Stability of Two Superposed Elasticoviscous Liquids in Plane Couette Flow

CHIN-HSIU LI
Department of Engineering Mechanics, The University of Michigan
Ann Arbor, Michigan
(Received 5 September 1968; final manuscript received 26 November 1968)

It is found that the elasticity in the liquid cannot only destabilize the flow, but it can also stabilize the flow for certain values of depth ratio, viscosity ratio, and elasticity ratio. The stabilizing or destabilizing effect of the elasticity in liquid is absent in the absence of viscosity stratification, and is only brought about when the viscosity varies.

I. INTRODUCTION

Oldroyd[1] proposed a mathematical model to describe the mechanical behavior of viscoelastic material. This model has proved very useful in characterizing the rheological properties of certain dilute polymer solutions at sufficiently small rates of strain. Using this model, Thomas and Walters,[2,3] Herbert,[4] and Lai[5] examined the stability of specific cases of flow. Recently, Chun and Schwarz[6] adopted Coleman and Noll's[7] model of a second-order fluid and considered the effect of slight viscoelasticity on the hydrodynamic stability of a plane Poiseuille flow. They all showed that the elastic behavior in elasticoviscous liquid destabilizes the flows. Yih[8] considered plane Couette–Poiseuille flow of two superimposed layers of Newtonian fluids of different viscosities. It was found that the viscosity stratification destabilizes the flow in some ranges of the depth and viscosity ratios, and stabilizes the flow in others.

Investigation of the plane Couette flow of two superimposed layers of elasticoviscous liquid of different viscosities and different elasticities is made. The prototype of liquid designed by Oldroyd[1] is considered. For this liquid, the equations of state are

$$S_{ik} = -pg_{ik} + p_{ik}, \tag{1}$$

and

$$\left(1 + \lambda_1 \frac{d_c}{dt}\right)p^{ik} = 2\eta_0\left(1 + \lambda_2 \frac{d_c}{dt}\right)\epsilon^{ik}, \tag{2}$$

in which S_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor of a fixed coordinate system, $\epsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i})$ the rate-of-strain tensor, η_0 a coefficient of viscosity, λ_1 the relaxation time, and λ_2 the retardation time. The coefficients η_0, λ_1, and $\lambda_2(<\lambda_1)$ are all positive. The symbol d_c/dt denotes the convective derivative of a tensor quantity in relation to the fluid in motion. For a contravariant tensor T^{ij}

$$\left(\frac{d_c}{dt}\right)T^{ij} = \left(\frac{\partial}{\partial t}\right)T^{ij} + T^{ij}_{,m}V^m - T^{mj}V^i_{,m} - T^{im}V^j_{,m}. \tag{3}$$

It is found that the elasticity in the liquid cannot only destabilize the flow, but it can also stabilize the flow for certain values of depth ratio, viscous ratio, and elasticity ratio.

II. THE PRIMARY FLOW

Consider two elasticoviscous liquids I and II between two parallel walls having the upper boundary moving with a constant velocity U_0, the lower boundary being stationary (see Fig. 1). Let $[(u_1)_k, (u_2)_k, (u_3)_k]$ denote the velocity components in the x_1, x_2, and x_3 directions, respectively, where x_1, x_2, and x_3 are Cartesian coordinates, $(p_{ij})_k$ denotes the $i-j$ component of stress tensor, and $(\lambda_1)_k$ and $(\lambda_2)_k$ denote the relaxation time and retardation time. The

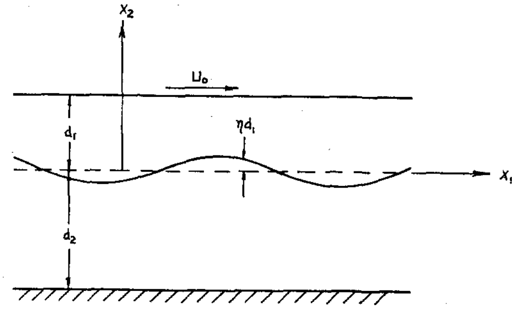

FIG. 1. Definition sketch.

