Cylindrical Couette Flow Experiments in the Transition Regime

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(Received 2 February 1970)

Density distributions were measured in rarefied argon contained between two concentric cylinders, the inner one rotating, the outer one stationary. The experiments were performed with a Mach number near unity, based on the surface speed and surface temperature of the rotating cylinder. Particular attention was focused on obtaining data in the slip and transition regimes where the Knudsen number, defined as the ratio of the mean free path to the gap size between the cylinders, varied from 0.04 to 1.07. The density distributions were measured by observing the gas luminescence induced by the passage of a narrow beam of high energy electrons through the gas. In addition to the density measurements, heat transfer and drag measurements were also made in order to estimate the values of the thermal accommodation and the tangential momentum accommodation coefficients. The experimental results were compared to solutions of the Navier–Stokes and Burnett equations as given by Schamberg and Lin and Street. The results of the Navier–Stokes and the Burnett equations were found to approximate the density distributions well at Knudsen numbers below ~0.05 and ~0.2, respectively. At higher Knudsen numbers the analytical and experimental results differ considerably.

I. INTRODUCTION

Owing to the difficulties involved in obtaining solutions to the Boltzmann equation every method of solution proposed thus far is based on simplifying approximations, the nature of which is generally demonstrated by applying the theoretical method to some simple problem. Two problems that have been investigated frequently are the problems of plane and cylindrical Couette flow of rarefied gases. In plane Couette flow shear flow is generated by two infinite parallel plates translating relative to each other, in cylindrical Couette flow shear flow is generated by two concentric cylinders rotating relative to each other. Thus, the Couette flow is one of the simplest situations which involves both mean flow of the gas and interaction between the gas and a solid boundary.

Although a large number of theoretical analyses are available on the plane and cylindrical Couette flow of rarefied gases,[1–17] there have been relatively few experiments dealing with these phenomena.[18–25] All of the previous experiments were performed using cylindrical geometry and in all but one investigation[25] only the drag between the concentric cylinders was measured. These drag measurements are valuable in that the data and the theoretical results agreed within the accuracies of the experiments.[1,14,18,26] Unfortunately, due to the scatter in the data and due to the fact that most theories yield drag values that differ only slightly, the drag measurements are not well suited for the critical evaluation of the accuracies of the various approximate methods of solution.[26] It has been pointed out by many previous investigators that the measurements of some local mean quantity such as velocity or density would be needed for further evaluation of existing, and also of forthcoming, theoretical analyses. Such measurements would be useful because the analyses differ more in their predictions of local velocity and density values than in their predictions of drag.[11]

The only measurement of a local quantity was reported by Malegue[25] who used a thermocouple probe to measure the temperature distribution in a gas contained between two concentric cylinders held at different temperatures. The cylinders were either stationary or underwent relative rotation. In the latter case, the relative surface speed was quite low (Mach number ~0.03) so that viscous heating of the gas was negligible. The experiments were limited to a narrow pressure range with the Knudsen number varying only between 0.2 and 0.4.

It is evident that little experimental data are available on the distribution of a mean quantity through a rarefied gas contained between two concentric cylinders rotating relative to each other. This investigation was undertaken, therefore, to measure local gas densities in cylindrical Couette flow using a monatomic gas (argon) as the test gas. Attention was focused on the transition and slip regimes, where the mean free path in the gas, λ, is of the same order of, or smaller than the characteristic geometric length L which, for this problem, is the size of the gap between the two cylinders. In the experiments the inner cylinder was rotating and the outer one was stationary (Fig. 1). The local gas density was determined by measuring the luminescence produced by the passage of a narrow beam

