathbf{v} = \mathbf{v}_0 + \text{Re}\left[\mathbf{v}_{10} + \frac{\tau}{\text{Re}}\mathbf{v}_{11}\right] + \frac{M_\infty^2}{\text{Re}}\mathbf{v}_2 + O(\text{Re}^2 \ln \text{Re});$$

Acrivos and Taylor Solution:

$$T = 1 + \text{Re}\,(\tau/\text{Re})h_0 + \text{Re}^2\,(\tau/\text{Re})\,\text{Pr}\,h_1 + O(\text{Re}^3 \ln \text{Re}); \qquad (50)$$

Proudman and Pearson Solution:

$$\mathbf{v} = \mathbf{v}_0 + \text{Re}\,\mathbf{v}_{10} + O(\text{Re}^2 \ln \text{Re}). \qquad (51)$$

It is clear that the present solutions reduce to those found previously if τ/Re, M_∞^2, and M_∞^2/Re are made sufficiently small.

The drag coefficient and Nusselt number are calculated by evaluating the stress and heat transfer at the particle surface. These quantities are found from the inner velocity, pressure, and temperature expansions.

If F_d is the drag force, the drag coefficient is defined as follows:

$$C_d = \frac{F_d}{\frac{1}{2}\rho_\infty u_\infty^2 \pi a^2} = -\frac{4}{\text{Re}}\left[\int_0^\pi p(1, \theta)\sin\theta\cos\theta\,d\theta + \int_0^\pi \mu(1, \theta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]_{r=1}\sin^2\theta\,d\theta\right]. \qquad (52)$$

If the integrals are evaluated, the result is

$$C_d = \frac{12}{\text{Re}}\left[1 + \text{Re}\left(\frac{3}{8} + \frac{13}{24}\frac{\tau}{\text{Re}}\right) + O(\text{Re}^2 \ln \text{Re}). \qquad (53)$$

The Nusselt number is defined by

$$\text{Nu} = \frac{-1}{\tau}\int_0^\pi \left[\frac{\partial T}{\partial r}\right]_{r=1}\sin\theta\,d\theta, \qquad (54)$$

and upon evaluation of the integral, one can derive the following:

$$\text{Nu} = 2 + \text{Re Pr}\left[1 - \frac{\tau}{\text{Re Pr}}\right] - \frac{M_\infty^2}{\tau}\text{Pr}\,(\gamma - 1)\,\frac{15}{8} + O(\text{Re}^3 \ln \text{Re}). \qquad (55)$$

VI. SOLUTIONS FOR THE CASE $\tau = 0(1)$

Physical considerations (i.e., see Table I) indicate that the zeroth-order inner variables are affected by the impressed temperature difference when $\tau = O(1)$. In essence, the problem is that of a compressible Stokes flow. The analysis in this section is devoted primarily to the solution of the zeroth-order equation, although the first-order temperature effect is also calculated.

Just as in previous solutions, it is clear that the zeroth-order outer solutions describe the free stream. Thus

$$\mathbf{V}_0 = \mathbf{i}_r\cos\theta - \mathbf{i}_\theta\sin\theta, \qquad P_0 = 0, \quad R_0 = 1, \quad T_0 = 1. \qquad (56)$$

Since a linear thermal conductivity–temperature relation has been assumed, the inner energy equation may be written as follows:

$$\nabla^2 t_0^2 = 0, \qquad (57)$$

with the boundary condition

$$t_0(1, \theta) = 1 + \tau. \qquad (58a)$$

The matching condition for t_0 requires that, for $r \to \infty$,

$$t_0 + \cdots \sim 1 + \cdots. \qquad (58b)$$

The solution of these equations is

$$t_0(r, \theta) = [1 + (K/r)]^{\frac{1}{2}}, \qquad (59)$$

where $K \equiv \tau(\tau + 2)$. The zeroth-order inner density may be found from the equation of state. Thus,

$$\rho_0 = (1/t_0) = [1 + (K/r)]^{-\frac{1}{2}}. \qquad (60)$$

The zeroth-order inner pressure and velocity components are found from the approximate continuity and momentum equations, which are as follows:

$$\nabla\cdot\rho_0\mathbf{v}_0 = 0,$$

$$0 = -\nabla p_0 + \tfrac{1}{3}\mu_0 \nabla(\nabla\cdot\mathbf{v}_0) + \mu_0\nabla^2\mathbf{v}_0 + [-\tfrac{2}{3}(\nabla\cdot\mathbf{v}_0)\nabla\mu_0 + \nabla(\mathbf{v}_0\cdot\nabla\mu_0) - (\mathbf{v}_0\cdot\nabla)\nabla\mu_0 + (\nabla\mu_0\cdot\nabla)\mathbf{v}_0]. \tag{61}$$