---

[1] J. G. Oldroyd, Proc. Roy. Soc. (London) **A245**, 278 (1958).
[2] R. H. Thomas and K. Walters, Proc. Roy. Soc. (London) **A274**, 371 (1963).
[3] R. H. Thomas and K. Walters, J. Fluid Mech. **18**, 33 (1964).
[4] D. M. Herbert, J. Fluid Mech. **17**, 353 (1963).
[5] W. Lai, Phys. Fluids **10**, 844 (1967).
[6] D. H. Chun and W. H. Schwarz, Phys. Fluids **11**, 5 (1968).
[7] B. D. Coleman and W. Noll, Arch. Ratl. Mech. Anal. **6**, 355 (1960).
[8] C.-S. Yih, J. Fluid Mech. **27**, 337 (1967).

subscript k is I for the upper liquid and II for the lower liquid since the equations of state and equations of motion governing liquid I and liquid II will be the same in the form. At this stage we drop the subscript for convenience, and consider the steady undisturbed flow

$$u_1 = \bar{u}(x_2), \quad u_2 = 0, \quad \text{and} \quad u_3 = 0.$$

For this flow, the constitutive equations can be written as follows:

$$\begin{aligned}
&\bar{p}_{11} + \lambda_1\left(\frac{\partial \bar{p}_{11}}{\partial t} + \frac{\partial \bar{p}_{11}}{\partial x_1}\bar{u} - 2\bar{p}_{11}\frac{\partial \bar{u}}{\partial x_2}\right) = -2\eta_0\lambda_2\left(\frac{\partial \bar{u}}{\partial x_2}\right)^2,\\
&\bar{p}_{12} + \lambda_1\left(\frac{\partial \bar{p}_{12}}{\partial x_1}\bar{u} - \bar{p}_{22}\frac{\partial \bar{u}}{\partial x_2}\right) = \eta_0\frac{\partial u}{\partial x_2},\\
&\bar{p}_{13} + \lambda_1\left(\frac{\partial \bar{p}_{13}}{\partial x_1}\bar{u} - \bar{p}_{23}\frac{\partial u}{\partial x_2}\right) = 0,\\
&\bar{p}_{22} + \lambda_1\frac{\partial \bar{p}_{22}}{\partial x_1}\bar{u} = 0,\\
&\bar{p}_{23} + \lambda_1\frac{\partial \bar{p}_{23}}{\partial x_1}\bar{u} = 0,\\
&\bar{p}_{33} + \lambda_1\frac{\partial \bar{p}_{33}}{\partial x_1}\bar{u} = 0.
\end{aligned} \tag{4}$$

The equations of motion can be written as

$$\begin{aligned}
0 &= -\frac{\partial \bar{p}}{\partial x_1} + \frac{\partial \bar{p}_{11}}{\partial x_1} + \frac{\partial \bar{p}_{21}}{\partial x_2} + \frac{\partial \bar{p}_{31}}{\partial x_3},\\
0 &= -\frac{\partial \bar{p}}{\partial x_2} + \frac{\partial \bar{p}_{12}}{\partial x_1} + \frac{\partial \bar{p}_{22}}{\partial x_2} + \frac{\partial \bar{p}_{32}}{\partial x_3} + \rho g,\\
0 &= -\frac{\partial \bar{p}}{\partial x_3} + \frac{\partial \bar{p}_{13}}{\partial x_1} + \frac{\partial \bar{p}_{23}}{\partial x_2} + \frac{\partial \bar{p}_{33}}{\partial x_3}.
\end{aligned} \tag{5}$$

Equations (4) and (5) admit the stress components of primary flow to be

$$\begin{aligned}
&\bar{p}_{13} = 0, \quad \bar{p}_{22} = 0, \quad \bar{p}_{23} = 0, \quad \bar{p}_{33} = 0,\\
&\bar{p} = \bar{p}(x_2), \quad \bar{p}_{11} = \bar{p}_{11}(x_2), \quad \text{and} \quad \bar{p}_{12} = \bar{p}_{12}(x_2).
\end{aligned} \tag{6}$$

We make all quantities nondimensional by letting

$$x_1 = xd_1, \quad y_1 = yd_1, \quad \bar{u} = UU_0, \quad \bar{p} = \rho_I U_0^2 P,$$

$$\bar{p}_{ij} = \rho_I U_0^2 P_{ij}, \quad \text{and} \quad t = \frac{d_1}{U_0}\tau.$$