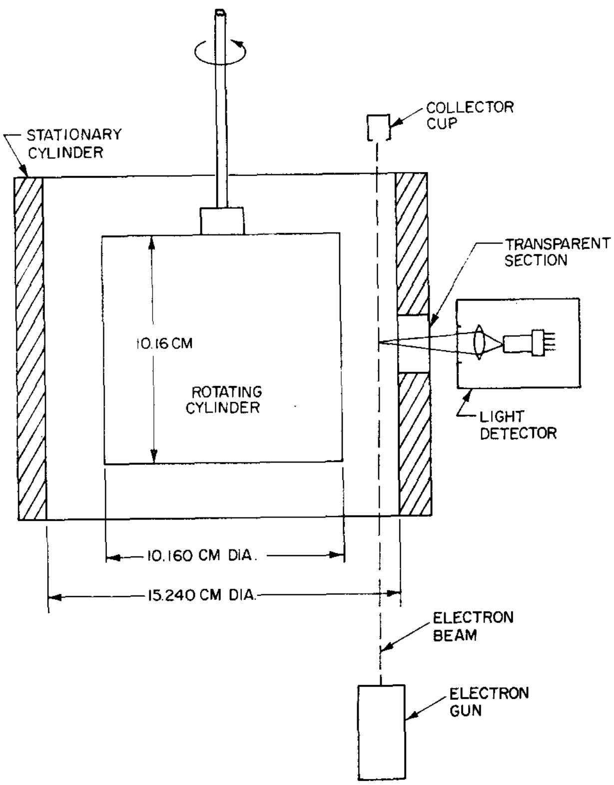

FIG. 1. Schematic of the experiment.

of high energy electrons through the test gas. This technique has recently been used by several investigators to measure density distributions in rarefied gas flows. In the present investigation the beam was directed through the gap between the cylinders and parallel to the axis of the cylinders. Gas densities at different radial positions in the gap were measured by moving the concentric cylinders relative to the beam.

The experiments were performed with Mach number near unity, based on the surface speed and surface temperature of the rotor. Such a high speed was chosen in order to obtain as large a variation of gas density across the gap as possible. The gap size L between the two cylinders was large (2.54 cm) compared with the diameter of the electron beam ($\sim$0.1 cm) so that density measurements could be made at several locations across the gap. Since the diameter of the inner cylinder D was about 10 cm, L was not negligibly small compared to D. For this reason, and because of the high Mach number, the data cannot be compared either to theoretical treatments of plane Couette flow problems[1–13] or to those cylindrical Couette flow solutions that are restricted to Mach numbers much less than unity.[13–15]

Analytical results that are not restricted to small gap size or to low Mach numbers were presented by Schamberg[16] and by Lin and Street,[17] who obtained solutions to the cylindrical Couette flow problem using the Navier–Stokes and Burnett levels of approximation in the Chapman–Enskog method of solution to the Boltzmann equation. Schamberg assumed constant viscosity and thermal conductivity, while Lin and Street's analysis included the variation of these properties with temperature. Here, the results of these analyses are compared to the data because, as pointed out above, these are the only ones presently available that are directly applicable to the problem under consideration. However, the experimental data is reported in sufficient detail to be useful in comparisons with other theoretical results which, hopefully, will be forthcoming shortly.

In comparing the data to theoretical results, one must specify the interaction between the gas and the solid boundaries. The gas–surface interaction may be conveniently described by the average thermal and momentum accommodation coefficients. In order to evaluate these coefficients heat transfer and drag measurements were performed in addition to the density measurements.

II. EXPERIMENTAL APPARATUS

The experimental arrangement used in the density distribution measurements is shown in Fig. 2. The apparatus consisted of (1) two concentric cylinders together with a high speed drive unit for producing the cylindrical Couette flow, (2) an electron gun

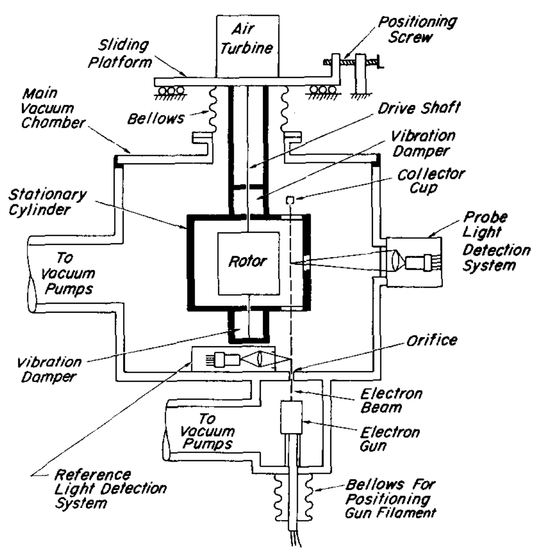

FIG. 2. Schematic of the apparatus.

system for generating the electron beam, (3) two optical systems for measuring the luminescence from the desired sections of the beam, and (4) a vacuum system. In addition, there were two separate apparatus; one for measuring heat transfer and one for measuring drag.