The boundary condition at $r = 1$ and the matching conditions for $r \to \infty$ are, as for the incompressible case,

$$\mathbf{v}_0(1, \theta) = 0, \qquad \mathbf{v}_0 + \cdots \sim \mathbf{V}_0 + \cdots, \qquad p_0 + \cdots \sim \mathrm{Re}\,(P_0 + \cdots). \tag{62}$$

The differential equations (61) are linear with variable coefficients depending on r. A dependence on θ enters only in the matching condition, in an especially simple way because of the form of $\mathbf{V}_0$. Solutions satisfying (61) and (62) can therefore be found by assuming v_{0r} and $v_{0\theta}$ to have the same dependence on θ as the components of $\mathbf{V}_0$. Thus

$$v_{0r} = u(r)\cos\theta, \quad v_{0\theta} = v(r)\sin\theta, \quad p_0 = p(r)\cos\theta, \tag{63}$$

where the form chosen for p_0 is consistent with (61).

The solution to Eqs. (61) and (62) is most easily obtained in terms of a Stokes stream function, $\psi(r, \theta)$, where

$$\rho_0 v_{0r} = -\frac{1}{r^2\sin\theta}\psi_\theta, \qquad \rho_0 v_{0\theta} = \frac{1}{r\sin\theta}\psi_r, \tag{64}$$

so that the first of Eqs. (61) is satisfied identically. Next, in view of the relationships given in Eqs. (63), $\psi(r, \theta)$ may be written in terms of a modified stream function, $\bar{a}(r)$

$$\psi(r, \theta) = \bar{a}(r)\sin^2\theta. \tag{65}$$

Substitution of Eqs. (63)–(65) into the second of Eq. (61) results in a fourth-order differential equation for $\bar{a}(r)$ (Ref. 11). The boundary conditions on $\bar{a}$ are found by transforming Eqs. (62). However, the solution is obtained more easily if the following transformation is employed:

$$\omega = K/2r, \qquad a(\omega) = \bar{a}(r). \tag{66}$$

Then the resulting equation for $a(\omega)$ is,

$$(1+2\omega)\omega^4 a^{\mathrm{IV}} + (12+29\omega)\omega^3 a''' + (32+104\omega)\omega^2 a'' + (8+68\omega)\omega a' - (8+32\omega)a = 0, \tag{67}$$

and $a(\omega)$ is subject to the boundary conditions

$$\lim_{\omega\to 0}\omega^2 a(\omega) = -\tfrac{1}{8}K^2, \qquad a(\tfrac{1}{2}K) = a'(\tfrac{1}{2}K) = 0. \tag{68}$$

A solution of equation (67) is

$$a = C_4/\omega^2, \qquad C_4 = \text{const}. \tag{69}$$

Hence, the governing equation for $a(\omega)$ can be reduced to a third-order equation,

$$(1 + 2\omega)\omega^3 y''' + (7 + 19\omega)\omega^2 y'' + (4 + 28\omega)\omega y' - (4 + 16\omega)y = 0, \tag{70}$$

where

$$a(\omega) = -\frac{1}{\omega^2}\int^{\omega} \sigma y(\sigma)\,d\sigma + \frac{C_4}{\omega^2}. \tag{71}$$

Finally, the first of the boundary conditions given in (68) becomes

$$\lim_{\omega\to 0}\omega^2 y(\omega) = 0. \tag{72}$$

Equation (70) has three regular singularities, at the points $\omega = -\frac{1}{2}, 0, \infty$. A power series solution about $\omega = 0$ has been obtained (see Ref. 11 for details), using a procedure outlined by Rainville.[14] In terms of $a(\omega)$, this solution is

$$\begin{aligned} a(\omega) &= -\frac{K^2}{8\omega^2} + C_2 a^{(2)}(\omega) - C_1 a^{(1)}(\omega) \\ &= -\frac{K^2}{8\omega^2} + C_2\Big\{-\sum_{n=0}^{\infty}\frac{A_n\omega^{n+1}}{(n+3)}\ln\omega + \frac{2}{\omega} - \frac{9}{2} + \sum_{n=0}^{\infty}\frac{1}{(n+3)}\Big(\frac{1}{n+3} - k_{n+2}\Big)A_n\omega^{n+1}\Big\} - C_1\sum_{n=0}^{\infty}\frac{A_n\omega^{n+1}}{n+3}, \end{aligned} \tag{73}$$

