the FEM patterns of exact twofold symmetry but also the patterns of nearly fourfold and threefold symmetry, respectively, like the W emitter tips.

Though the present FEM observations were not carried out in ultrahigh vacuum but in conventional high vacuum, the experimental facts discussed here may offer some information about the oxidation on tungsten surfaces.

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# Oscillations of a Gas Bubble in Viscoelastic Liquids Subject to Acoustic and Impulsive Pressure Variations*

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An analysis is presented of the forced oscillations of a gas-filled bubble at rest in a large body of a linear viscoelastic fluid. Two types of forcing are considered. In the first, oscillations are induced by a pressure surge on the system. For the three-parameter fluid model employed, numerical computations show that for a given ratio of the fluid's elastic modulus to the pressure surge, the damping of the bubble motion exhibits a maximum as a function of the fluid relaxation time at a value of this parameter equal approximately to one-fifth the natural period of oscillation. At very high or very low relaxation times, the damping becomes insignificant. As the second type of forced oscillation, we consider the motion induced by the application of ultrasonic waves to the system. Here, damping is found to depend strongly on the product of impressed frequency and fluid relaxation time.

## I. INTRODUCTION

Previous studies on bubble oscillation and collapse in liquids have primarily dealt with bubble motion in Newtonian fluids.[1-3] Although there have been studies carried out on bubble oscillation in "non-Newtonian" liquids,[4,5] these analyses have been limited to fluids belonging to the Stokesian, i.e., viscous, group in the rheological classification of materials.[6] It is of both practical and theoretical interest to consider the possible reduction or suppression of acoustical or flow-induced cavitation by the presence of viscoelasticity in the ambient liquid.

The motion of bubbles in fluids for which the stress–strain functionality involves memory effects or dependence on the history of the fluid motion has only been treated in a few limiting cases. Fogler and Goddard[7] have presented an analysis of the collapse of spherical voids in viscoelastic fluids under the action of a "step-function" pressure surge. For the case of a particular linear viscoelastic fluid model it was shown, among other things, that the presence of shear elasticity could significantly retard the collapse of voids in liquids having relaxation times comparable to the classical Rayleigh collapse time. Also, some speculations were made concerning the effects that liquid-phase elasticity might have on the motion of gas-filled bubbles and, in a later work, Tanasawa and Yang,[8] have addressed themselves to this problem. However, their analysis[8] relating to bubble collapse, induced by a sudden pressure surge, fails to reveal some important aspects of the collapse phenomenon, which appears to result from the inappropriate mechanical analogue they propose.

Apart from a somewhat more careful analysis of this type of oscillation, the present work will treat

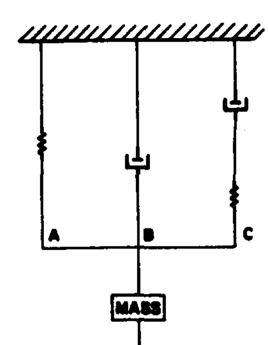


FIG. 1. Mechanical analogue of the (A) gas pressure, (B) viscous, and (C) viscoelastic effects in a gas-filled, oscillating bubble. The mass represents the effect of fluid inertia.

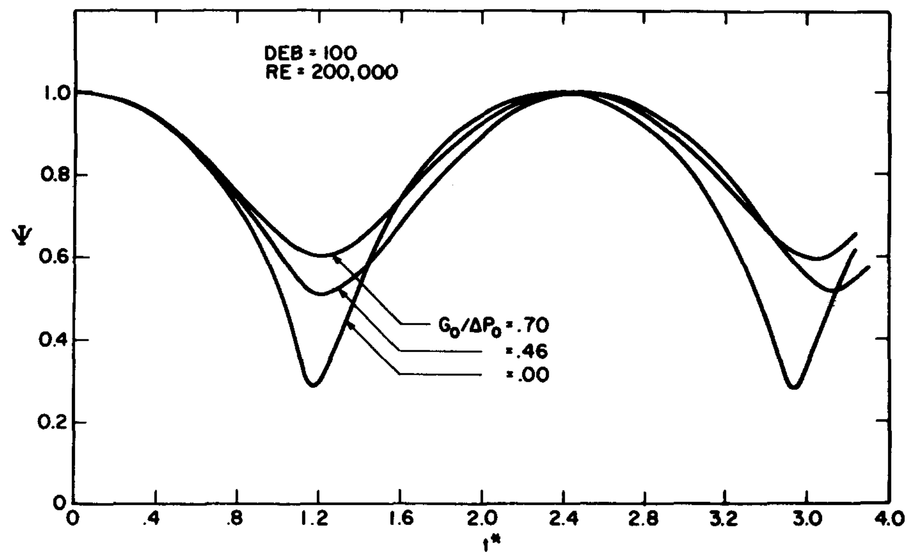


FIG. 2. Dimensionless radius–time curves for various values of the ratio of elastic modulus to the amplitude of the pressure surge.