The concentric cylinders were made of 7075-T6 aluminum. The 10.160 cm diam and 10.16 cm long inner cylinder (rotor) rotated at 60 800 rpm (surface speed: 32 327 cm/sec). The stationary outer cylinder (15.240 cm diam) was provided with a 0.63 cm wide and 2.54 cm long Plexiglas window for viewing the electron beam. The rotor was completely enclosed by plates placed on both ends of the outer cylinder. There was a 1 in. diam hole in each plate to allow the electron beam to pass through them (Fig. 2). In order to move the concentric cylinders relative to the electron beam the cylinders were mounted on a movable platform. The position of this platform could be adjusted to within 0.0025 cm by a screw mechanism. The rotor was driven by a variable speed air turbine. The speed was measured by a magnetic pickup and an electronic counter. The rotor was connected to the turbine with a 0.47 cm diam and 61 cm long shaft. In order to prevent whirling of this shaft, vibration dampers immersed in circulating oil were mounted just above and below the rotor. Oil seals prevented the escape of oil from the dampers into the vacuum chamber.

The electron beam system consisted of an electron gun capable of producing a narrow, high energy electron beam and a collector cup which was used to measure the electron beam current. The electron gun was placed in a vacuum chamber in which the pressure could be maintained below 1×10^{-4} mm Hg throughout the experiments. There was a small (0.2 cm diam) hole between the test chamber and the vacuum chamber housing the gun. The electron gun was the type commonly used in television tubes, but the commercial cathode was replaced with a tungsten filament. The electron beam was collected in a shielded cup and the electron beam current (in the range of 1–2 μA) was measured with a microammeter. Two sets of electrostatic deflection plates were also provided to adjust the beam position, so that the beam passed through the hole between the gun chamber and test chamber, and that it also entered the 0.1 cm hole in the collector cup. Since these two holes were aligned parallel to the axis of the cylinders, the electron beam was also parallel to this axis.

Two light detection systems were used in making the density measurements, each system measuring light intensity from a different length section of the beam. For the main detector, this length section was 1 cm long, located midway between the top and bottom of the rotor (Fig. 2). For the reference detector this length section was 3 cm long, and was located in the test chamber about 12 cm below the rotor (Fig. 2). The purpose of the reference detector was to record any small fluctuations ($\pm 2\%$) in the chamber pressure. This information was then used to correct the final data for these pressure fluctuations. Each detector had a lens system which focused an image of the luminescent beam on a slit which served as an optical stop. Directly behind the slit was a model 6655A RCA photomultiplier tube, whose output was measured with an electrometer and recorded on a strip chart recorder.

The vacuum system (Fig. 2) was constructed of type 304 stainless steel. The test chamber (45 cm diam, 45 cm high) and the gun chamber (12.7 cm diam, 25 cm high) were equipped with separate diffusion and mechanical pumps. On the side of the test chamber was a 6.35 cm diam window for the optical system. The lowest pressure attainable in the test section was about 1×10^{-5} mm Hg with the rotor up to speed. The test gas (commercially pure argon) was admitted into the test chamber through adjustable leaks. The test chamber was equipped with an ionization gauge, a thermocouple gauge, and two precision McLeod gauges. The electronic gauges could be read continuously, and, therefore, these were used for checking that the chamber pressure remained at a constant value during the experiments. However, the actual value of the chamber pressure was always measured with the McLeod gauges. The pressure in the gun chamber was monitored by an additional ionization gauge. No part of the experimental apparatus was baked during the experiments.

Heat transfer and drag measurements were also made in order to evaluate the thermal and tangential momentum accommodation coefficients. Ideally, these measurements should be made with the same apparatus as used for the density distribution measurements. From a design standpoint, this would be extremely difficult. Therefore, the heat transfer and drag measurements were performed with two separate apparatus, each placed in the main vacuum chamber and each constructed from the same material and with the same surface finish as the concentric cylinders used in measuring density distribution.