where

$$A_0 = \tfrac{72}{5},$$

$$A_n = -\frac{(2n^3 + 13n^2 + 13n + 16)}{n(n+2)(n+5)}A_{n-1}, \qquad n \geq 1,$$

$$k_2 = \tfrac{661}{720},$$

$$k_{n+2} = \frac{6n^2 + 26n + 13}{2n^3 + 13n^2 + 13n + 16} - \frac{3n^2 + 14n + 10}{n(n+2)(n+5)} + k_{n+1}, \qquad n \geq 1.$$

Due to the singularity at $\omega = -\frac{1}{2}$, the above series solution is valid only when $0 \leq \omega < \frac{1}{2}$. Hence, since $1 \leq r < \infty$, the constants C_1 and C_2 can be evaluated from the boundary conditions on $a(\omega)$ only for values of K in the range $0 \leq K < 1$, which correspond to values of τ in the range $0 \leq \tau < 0.414$.

Series solutions could be obtained for larger values of $K(\tau)$ by developing expansions about some ordinary point beyond $\omega = \frac{1}{2}$. However, for the purposes of obtaining numerical results, it is more convenient to resort to a numerical solution of Eqs. (67) and (68). The calculations were made by using a fifth-order Runge–Kutta

[14] E. Rainville, *Intermediate Differential Equations* (The Macmillan Company, New York, 1964), 2nd ed.

TABLE II. Summary of numerical results.

τ	C_1	C_2	ω	$a^{(1)}(\omega)$	$a^{(2)}(\omega)$
0.095	0.404275	0.419474 × 10^{-1}	0.1	0.458360	0.163372 × 10^2
0.183	0.123662	0.925079 × 10^{-1}	0.2	0.880913	0.658253 × 10^1
0.265	0.266352 × 10^{-2}	0.151344	0.3	0.127581 × 10^1	0.332619 × 10^1
0.342	−0.846962 × 10^{-1}	0.218198	0.4	0.164856 × 10^1	0.165159 × 10^1
0.414	−0.162767	0.292868	0.5	0.200308 × 10^1	0.594004
0.483	−0.239360	0.375190	0.6	0.234223 × 10^1	−0.161614
0.549	−0.317672	0.465022	0.7	0.266825 × 10^1	−0.747554
0.612	−0.399226	0.562255	0.8	0.298286 × 10^1	−0.122869 × 10^1
0.673	−0.484813	0.666783	0.9	0.328745 × 10^1	−0.164041 × 10^1
0.732	−0.574868	0.778508	1.0	0.358312 × 10^1	−0.200361 × 10^1
0.789	−0.669638	0.897344	1.1	0.387082 × 10^1	−0.233138 × 10^1
0.844	−0.769304	0.102325 × 10^1	1.2	0.415127 × 10^1	−0.260331 × 10^1
0.897	−0.873956	0.115615 × 10^1	1.3	0.442514 × 10^1	−0.291257 × 10^1
0.949	−0.983663	0.129603 × 10^1	1.4	0.469302 × 10^1	−0.317611 × 10^1
1.000	−0.109845 × 10^1	0.144283 × 10^1	1.5	0.495540 × 10^1	−0.342610 × 10^1
1.049	−0.121833 × 10^1	0.159648 × 10^1	1.6	0.521270 × 10^1	−0.366481 × 10^1
1.098	−0.134332 × 10^1	0.175696 × 10^1	1.7	0.546528 × 10^1	−0.389401 × 10^1
1.145	−0.147335 × 10^1	0.192416 × 10^1	1.8	0.571349 × 10^1	−0.411502 × 10^1
1.191	−0.160840 × 10^1	0.209802 × 10^1	1.9	0.595759 × 10^1	−0.432894 × 10^1
1.236	−0.174847 × 10^1	0.227850 × 10^1	2.0	0.619786 × 10^1	−0.453665 × 10^1

scheme on an IBM 7090 computer. In order to start the solution, the linearly independent solutions in Eq. (73) were evaluated at an initial point $\omega = 0.1$. Then, values of $a^{(1)}$ and $a^{(2)}$ were generated for $\omega \leq 2$. Finally, the constants C_1 and C_2 were evaluated, using Eqs. (68), for $0.2 \leq K \leq 4$, which corresponds to $0.095 \leq \tau \leq 1.236$.