The nondimensional forms of the first two equations in (4) and (5) are then

$$(P_{11})_i - 2(M_1)_i(P_{12})_i\frac{dU_i}{dy} = -2\beta_i\frac{(M_2)_i}{R}\left(\frac{dU_i}{dy}\right)^2, \tag{7a}$$

$$(P_{12})_i = \beta_i\frac{1}{R}\left(\frac{dU_i}{dy}\right), \tag{7b}$$

$$0 = \frac{d}{dy}(P_{21})_i, \tag{7c}$$

$$0 = -\frac{d}{dy}P_i - \delta_i F^{-2}, \tag{7d}$$

in which $R = [U_0\rho_I d_1/(\eta_0)_I]$ is the Reynolds number, $(M_1)_i = [U_0(\lambda_1)_i/d_1]$, $(M_2)_i = [U_0(\lambda_2)_i/d_1]$, and $F^2 = U_0^2/gd_1$ is the Froude number. The subscript i is taken to be I and II for the upper and the lower layer of liquids, respectively,

$$\beta_I = 1, \qquad \beta_{II} = m_\eta, \qquad m_\eta = \frac{(\eta_0)_{II}}{(\eta_0)_I} \tag{8}$$

is the ratio of viscosity, and

$$\delta_I = 1, \qquad \delta_{II} = \gamma, \qquad \gamma = \frac{\rho_{II}}{\rho_I} \tag{9}$$

is the ratio of density. From Eqs. (7b) and (7c) the equations governing the primary flow can be obtained. These are

$$\frac{d^2U_I}{dy^2} = 0 \quad \text{and} \quad \frac{d^2U_{II}}{dy^2} = 0. \tag{10}$$

Subject to the boundary conditions that U_I is equal to a specified U_0 on the upper boundary and U_{II} is zero on the lower boundary, and that U_I and U_{II}, and $(P_{12})_I$ and $(P_{12})_{II}$ must be continuous at the interface, Eqs. (10) can be solved to yield the solutions

$$U_I = a_1y + b \quad \text{and} \quad U_{II} = a_2y + b, \tag{11}$$

in which

$$a_1 = \frac{m_\eta}{m_\eta + n}, \qquad b = \frac{n}{m_\eta + n},$$

$$a_2 = \frac{1}{m_\eta + n}, \qquad n = \frac{d_2}{d_1}.$$

The primary flow has the same velocity profile as that in the Newtonian fluid. As soon as we have U_I and U_{II}, from Eqs. (7a–d) we obtain

$$\begin{aligned}
&(P_{12})_I = a_1/R, \quad (P_{11})_I = (2/R)[(M_1)_I - (M_2)_I]a_1^2,\\
&\frac{dP_I}{dy} = -F^{-2}, \quad (P_{12})_{II} = m_\eta\frac{a_2}{R} = (P_{12})_I,\\
&(P_{11})_{II} = \frac{2}{R}m_\eta[(M_1)_{II} - (M_2)_{II}]a_2^2,
\end{aligned} \tag{12}$$

and

$$\frac{dP_{II}}{dy} = -\gamma F^{-2}.$$

III. DIFFERENTIAL SYSTEM GOVERNING STABILITY

Two-dimensional infinitesimal disturbances are considered in this investigation for simplicity, but this does not imply that Squire's theorem is applicable. Hence, the stability against the three-dimensional disturbances should be studied in a separate investigation.

Let $u_1 = U + u'$, $u_2 = V'$, $p = P + p'$, $p_{11} = P_{11} + \sigma_{11}$, $p_{12} = P_{12} + \sigma_{12}$, $p_{22} = \sigma_{22}$, $\epsilon_{11} = \epsilon'_{11}$, $\epsilon_{12} = E_{12} + \epsilon'_{12}$, $\epsilon_{22} = \epsilon'_{22}$, in which the strain components are nondimensionalized by the unit U_0/d_1, and σ_{11}, σ_{12}, σ_{22}, and the quantities denoted by a prime indicate the small perturbation from the primary flow. Substituting these quantities into the general equations of motion and the constitutive equations, and neglecting terms of higher orders, we obtain the linearized equations governing the perturbation flow. The linearized equations of motion are

$$\frac{\partial u'}{\partial \tau} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial p'}{\partial x} + \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{21}}{\partial y},$$

$$\frac{\partial v'}{\partial \tau} + U\frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y}, \tag{13}$$