The heat transfer apparatus was a plane layer type thermal conductivity cell similar in design to the ones described in detail by Teagan and

Springer.[27,28] The drag measuring apparatus was described in Ref. 29 and consisted of two concentric cylinders, the inner one rotating and the outer one suspended on a long, thin tungsten wire. The drag was determined from the measured angle of rotation of the outer cylinder.

Further details of the experimental apparatus may be found in Ref. 30.

III. EXPERIMENTAL PROCEDURE

During the density distribution measurements the test chamber pressure P and the rotor surface speed V were held constant, while the radial position r of the beam relative to the gap between the cylinders was changed. The surface temperatures of the inner and outer cylinders (T_i and T_o) remained constant during the experiments, thus, the Mach number M defined in terms of V and T_i was also constant. The closest distance between the beam and the walls of either cylinder was 0.508 cm. When the beam was closer to the walls experimental accuracy decreased because wall fluorescence became too strong. The light intensity was measured as a function of r at a given chamber pressure and Mach number by moving the beam first toward the outer cylinder and then, as a check, in the opposite direction. Since the Mach number was constant ($M \cong 1$) during all these experiments, these light intensities are denoted by $I(P, M = 1, r)$. Light intensities were also measured with the rotor stopped ($M = 0$); the resulting light intensities are denoted by $I(P, M = 0, r)$.

The density distribution between the cylinders was evaluated as follows. In the pressure range of our experiments ($P < 0.05$ mm Hg) the light intensity is linearly proportional to density,[28] i.e.,

$$I = C_1 + C_2\rho. \qquad (1)$$

In the present investigation C_1 and C_2 are constants that depend only on the beam position r. The reason for this dependence is that the optical system allowed some stray light to reach the photomultiplier tube. From Eq. (1) the following dimensionless density ratio is formed:

$$\bar{\rho} = \frac{\rho(P, M = 1, r)}{\rho(P, M = 0, r)} = \frac{I(P, M = 1, r) - C_1(r)}{I(P, M = 0, r) - C_1(r)}. \qquad (2)$$

$I(P, M = 1, r)$ and $I(P, M = 0, r)$ are measured experimentally, therefore, $\bar{\rho}$ can be calculated once $C_1(r)$ is known. In order to evaluate $C_1(r)$ the light intensity for $M = 0$ was measured not only at P but also at a pressure P_1 which was somewhat different from $P(0.6 < P/P_1 < 1.5)$. Equation (1) is now written as

$$\frac{I(P, M = 0, r) - C_1(r)}{I(P_1, M = 0, r) - C_1(r)} = \frac{\rho(P, M = 0, r)}{\rho(P_1, M = 0, r)}, \qquad (3)$$

where $\rho(P, M = 0, r)$ and $\rho(P_1, M = 0, r)$ are the local gas densities at $M = 0$ and at chamber pressures P and P_1, respectively. C_1 could be calculated from Eq. (3) if the density ratio were known. This would be the case if the surface temperatures T_i and T_o were equal, since then the density across the gap would be constant and equal to the density in the test chamber. In the apparatus of the present investigation the wall temperatures were slightly different ($T_o/T_i = 1.026$) and, therefore, we do not experimentally know the density ratio. The gas densities may be estimated by the analysis proposed by Lees and Liu[31] for calculating the heat transfer through rarefied gases contained between two stationary concentric cylinders. The density ratio was calculated by this method for $\alpha = 1$ and for the surface temperatures of our experiments. The results of the calculation show that $\rho(P, M = 0, r)/\rho(P_1, M = 0, r) = (1 \pm 0.0005)(P/P_1)$, when the pressure ratio P/P_1 varies in the range 0.6 to 1.5. Thus, with the approximation of $\rho(P, M = 0, r)/\rho(P_1, M = 0, r) \cong P/P_1$ the constant C_1 can be determined from Eq. (3).