another type as well. In this latter type, we shall again consider a gas bubble initially at rest in a large body of viscoelastic liquid; however, the bubble motion will now be induced by the application of acoustic pressure variations to the system.

## II. PROBLEM FORMULATION

Consider a spherical gas-filled bubble in a large body of an incompressible liquid which initially, for all time $t<0$, is at rest with a radius $R_0$, a uniform pressure on the liquid system, $P_\infty = P_{0e}$, and with a gas pressure inside the bubble $P_{g0}$; the latter is determined from the relation

$$P_{g0} = P_{0e} + 2\sigma/R_0, \tag{1}$$

where $\sigma$ represents the surface tension. We wish to analyze motions which have been induced by two different methods. In the first, the motion is induced by a sudden surge of pressure on the system, i.e.,

$$\text{at } t<0, \quad P_\infty = P_{0e}$$
$$t>0, \quad P_\infty = P_0. \tag{2}$$

Whereas in the second, it results from the application of acoustic waves to the system. In this case, the pressure at a large distance from the bubble, for any time $t>0$, will be given by

$$P_\infty = P_{0e} - P_a \sin\omega t, \tag{3}$$

where $P_a$ is the "acoustic pressure" and $\omega$ is the angular frequency of oscillation.

In either case, the general equation describing the spherically symmetric motion of a bubble containing a uniform gas phase, in which there is no condensation or evaporation of fluid, has been shown[7] to reduce to

$$R\cdot\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P_g - P_\infty}{\rho} - \frac{2\sigma}{\rho R} - \frac{3}{\rho}\int_R^\infty \frac{\tau_{rr}dr}{r}, \tag{4}$$

where $\tau_{rr}$ is the radial component of deviatoric compressive stress in the liquid phase. As a rheological constitutive equation, relating stress $\tau_{rr}(t)$, at a fluid particle to the past history of the deformation rate $e_{rr}(t')$, $0\leq t'\leq t$, we adopt the linear viscoelastic model used in our previous work:

$$\tau_{rr}(t) = -2\int_0^t N(t-t')e_{rr}(t')dt', \tag{5}$$

with

$$N(t) = \mu\delta(t) + G_0\exp(-t/\lambda), \tag{6}$$

where, as constant parameters, $\mu$ is a viscosity, $G_0$ an elastic modulus, and $\lambda$ a relaxation time for the fluid, and where $\delta(t)$ denotes the delta function.

We note, incidentally, that this fluid model is *identical*[8] with the "linear Oldroyd model"

$$\tau_{rr} + \lambda_1(D\tau_{rr}/Dt) = -2\eta_0[e_{rr} + \lambda_2(De_{rr}/Dt)] \tag{7}$$

of the form employed by Tanawasa and Yang (which follows directly by an elementary integration of (7) together with the transformation of the three parameters:

$$\lambda_1 = \lambda, \quad \eta_0 = \mu + \lambda G_0, \quad \lambda_2 = \lambda\mu/\eta_0.$$

From a consideration of the liquid-phase velocity field for spherically symmetric motion, it can be shown[7] that Eqs. (4)–(6) combine to give the complete