and the linearized equations of state are

$$\sigma_{11} + M_1\left[\frac{\partial \sigma_{11}}{\partial \tau} + \frac{\partial \sigma_{11}}{\partial x}U + v'\frac{\partial P_{11}}{\partial y} - 2\left(\frac{\partial u'}{\partial x}P_{11} + \frac{\partial u'}{\partial y}P_{12} + \sigma_{12}\frac{dU}{dy}\right)\right] = \frac{2}{R}\epsilon'_{11} + \frac{2M_2}{R}\left[\frac{\partial \epsilon'_{11}}{\partial \tau} + \frac{\partial \epsilon'_{11}}{\partial x}U - \left(\frac{\partial u'}{\partial y} + 2\epsilon'_{12}\right)\frac{dU}{dy}\right],$$

$$\sigma_{12} + M_1\left(\frac{\partial \sigma_{12}}{\partial \tau} + \frac{\partial \sigma_{12}}{\partial x}U + v'\frac{\partial P_{12}}{\partial y} - \sigma_{22}\frac{dU}{dy} - \frac{\partial v'}{\partial x}P_{11}\right) = \frac{2}{R}\epsilon'_{12} + \frac{2M_2}{R}\left(\frac{\partial \epsilon'_{12}}{\partial \tau} + \frac{\partial \epsilon'_{12}}{\partial x}U + \frac{v'}{2}\frac{d^2U}{dy^2} - \epsilon'_{22}\frac{dU}{dy}\right), \tag{14}$$

$$\sigma_{22} + M_1\left(\frac{\partial \sigma_{12}}{\partial \tau} + \frac{\partial \sigma_{22}}{\partial x}U - 2\frac{\partial v'}{\partial x}P_{12}\right) = \frac{2}{R}\epsilon'_{22} + \frac{2M_2}{R}\left(\frac{\partial \epsilon'_{22}}{\partial \tau} + \frac{\partial \epsilon'_{22}}{\partial x}U - \frac{\partial v'}{\partial x}\frac{dU}{dy}\right).$$

In addition to these, we have the equation of continuity

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{15}$$

In order to retain the meaning of R and nondimensionalize the pressure by the same unit $\rho_I U_0^2$ for both layers of liquid, a factor $1/\gamma$ will arise on the right-hand sides of Eq. (13) and another factor m_η will arise in Eqs. (14) for those terms which contain the Reynolds number R, when the lower layer is considered.

Equation (15) permits the use of a stream function ψ, in terms of which

$$u' = \frac{\partial \psi}{\partial y}, \qquad v' = -\frac{\partial \psi}{\partial x}. \tag{16}$$

As is customary in stability analyses, we assume, that all perturbation quantities contain an exponential time factor, and study the behavior of a spatially periodic disturbance. Thus,

$$(\psi, p', \sigma_{11}, \sigma_{12}, \sigma_{22}) = [\phi(y), f(y), F_1(y), F_2(y), F_3(y)] \cdot \exp[i\alpha(x - c\tau)], \tag{17}$$

in which c is the eigenvalue sought. The stability or instability is determined by the sign of the imaginary part of c, which can be written as $c_r + ic_i$. The flow is stable when c_i is negative and unstable when it is positive. It is understood that more general disturbances can be decomposed into Fourier components, each of which has the form assumed in Eq. (17).

Substituting Eqs. (17) and (16) into Eqs. (13) we eliminate p' and obtain, for the two layers in turn,

$$i\alpha[(U_I - c)(\phi''_I - \alpha^2\phi_I)] = i\alpha(F_1)'_I + (F_2)''_I + \alpha^2(F_2)_I - i\alpha(F_3)'_I,$$

$$i\alpha\gamma[(U_{II} - c)(\phi''_{II} - \alpha^2\phi_{II})] = i\alpha(F_1)'_{II} + (F_2)''_{II} + \alpha^2(F_2)_{II} - i\alpha(F_3)'_{II}. \tag{18}$$