As mentioned in the foregoing, the temperatures of the inner and outer cylinders were slightly different. As was shown by Kuhlthau[20] viscous heating would not cause such a temperature difference, but in our experiments there was the added effect of the circulation of hot oil through the vibration dampers. The outer cylinder was in better thermal contact with the dampers than the inner one, resulting in the temperature difference between the two cylinders. From measurements made with thermocouples while the rotor was stationary the wall temperatures were estimated to be $T_1 = 303°$K and $T_o - T_i = 8°$K $\pm$ 1°K. It is noted here that $\bar{\rho}$ [see Eq. (2)] varies much less with T_i/T_o than does the actual density ρ. The slight difference in T_i and T_o and the uncertainty in these measured values do not affect $\bar{\rho}$ significantly.

The thermal accommodation coefficients α were determined from the heat transfer measurements made in the plane layer type thermal conductivity cell, in the Knudsen number range 0.1–5.0 (based on the separation between the plates). The thermal accommodation coefficients were calculated from the result of Lees' moment method[32]

$$Q/Q_{FM} = [1 + (4/15)(H/\lambda)\alpha(2 - \alpha)]^{-1}, \qquad (4)$$

where Q is the heat conducted between the two parallel plates placed a distance H apart, and Q_{FM} is the heat conducted in the free molecule limit $(H/\lambda \rightarrow 0)$.[33] Using the data of seven measurements, the average value of α was found to be 0.99.

The tangential momentum accommodation coefficients σ were determined from the drag measurements. The procedures for these measurements followed closely those given by Kuhlthau[20] and Bowyer and Talbot.[18] The measurements were made at Knudsen numbers $[\mathrm{Kn} = \lambda/(b - a)]$ greater than seven, where the following expression for drag obtained from free molecule considerations[29] is expected to be reasonably accurate:

$$F = 0.5\rho V(2RT/\pi)^{1/2}(a/b)^2[\sigma/(2 - \sigma)], \qquad (5)$$

and where F is the drag per unit area and T is the gas temperature ($T_i = T_o = T$). Values of σ were calculated from Eq. (5) using the data of 17 different measurements. The average value of σ was found to be 0.97. When the drag data reported by Bowyer and Talbot for argon are substituted into Eq. (5), a value of σ is obtained which agrees to within 2% with the above σ value.

IV. RESULTS

The experimentally determined density distributions may be compared to results of theoretical methods proposed for the solution of the rarefied cylindrical Couette flow problem. In this investigation the density distributions measured in argon were compared to the results given by the Navier–Stokes and Burnett levels of approximations in the Chapman–Enskog method of solution of the Boltzmann equation. Such results were obtained by Schamberg[16] who assumed constant viscosity μ and constant thermal conductivity K, and by Lin and Street[17] who assumed a linear dependence of these properties upon temperature. In order to evaluate how much effect μ and K have on the density distributions, density values were calculated according to both Schamberg's (μ = const, K = const) and Lin–Street's ($\mu \sim T$, $K \sim T$) analyses. The results of these calculations indicate (Fig. 3) that the densities obtained for these two cases agree very closely. Therefore, all analytical results given in the following are for $\mu \sim T$ and $K \sim T$.

The analytical solutions of Schamberg and Lin–Street are based on the assumptions that the flow between the cylinders is steady, stable, and two dimensional, i.e., the streamlines are circles, and end effects are negligible. In order to compare the data to the results of these analyses the experiment must approximate the above conditions. In the present experiments the density was measured at a point half-way between the top and bottom of the

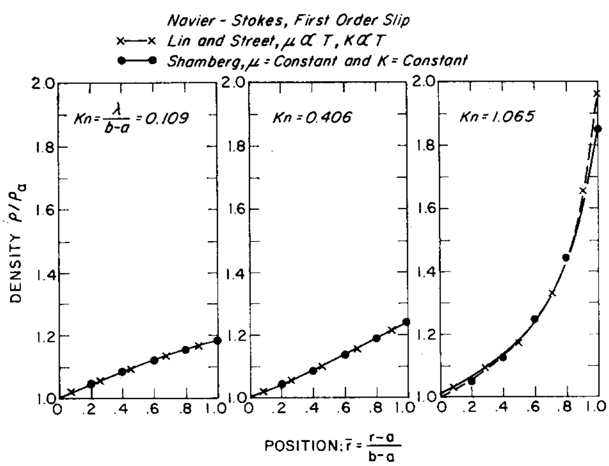

FIG. 3. The variation of density between the cylinders calculated from the Navier–Stokes equation with first-order slip boundary conditions. (Argon, T_i = 303°K, T_o = 311°K, V = 32 327 cm/sec, M = 0.9917, Pr = 2/3, γ = 5/3, α = 0.99, σ = 0.972.)