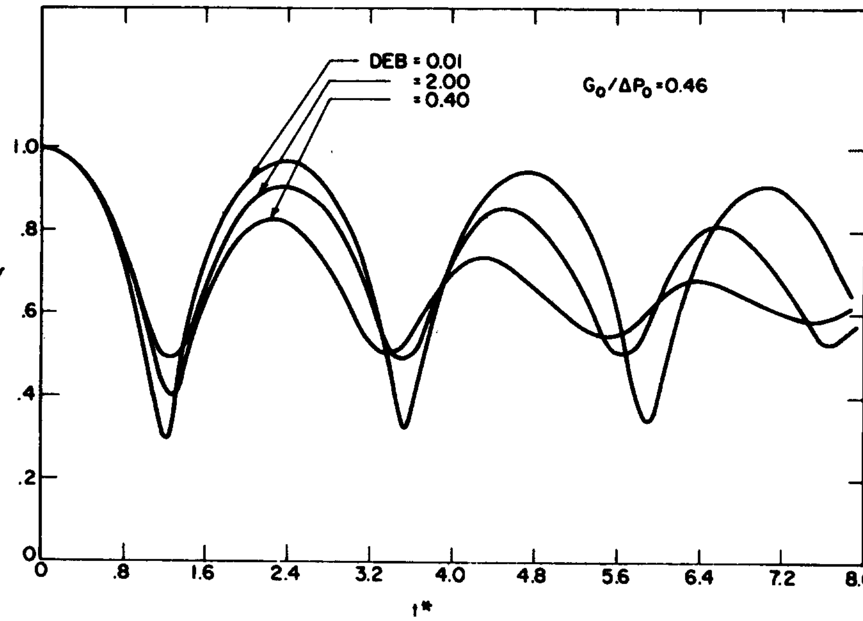


FIG. 3. Effect of the Deborah number (fluid relaxation time/Rayleigh collapse time) on bubble motion.

dynamical equation for the bubble motion:

$$\psi\cdot\ddot{\psi}+\tfrac{3}{2}\dot{\psi}^2=(P_g/P_0)-(4\dot{\psi}/\mathrm{Re}\,\psi)-(12\,\mathrm{El}/\mathrm{Re})$$
$$\times\int_0^{t^*}\left[\exp\left(-\frac{(t^*-t_1)}{\mathrm{De}}\right)\right]\frac{\dot{\psi}_1\psi_1^2\ln(\psi_1/\psi)}{\psi_1^3-\psi^3}\,dt_1$$
$$-2/\mathrm{We}\,\psi-P_\infty/P_0 \quad (8)$$

with

$$P_g=P_{g0}\cdot\psi^{-3},\qquad \psi(0)=1 \quad\text{and}\quad \dot{\psi}(0)=0,$$

where

$$\mathrm{De}=\lambda/t_c \text{ (a Deborah number)},$$
$$\mathrm{Re}=\rho R_0^2/\mu t_c \text{ (a Reynolds number)},$$
$$\mathrm{El}=G_0t_c/\mu \text{ (an elastic number)},$$
$$\mathrm{We}=\rho R_0^3/t_c\sigma \text{ (a Weber number)}, \quad (10)$$

and $\psi=R/R_0$, $t^*=t/t_c$, and $\psi_1=\psi(t_1)$. Also, $t_c=R_0(\rho/P_0)^{1/2}$ is a characteristic ("Rayleigh") collapse time, with $P_0$ being the initial pressure.

To provide an intuitive appreciation of the system, a mechanical analogue representing the effect of the first three terms on the right-hand side of Eq. (8) is shown in Fig. 1. These terms, representing the effects of gas compressibility, liquid viscosity, and liquid viscoelasticity correspond to the elements, A, B, and C, respectively, in a spring-dashpot assemblage. One can also note that the conceptual model in Fig. 1 differs from that proposed in Ref. 8, in that the element representing the effect of gas pressure has been placed in parallel here with the other elements rather than in series. The latter arrangement would suggest that the bubble could collapse to zero radius as the fluid viscosity and relaxation time both approach zero, which of course is not possible as long as there is a noncondensable gas inside the bubble.

As discussed in our previous work, the rather large number of parameters, even in this relatively simple fluid model, requires one practically to consider some special limiting cases. In all the calculations made in the present work, we have adopted fixed values, $\mu=1$ cp, $\rho=1$ g/cm$^3$, and $\sigma=72$ dyn/cm (corresponding to the equivalent values for water). These values correspond generally to large values of the Reynolds and Weber numbers, defined above, and imply that the "purely viscous" and surface tension effects, as represented by the second and fourth terms on the right-hand side of (8), are small. This is not a severe restriction since it turns out that the parameters $\mu$ and $\sigma$ could easily be changed by a factor of ten to a hundred without rendering these effects important.