The results of the numerical calculation are presented in Table II, where C_1 and C_2 are given for various values of τ and $a^{(1)}(\omega)$ and $a^{(2)}(\omega)$ are given for various values of ω. These results have been used in Fig. 1, where the velocity component $u(r)$, for $K = 4$, is compared with the Stokes[1] profile, which corresponds to $K = 0$. For intermediate values of K, the result will fall between the two curves. Chang's[10] velocity profile would be very close to the Stokes curve because his calculation was carried out for a small value of τ.

The equation for the first-order outer temperature, obtained from the outer energy equation, is

$$\Pr \mathbf{V}_0 \cdot \nabla T_1 = \nabla^2 T_1. \tag{74}$$

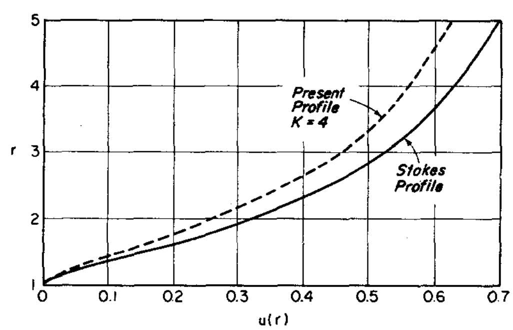

FIG. 1. Zeroth-order radial velocity profile, $u(r)$, for $\tau = O(1)$.

The boundary condition for $s \to \infty$ and the matching condition as $s \to 0$ are

$$T_1(\infty, \theta) = 0, \quad t_0 + \cdots \sim 1 + e_1 T_1 + \cdots . \tag{75}$$

The solution of the above equations is found in a manner similar to that employed in the analogous $\tau = O(\mathrm{Re})$ case and is given by

$$T_1 = (K/2 \Pr s) e^{\frac{1}{2} s \Pr (\cos\theta - 1)}, \qquad e_1 = \mathrm{Re} \Pr. \tag{76}$$

Thus, the order of magnitude of the first-order outer temperature perturbation is larger than that for the corresponding $\tau = O(\mathrm{Re})$ case.

Substitution of the expansions in the inner energy equation indicates that $\epsilon_1 = \mathrm{Re}$. Thus, the equation for t_1 is

$$\nabla^2 t_0 t_1 = \Pr \rho_0 \mathbf{v}_0 \cdot \nabla t_0 = \tfrac{1}{2} \Pr K G(r) \cos\theta, \tag{77}$$

where

$$G(r) = -u(r)/r(r + K),$$

and t_1 is subject to the following boundary condition and matching condition (as $r \to \infty$):

$$t_1(1, \theta) = 0, \qquad t_0 + \mathrm{Re}\, t_1 + \cdots$$
$$\sim 1 + \mathrm{Re} \Pr T_1 + \cdots . \tag{78}$$

Now, the function $G(r)$ cannot be written in closed form since there is no analytical expression for $u(r)$. However, the solution for t_1 can be written in terms of integrals of $G(r)$. This formulation is useful for matching since an asymptotic form of $u(r)$ for large r can be found from Eqs. (63)–(65), (68), and (73). Thus, in terms of these integrals, the solution is,

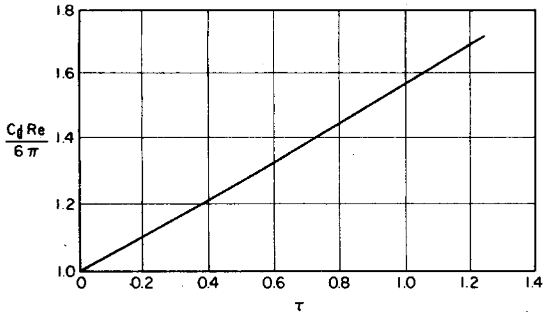

FIG. 2. Zeroth-order drag coefficient for $\tau = O(1)$.

$$t_0 t_1 = \frac{\Pr K}{4}\left(\frac{1}{r} - 1\right) - \frac{\Pr K}{6}\left[r \int_r^\infty G(\sigma)\, d\sigma + \frac{1}{r^2}\left(\int_1^r G(\sigma)\sigma^3\, d\sigma - \int_1^\infty G(\sigma)\, d\sigma\right)\right] \cos\theta. \quad (79)$$

While the integral terms could be calculated numerically, using the results of the previous numerical calculation for $a(\omega)$, such an evaluation is unnecessary for this analysis, since the terms multiplied by $\cos\theta$ in Eq. (79) do not contribute to the average Nusselt number.