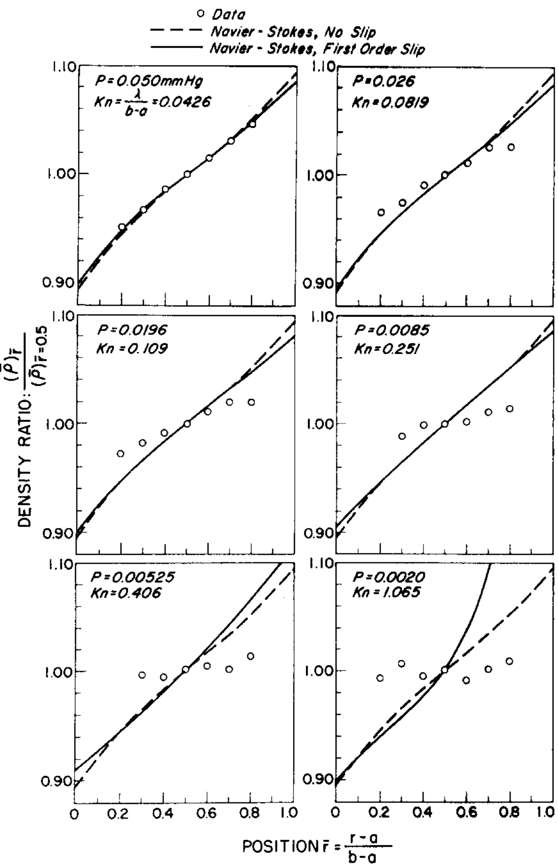

FIG. 4. Density ratio versus position. Comparison between experimental data and analytical results (conditions given in Fig. 3).

cylinders (Fig. 2) for six Knudsen numbers ranging from 0.0426 to 1.065 (Fig. 4). Although the effects of finite cylinder lengths at this point could not be evaluated directly from the data, an estimate of the end effects can be made based on the experimental results of Kuhlthau[20] and Bowyer–Talbot.[18] These investigators evaluated the range in which end effects are significant from drag measurements made with cylinders of varying length. The results of Kuhlthau and Bowyer–Talbot indicate that end effects do not extend beyond a maximum distance of about 0.2λ from the ends of the cylinders, even for Knudsen numbers as high as 60. In our experiments the maximum mean free path was ~2.5 cm (Kn ≅ 1.0) and, therefore, it is felt that the measured density values were not affected significantly by the cylinder ends.

In the concentric cylinder apparatus used in the experiments instabilities may give rise to Taylor vortices. The onset of these instabilities was investigated experimentally by Kuhlthau.[21] His results indicate that for the geometry and conditions of our experiments the onset of instabilities is at ~0.02 mm Hg of pressure. Thus, instabilities could be expected to be present in our measurements that were made at the lowest two Knudsen numbers (Kn = 0.0426 and Kn = 0.0819, Fig. 4). However, the excellent agreement between the theoretical results and the data at Kn = 0.0426 suggest that instabilities did not measurably influence the experimental results.

Density distributions measured in argon are shown in Fig. 4. The parameter $\bar{\rho}$ is defined by Eq. (2) and the data have been further normalized with respect to $\bar{\rho}$ evaluated at a point halfway between the inner and outer cylinders [$\bar{r} = (r - a)/(b - a) = 0.5$]. In Fig. 4 are also given solutions to the Navier–Stokes equations for no slip boundary conditions (continuum solution) and for first-order slip boundary conditions.[17] The Knudsen numbers shown are average values based on the test chamber pressure P and the rotor temperature T_i. The analytical results in Fig. 4 were computed by choosing a value of the gas density at the surface of the inner cylinder such that the calculated average density in the gap was equal to the density corresponding to P and T_i. A detailed error analysis of the data has been made indicating that the systematic errors in the experiments were less than about ±1%. However, the good agreement between the analysis and the data at Kn = 0.0426 suggests that the actual systematic error was less than this estimated maximum. It can be seen from Fig. 4 that at Kn = 0.0426 the