## III. OSCILLATION INDUCED BY A SUDDEN PRESSURE SURGE

With the initial conditions expressed mathematically in Eq. (9) and the pressure surge given by Eq. (2), Eq. (8) was solved numerically for $\psi(t^*)$ by a slightly improved version of our previous integration technique.[7] Figure 2 portrays the resulting oscillations, for various values of the ratio of the elastic modulus to incremental pressure surge, $\Delta P=(P_0-P_{0e})$. The radius–time curves in this figure correspond to fluids in which the relaxation time is much greater than either the "natural" period of oscillation or the Rayleigh collapse time for an ideal fluid. For this condition, i.e., large "Deborah

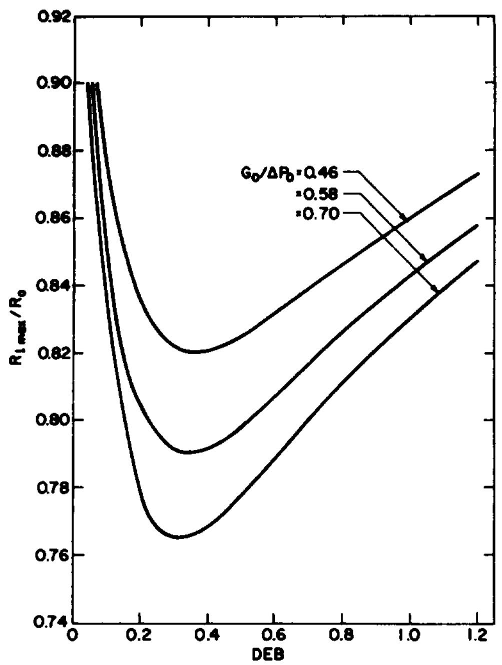


FIG. 4. Maximum value of the first rebound radius as a function of the Deborah number for various values of $G_0/\Delta P$.

numbers", the amplitude of oscillation decreases with increasing elastic modulus $G_0$. In addition, it is noted that as time proceeds, a phase shift develops between the viscoelastic oscillation, $G_0>0$, and the "purely viscous" oscillation, for which $G_0=0$.

The bubble motion is shown in Fig. 3 for various Deborah numbers and for the fixed value 0.46 of $G_0/\Delta P$. One observes in this figure that the amplitude of oscillations for a Deborah number of 0.4 is less than that for a Deborah number of either 0.01 or 2.0. In other words, for a given $G_0/\Delta P$, there appears to be a minimum amplitude of oscillation as the Deborah number is increased from zero.

One possible parameter for characterizing the degree of damping for this type of oscillation is the maximum radius reached after initial rebound of the bubble, $R_{1\,max}$. There are of course other, more standard, methods of specifying the damping in oscillating systems; however, they usually involve several cycles of bubble collapse and rebound. For the system discussed here, the calculation of numerous cycles would require excessive computational time with perhaps little, if any, additional information gained. Hence, we shall use the first maximum rebound radius as a measure of damping, and this quantity is shown as a function of the Deborah number in Fig. 4. On this basis, one concludes, by a rough extrapolation, that damping of the bubble motion at high Reynolds numbers would be small at Deborah numbers greater than 3 or less than 0.01.

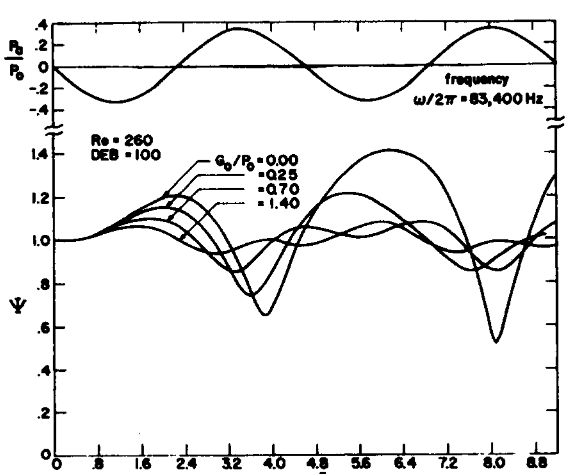


FIG. 5. Effect of elastic modulus on bubble motion induced by acoustic pressure waves.