Similarly, substituting Eq. (17) into Eqs. (14) for the two layers in turn, one obtains the linearized equations of state

$$(F_1)_i[1 + i\alpha(M_1)_i(U_i - c)] = (M_1)_i\{i\alpha(P_{11})'_i\phi_i + 2[i\alpha(P_{11})_i\phi'_i + (P_{12})_i\phi''_i + (F_2)_iU'_i]\} + i\alpha\beta_i\frac{2}{R}\phi'_i - \beta_i\frac{2(M_2)_i}{R} \cdot [\alpha^2\phi'_i(U_i - c) + U'_i(2\phi''_i + \alpha^2\phi_i)],$$

$$(F_2)_i[1 + i\alpha(M_1)_i(U_i - c)]$$
$$= (M_1)_i[i\alpha(P_{12})_i'\phi_i + (F_3)_iU_i' + \alpha^2\phi_i(P_{11})_i]$$
$$+ \beta_i \frac{1}{R}(\phi_i'' + \alpha^2\phi_i) + i\alpha\beta_i \frac{(M_2)_i}{R}$$
$$\cdot[(\phi_i'' + \alpha^2\phi_i)(U_i - c) - \phi_iU_i'' + 2\phi_i'U_i'],$$
$$(F_3)_i[1 + i\alpha(M_1)_i(U_i - c)]$$
$$= 2(M_1)_i(P_{11})_i\alpha^2\phi_i - i\alpha\beta_i\frac{2}{R}\phi_i'$$
$$+ \beta_i\frac{2(M_2)_i}{R}\alpha^2[\phi_i'(U_i - c) - U_i'\phi_i], \tag{19}$$

in which the subscript i is taken to be I and II as before for the upper layer and the lower layer of fluids, respectively.

On the boundaries, the zero normal velocity and nonslip condition demand that

$$\phi_{\mathrm{I}}(1) = 0, \qquad \phi_{\mathrm{I}}'(1) = 0, \qquad \phi_{\mathrm{II}}(-n) = 0, \qquad \phi_{\mathrm{II}}'(-n) = 0. \tag{20a, b, c, d}$$

The continuity of v' at the interface demands

$$\phi_{\mathrm{I}}(0) = \phi_{\mathrm{II}}(0). \tag{20e}$$

The kinematic boundary condition at the interface is

$$\left(\frac{\partial}{\partial\tau} + U_{\mathrm{I}}\frac{\partial}{\partial x}\right)\eta = v' = -i\alpha\phi_{\mathrm{I}}(0)\exp[i\alpha(x - c\tau)].$$

From this we find

$$\eta = \frac{\phi_{\mathrm{I}}(0)}{c'}\exp[i\alpha(x - c\tau)], \tag{21}$$

in which η is the deviation of interface from its mean position, and $c' = c - U_{\mathrm{I}}(0)$ (or $c' = c - b$). The continuity in u' at interface then demands

$$\phi_{\mathrm{I}}'(0) + \frac{\phi_{\mathrm{I}}(0)}{c'}U_{\mathrm{I}}'(0) = \phi_{\mathrm{II}}'(0) + \frac{\phi_{\mathrm{II}}(0)}{c'}U_{\mathrm{II}}'(0). \tag{20f}$$

The continuity of shear stress at interface is expressed by $[\sigma_{12} + (dP_{12}/dy)\eta]_{\mathrm{I}} = [\sigma_{12} + (dP_{12}/dy)\eta]_{\mathrm{II}}$, evaluated at $y = 0$. Since $dP_{12}/dy = 0$ from Eq. (7c), this boundary condition is simply

$$(F_2)_{\mathrm{I}} = (F_2)_{\mathrm{II}} \quad \text{at} \quad y = 0. \tag{20g}$$

The continuity of normal stress at interface is expressed by

$$\left(-p' - \frac{dP}{dy}\eta + \sigma_{22}\right)_{\mathrm{I}} - \left(-p' - \frac{dP}{dy}\eta + \sigma_{22}\right)_{\mathrm{II}} = -S\frac{\partial^2\eta}{\partial x^2},$$

in which $S = T/\rho_{\mathrm{I}}U_0^2d_1$, and T is the surface tension, and again, variables are evaluated at $y = 0$. We utilize the first equation in (13) to evaluate f and hence p' for either liquid. With the results so obtained, and with η evaluated from Eq. (21), and dP_{I}/dy and dP_{II}/dy from Eq. (12), the normal stress condition can be rewritten as

$$-\alpha(c'\phi_{\mathrm{I}}' + a_1\phi_{\mathrm{I}}) - \alpha(F_1)_{\mathrm{I}} + i(F_2)_{\mathrm{I}}'$$
$$+ \alpha(F_3)_{\mathrm{I}} + \alpha\gamma(c'\phi_{\mathrm{II}}' + a_2\phi_{\mathrm{II}})$$
$$+ \alpha(F_1)_{\mathrm{II}} - i(F_2)_{\mathrm{II}}' - \alpha(F_3)_{\mathrm{II}}$$
$$= \alpha[(\gamma - 1)F^{-2} + \alpha^2 S]\frac{\phi_{\mathrm{I}}(0)}{c'}, \tag{20h}$$

evaluated at $y = 0$.