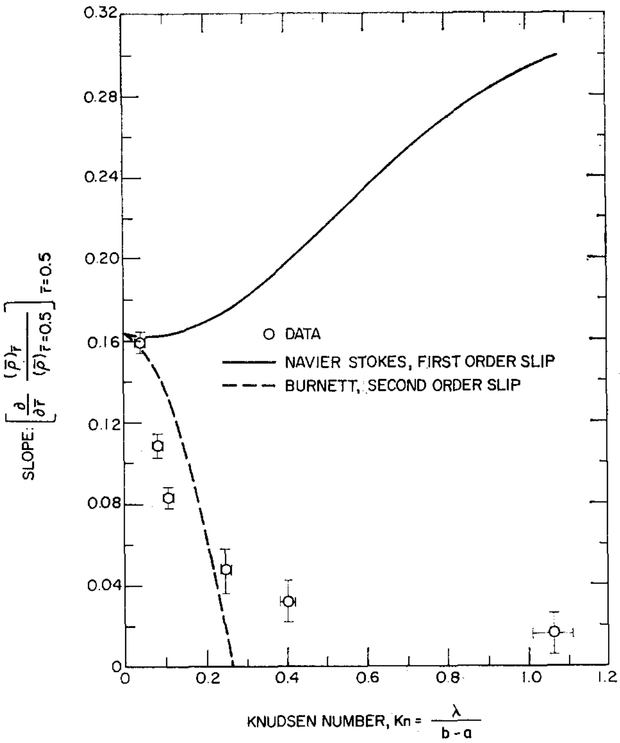

FIG. 5. Density gradient versus Knudsen number. Comparison between experimental data and analytical results (conditions given in Fig. 3).

data agree within 0.1% with the results of the Navier–Stokes equations with first-order slip boundary conditions. As the Knudsen number increases the differences between the data and the analytical results increase, and above Knudsen numbers of ~0.5 the experimental and analytical results differ considerably.

A further rather striking demonstration can be made of the differences between the analysis and the data by plotting the density gradient at the center of the gap ($\bar{r} = 0.5$) as a function of the Knudsen number (Fig. 5). The points labeled as data on this plot were obtained by fitting the data points in Fig. 4 with a straight line according to the method of least squares and by taking the slope of this line (note that this line is not shown in Fig. 4). The estimated errors are also indicated in Fig. 5. Again, it can be seen that the Navier–Stokes first-order slip solution of Lin–Street breaks down above Knudsen numbers of about 0.05.

Owing to the approximations introduced in the Navier–Stokes level of solution one expects the results of these solutions to describe well the density variation at low Knudsen numbers only. The question may be raised now as to how accurate are the results for other mean quantities such as the pressure, temperature, and velocity across the gap

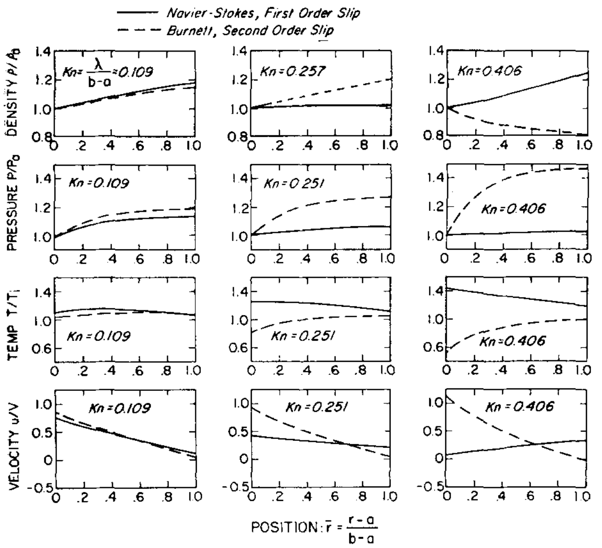

FIG. 6. The variations of density, pressure, temperature, and velocity between the cylinders as calculated from the Navier–Stokes and Burnett equations (conditions given in Fig. 3).