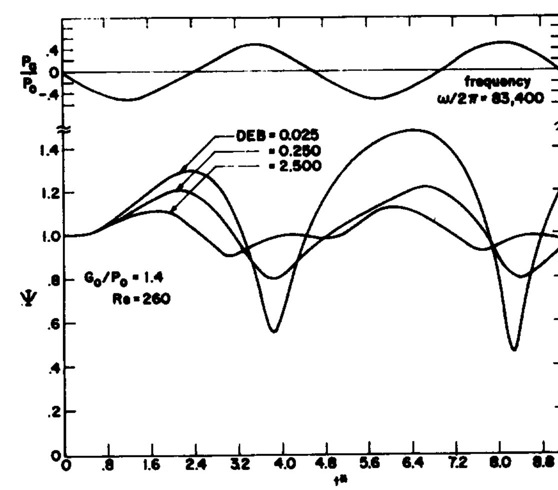


FIG. 6. Effect of Deborah number on bubble motion induced by acoustic pressure waves.

In terms of the conceptual analogue above, one sees that a high Deborah number corresponds effectively to an immobile dashpot in element C, while a very low Deborah number corresponds to a "frictionless" dashpot in C. In the former case, we are left with two springs in parallel with a "weak" dashpot B (for high Re), while the latter case results in one spring in parallel with the same dashpot, B. In this manner, it is easy to understand the occurrence of a minimum in damping as we vary fluid relaxation time. As a final remark on these calculations, we note that for all the curves in Fig. 4, the minimum value of $R_{1\,max}$ occurs at a value of the Deborah number corresponding roughly to a relaxation time equal to one-fifth the natural period of oscillation. One also observes from this plot that the minimum value of $R_{1\,max}$ decreases with increasing elastic modulus.

While the elementary fluid model used here would probably not provide an exact description of real viscoelastic liquids, it may not be too implausible to expect that they might exhibit a qualitative relation between damping and relaxation effects of the type presented here.

## IV. OSCILLATIONS INDUCED BY ACOUSTIC WAVES

The application of ultrasonic waves to liquids has been observed experimentally to produce a number of unusual and interesting phenomena: (1) sonoluminescence, (2) erosion, (3) rectified diffusion, and (4) increased chemical reaction rates.[1] These phenomena are usually attributed to acoustically induced cavitation. Hence, it is of interest to consider the effect which fluid viscoelasticity might have on these phenomena, through its specific effect on bubble motion. In the present model for oscillation, induced by this method, the ambient pressure is given by Eq. (3). Again, a numerical solution to the integro-differential equation of (8) is required. As in the previous case, an extended calculation of bubble motion would require prodigious amounts of computation time. Consequently, we have limited our calculation to the first few cycles of oscillation.

A number of radius–time curves for bubble motion in Newtonian fluids have already been presented by Flynn.[1] For purposes of comparison, one of Flynn's curves ($G_0=0$) was recalculated by our numerical procedure and is presented in Fig. 5, along with the radius–time curves of the present work. For large Deborah numbers one observes from this figure that the damping of bubble motion increases systematically with increasing elastic modulus. In Fig. 6, the radius–time curves show that the damping of the motion also increases with increasing Deborah number. One can note that elasticity in a fluid may or may not have significant effects on the motion. Indeed, it appears that, for the relatively large amplitude acoustic waves considered here ($P_a/P_0>1/3$), the bubble oscillation can be significantly damped by "elastic" response at the higher frequencies, where $\lambda\omega$ is order unity or greater. However, the same fluid subjected to ultrasonic waves of much lower frequency ($\lambda\omega\ll1$) may show insignificant elastic damping.

## V. SUMMARY

We have considered briefly here the oscillations of gas-filled bubbles in idealized viscoelastic liquids induced by two different methods: (1) single pressure surge, and (2) acoustic pressure waves. In the first case, it is observed that for a given ratio of the elastic modulus to the amplitude of the pressure surge, the damping effect on the bubble motion exhibits a maximum, as a function of the Deborah number. In the second case, it is observed that the extent of damping depends strongly on the product of applied frequency and relaxation time for the fluid.

In conclusion, we have attempted to show in our calculations the importance of the parametric regime to the occurrence of significant elastic effects. It can be reasonably expected that similar care would have to be exercised in any further calculations, based on other viscoelastic fluid models, or in any experimental explorations of such effects in real fluids.

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In view of this, it is difficult to appreciate the statement in Ref. 8 that our previous work[7] employed a "two-parameter" model as well as the implication that Eq. (7) is more general than Eq. (5).