The differential system governing the stability consists of Eqs. (18), (19), and (20a–h).

IV. SOLUTION OF THE DIFFERENTIAL SYSTEM

The regular perturbation technique is used to solve the eigenvalue problem for long waves, or for $\alpha \ll 1$. From Eqs. (19) we obtain as Lai[5] did,

$$(F_1)_{\mathrm{I}} = \frac{4\,\Delta M}{R}U_{\mathrm{I}}'\phi_{\mathrm{I}}'' + O(\alpha),$$
$$(F_2)_{\mathrm{I}} = \frac{1}{R}\phi_{\mathrm{I}}'' + i\alpha\frac{\Delta M}{R}[U_{\mathrm{I}}''\phi_{\mathrm{I}} - 2\phi_{\mathrm{I}}'U_{\mathrm{I}}'$$
$$- \phi_{\mathrm{I}}''(U_{\mathrm{I}} - c)] + O(\alpha^2),$$
$$(F_3)_{\mathrm{I}} = -i\alpha\frac{2}{R}\phi_{\mathrm{I}}' + O(\alpha^2),$$
$$(F_1)_{\mathrm{II}} = \frac{4\,\Delta M}{R}m_\eta m_\lambda U_{\mathrm{II}}'\phi_{\mathrm{II}}'' + O(\alpha), \tag{22}$$
$$(F_2)_{\mathrm{II}} = \frac{1}{R}m_\eta\phi_{\mathrm{II}}'' + i\alpha\frac{1}{R}m_\eta m_\lambda\,\Delta M$$
$$\cdot[U_{\mathrm{II}}''\phi_{\mathrm{II}} - 2\phi_{\mathrm{II}}'U_{\mathrm{II}}' - \phi_{\mathrm{II}}''(U_{\mathrm{II}} - c)] + O(\alpha^2),$$
$$(F_3)_{\mathrm{II}} = -i\alpha\frac{2}{R}m_\eta\phi_{\mathrm{II}}' + O(\alpha^2),$$

in which

$$\Delta M = (M_1)_{\mathrm{I}} - (M_2)_{\mathrm{I}}, \qquad m_\lambda = \frac{(M_1)_{\mathrm{II}} - (M_2)_{\mathrm{II}}}{(M_1)_{\mathrm{I}} - (M_2)_{\mathrm{I}}} = \frac{(\lambda_1)_{\mathrm{II}} - (\lambda_2)_{\mathrm{II}}}{(\lambda_1)_{\mathrm{I}} - (\lambda_2)_{\mathrm{I}}}. \tag{23}$$

Substituting Eqs. (22) into Eqs. (18) and (20a–h), we have, to the first power of α

$$\phi_{\mathrm{I}}'''' - i\alpha R(U_{\mathrm{I}} - c)\phi_{\mathrm{I}}'' + i\alpha\,\Delta M\,\{4(U_{\mathrm{I}}'\phi_{\mathrm{I}}'')' + [U_{\mathrm{I}}''\phi_{\mathrm{I}} - 2\phi_{\mathrm{I}}'U_{\mathrm{I}}' - \phi_{\mathrm{I}}''(U_{\mathrm{I}} - c)]''\} = 0, \tag{24}$$

$$\phi_{II}'''' - i\alpha R \frac{\gamma}{m_\eta} (U_{II} - c)\phi_{II}'' + i\alpha\, \Delta M\, m_\lambda\{4(U_{II}'\phi_{II}'')' + [U_{II}''\phi_{II} - 2\phi_{II}'U_{II}' - \phi_{II}''(U_{II} - c)]''\} = 0;$$