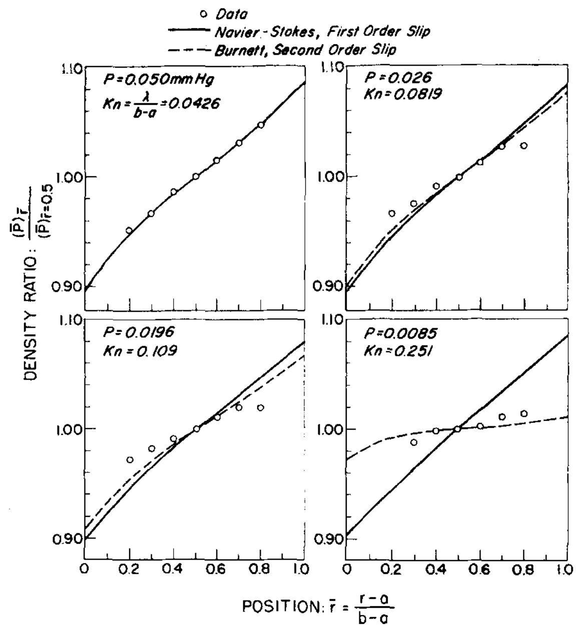

FIG. 7. Density ratio versus position. Comparison between experimental data and analytical results (conditions given in Fig. 3).

between the two cylinders. In order to gain an insight into this question, in addition to the density distributions, pressure, temperature, and velocity distributions across the gap were also computed for three of the experimental Knudsen numbers. The results given in Fig. 6 show that at Kn = 0.406 the Navier–Stokes first-order slip solution yields the physically unrealistic result that the velocity is lower near the rotating inner cylinder than near the stationary outer one. Although it is not shown in Fig. 6, it was also found that at even higher Knudsen numbers (Kn = 0.44) the analysis gives a negative mean velocity at the surface of the rotating inner cylinder. Similar calculations were also performed using Lin–Street's[17] solution to the Burnett equations with second-order slip boundary conditions. The results of these calculations (Fig. 6) yield unrealistic gas velocities and temperatures at Knudsen numbers above about 0.2. For instance, at the surface of the inner cylinder ($\bar{r} = 0$) at Kn = 0.251 the calculated mean gas temperature is lower than the surface temperature, and at Kn = 0.406 the mean gas velocity is higher than the surface velocity.

In Figs. 5 and 7 comparisons between the data and the Burnett solutions are presented, as derived by Lin and Street. In Fig. 7 the comparisons are made only at Knudsen numbers below 0.251, where the analytical results are expected to be at all reasonable, as discussed above.

At Kn = 0.0426 both the Burnett equations and the Navier–Stokes equations (with no slip and first-order slip boundary conditions) agree very closely with the data (Fig. 4, 5, and 7). Data were not taken below Kn = 0.0426. Measurements made at lower Knudsen numbers would not have been of help in evaluating the results of the Navier–Stokes and Burnett equations because at Knudsen numbers below 0.04 the densities $\bar{\rho}$ calculated by these equations differ only by a maximum of ~0.3% ($0.2 < \bar{r} < 0.8$). In the Knudsen number range of 0.04–0.25 the results of the Burnett equations with second-order slip boundary conditions approximate the data better than the results of the Navier–Stokes equations with first-order slip boundary conditions. At Knudsen numbers above 0.25 the results of both the Navier–Stokes and Burnett levels of solutions deviate from the data considerably. The failure of the Navier–Stokes and Burnett equations to give adequate results at higher Knudsen numbers is not surprising, in view of the well-known approximations that form the bases of these equations and their solutions. It is interesting to note, however, that in the transition regime (Kn ~1), the Navier–Stokes equations yield less accurate results for the cylindrical Couette flow problem than, for example, for the problem of sound propagation through a gas.[34,35]

ACKNOWLEDGMENTS

The authors wish to thank Mr. R. Flagan for his help in the experiments.

This work was supported by the National Science Foundation under Grant No. GK-1745.

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