The zeroth-order drag force is calculated by means of a momentum balance far from the body, as was done by Chang. Since the zeroth-order flow field is symmetric with respect to the equatorial plane, convective terms do not contribute to the momentum balance. Hence, the drag coefficient can be defined as

$$C_d = \lim_{r\to\infty} \frac{4}{\mathrm{Re}} \int_0^\pi [-(\tau_{rr} + \tau_{\theta r} + p)\cos\theta + (\tau_{\theta r} + \tau_{\theta\theta})\sin\theta] r^2 \sin\theta\, d\theta, \quad (80)$$

where τ_{rr}, $\tau_{\theta r}$, and $\tau_{\theta\theta}$ are the shear stresses (e.g. see Ref. 11). The shear stress components and the pressure can be evaluated for large r by using an asymptotic form of $a(\omega)$ [Eq. (73)]. The pressure is calculated from the θ component of the momentum equation [Eq. (61)]. In calculating the asymptotic forms, it is assumed that $\omega \to 0$ (i.e., $r \to \infty$), for arbitrary K. The resulting shear stress components and pressure are as follows:

$$(\tau_{rr})_0 = -\mu_0\left[\left(\frac{16C_2}{K} - \frac{2K}{3}\right)\frac{1}{r^2} + O\left(\frac{1}{r^3}\right)\right]\cos\theta,$$

$$(\tau_{\theta r})_0 = -\mu_0\left[\frac{K}{2r^2} + O\left(\frac{1}{r^3}\right)\right]\sin\theta, \quad (81)$$

$$(\tau_{\theta\theta})_0 = \mu_0\left[\frac{8C_2}{Kr^2} + O\left(\frac{1}{r^3}\right)\right]\cos\theta,$$

$$p_0 = \left[-\frac{1}{r^2}\left(\frac{K}{6} + \frac{8C_2}{K}\right) + O\left(\frac{1}{r^3}\right)\right]\cos\theta.$$

Finally, then, the drag coefficient may be calculated:

$$C_d = \frac{12}{\mathrm{Re}}\left[\frac{16C_2}{3K} - \frac{K}{3}\right], \quad (82)$$

where, again, $K = \tau(\tau + 2)$ and the values of $C_2(K)$ are found in Table II. In Fig. (2), Eq. (82) is used to plot C_d vs τ.

The Nusselt number is calculated by substituting Eqs. (59) and (79) in Eq. (54). The result is

$$\mathrm{Nu} = [(\tau + 2)/2(\tau + 1)] \cdot [2 + \mathrm{Re}\Pr + O(\mathrm{Re}^2)]. \quad (83)$$

VII. DISCUSSION OF RESULTS

It is seen from Eq. (53) that when $\tau = O(\mathrm{Re})$ the drag coefficient differs from its incompressible counterpart by a term of order τ. This increase in drag coefficient (for $\tau > 0$) may be traced directly to the inclusion of variable density and viscosity in the analysis. The Mach number effect does not contribute to the drag, to the order calculated, because it produces a symmetric change in the stress and pressure field.

The Nusselt number for the case $\tau = O(\mathrm{Re})$ [Eq. (55)] is smaller than the previously calculated values, by a term of order τ. This reduction arises solely from the inclusion of variable thermal conductivity. A term involving the Mach number also appears, due to the effects of viscous dissipation.

The drag coefficient and Nusselt number for the case $\tau = O(1)$ are seen to be greatly affected by this large impressed temperature difference. The zeroth-order density and viscosity both vary in this case, causing pressure and viscous stress distributions in the flow field which are altered considerably from the corresponding incompressible values. As a result, the drag coefficient increases almost linearly with τ; its magnitude when τ is near unity is substantially greater than the classical Stokes value. The zeroth-order Nusselt number also varies significantly with τ. As τ increases, Nu decreases, tending toward one-half the incompressible value when τ becomes large.

ACKNOWLEDGMENT

This work was supported in part by the Ballistic Missile Radiation Analysis Center, a contract funded through Project Defender and sponsored by the Advanced Research Projects Agency of the Department of Defense, under Contract SD-91.