$$\phi_I(1) = 0, \qquad \phi_I'(1) = 0, \qquad (25a, b)$$

$$\phi_{II}(-n) = 0, \qquad \phi_{II}'(-n) = 0, \qquad (25c, d)$$

$$\phi_I(0) = \phi_{II}(0), \qquad (25e)$$

$$\phi_I'(0) - \phi_{II}'(0) + \frac{\phi_I(0)}{c'} [U_I'(0) - U_{II}'(0)] = 0, \qquad (25f)$$

$$\phi_I'' + i\alpha\, \Delta M\, [U_I''\phi_I - 2\phi_I'U_I' - \phi_I''(U_I - c)] - m_\eta\phi_{II}'' - i\alpha\, \Delta M\, m_\eta m_\lambda \cdot [U_{II}''\phi_{II} - 2\phi_{II}'U_{II}' - \phi_{II}(U_{II} - c)] = 0, \qquad (25g)$$

$$\phi_I''' + i\alpha R(c'\phi_I' + a_1\phi_I) + i\alpha\, \Delta M \cdot \{4U_I'\phi_I'' + [U_I''\phi_I - 2\phi_I'U_I' - \phi_I''(U_I - c)]'\} - m_\eta\phi_{II}''' - i\alpha R\gamma(c'\phi_{II}' + a_2\phi_{II}) - i\alpha\, \Delta M\, m_\eta m_\lambda\{4U_{II}'\phi_{II}'' + [U_{II}''\phi_{II} - 2\phi_{II}'U_{II}' - \phi_{II}''(U_{II} - c)]'\} + i\alpha R(\gamma - 1)F^{-2} \frac{\phi_I(0)}{c'} = 0, \qquad (25h)$$

In Eqs. (25g) and (25h), all variables are evaluated at $y = 0$, where primes, except the prime on c, indicate the derivative with respect to y. All terms containing α^2 and higher orders of α are ignored since this investigation will include only the first-order approximation. In fact, the method of regular perturbation adopted here can accommodate as high an order of approximation as we desire by including the higher order terms of α.

Following the approach of Yih[9] by expanding the eigenfunctions and eigenvalue in a power series of wavenumber α

$$\phi_I = \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + \cdots,$$
$$\phi_{II} = \chi_0 + \alpha\chi_1 + \alpha^2\chi_2 + \cdots,$$
$$c = c_0 + \alpha c_1 + \alpha^2 c_2 + \cdots.$$

A. The Zeroth-Order Approximation

The zeroth approximation gives us the governing equations

$$\phi_0'''' = 0, \qquad \chi_0'''' = 0; \qquad (26)$$

and the boundary conditions

$$\phi_0(1) = 0, \qquad \phi_0'(1) = 0,$$
$$\chi_0(-n) = 0, \qquad \chi_0'(-n) = 0,$$
$$\phi_0(0) - \chi_0(0) = 0,$$
$$\phi_0'(0) - \chi_0'(0) + \frac{\phi_0(0)}{c_0'} (a_1 - a_2) = 0,$$
$$\phi_0''(0) - m_\eta\chi_0''(0) = 0,$$
$$\phi_0'''(0) - m_\eta\chi_0'''(0) = 0.$$

Solution of the differential system is found to be

$$\phi_0 = 1 + A_1y + A_2y^2 + A_3y^3, \qquad \chi_0 = 1 + B_1y + B_2y^2 + B_3y^3, \qquad (27)$$

in which

$$A_1 = -\frac{m_\eta + 3n^2 + 4n^3}{2n^2(1 + n)}, \qquad B_1 = \frac{n^3 + m_\eta(4 + 3n)}{2m_\eta n(1 + n)},$$

$$B_2 = \frac{n^3 + m_\eta}{n^2 m_\eta(1 + n)}, \qquad B_3 = \frac{n^2 - m_\eta}{2n^2 m_\eta(1 + n)},$$

$$A_2 = m_\eta B_2, \qquad A_3 = m_\eta B_3.$$

The eigenvalue c_0 is determined by

$$c_0' = c_0 - b = -\frac{a_1 - a_2}{A_1 - B_1} = \frac{2n^2 m_\eta(1 + n)(a_1 - a_2)}{m_\eta^2 + 2nm_\eta(2 + 3n + 2n^2) + n^4}. \qquad (28)$$

The solution of the zeroth-order approximation is the same as that given by Yih[8] for Newtonian fluids.

B. The First-Order Approximation

Having obtained the eigenvalue c_0 and eigenfunctions ϕ_0 and χ_0, we put them into the equations governing the first-order approximation and obtain

$$\phi_1'''' = iR[6a_1A_3y^2 + 2(a_1A_2 - 3A_3c_0')y - 2c_0'A_2],$$
$$\chi_1'''' = iR \frac{\gamma}{m_\eta} [6a_2B_2y^2 + 2(a_2B_2 - 3B_3c_0')y - 2c_0'B_2]. \qquad (29)$$

The general solutions of Eq. (29) can be written as

$$\phi_1 = \Delta A_1y + \Delta A_2y^2 + \Delta A_3y^3 + iRH_1(y),$$
$$\chi_1 = \Delta B_1y + \Delta B_2y^2 + \Delta B_3y^3 + iR \frac{\gamma}{m_\eta} H_2(y), \qquad (30)$$

in which

$$H_1(y) = \frac{a_1A_3}{60} y^6 + \frac{a_1A_2 - 3A_3c_0'}{60} y^5 - \frac{c_0'A_2}{12} y^4,$$

[9] C.-S. Yih, Phys. Fluids 6, 321 (1963).

and

$$H_2(y) = \frac{a_2B_3}{60}y^6 + \frac{a_2B_2 - 3B_3c_0'}{60}y^5 - \frac{c_0'B_2}{12}y^4.$$

The six constants of integration and hence the eigenvalue will be determined from the boundary conditions. The boundary condition (22h), in the first-order approximation, can be simplified by using the relations in the zero-order approximation, and the simplified form is

$$\phi_1'''(0) - m_\eta\chi_1'''(0) + iR\left[(\gamma - 1)\left(\frac{1}{c_0'F^2} - c_0'A_1 - a_1\right)\right] + 2i\,\Delta M\,[a_1A_2 + 3c_0'A_3 - m_\eta m_\lambda(a_2B_2 + 3c_0'B_2)] = 0. \quad (31)$$

The first-order approximation of the boundary condition (25e) is automatically satisfied by (30) because we take the terms of zero power in y for ϕ_1 and χ_1 to be zero by following Yih's[9] argument. The first-order approximations of the boundary conditions can be derived straightforwardly from Eqs. (25). Applying the boundary conditions on ϕ_1 and χ_1, we obtain

$$\begin{aligned}
\Delta A_1 + \Delta A_2 + \Delta A_3 + iRH_1(1) &= 0,\\
\Delta A_1 + 2\,\Delta A_2 + 3\,\Delta A_3 + iRH_1'(1) &= 0,\\
n\,\Delta B_1 - n^2\,\Delta B_2 + n^3\,\Delta B_3 - iR\gamma\frac{1}{m_\eta}H_2(-n) &= 0,\\
\Delta B_1 - 2n\,\Delta B_2 + 3n^2\,\Delta B_3 + iR\gamma\frac{1}{m_\eta}H_2'(-n) &= 0,\\
\Delta A_1 - \Delta B_1 + \frac{a_2 - a_1}{(c_0')^2}c_1 &= 0,\\
\Delta A_2 - m_\eta\,\Delta B_2 + i\,\Delta M\,K_1 &= 0,\\
\Delta A_3 - m_\eta\,\Delta B_3 + iRK_2 + i\,\Delta M\,K_3 &= 0,
\end{aligned} \quad (32)$$

in which

$$K_1 = a_1A_1 - c_0'A_2 - m_\eta m_\lambda(a_2B_1 - c_0'B_2),$$

$$K_2 = \tfrac{1}{6}(\gamma - 1)\left(\frac{1}{c_0'F^2}c_0'A_1 - a_1\right),$$

and

$$K_3 = \tfrac{1}{3}[a_1A_2 + 3c_0'A_3 - m_\eta m_\lambda(a_2B_2 + 3c_0'B_3)].$$

We note that the non-Newtonian effect is felt only through the conditions of continuity of shear and normal stresses at the interface. Separating the eigenvalue c_1 into real and imaginary as $c_1 = (c_1)_r + i(c_1)_i$, we obtain from Eqs. (32) the results $(c_1)_r = 0$ and

$$(c_1)_i = RJ_1(\gamma, n, m_\eta) + \Delta MJ_2(n, m_\eta, m_\lambda), \quad (33)$$

in which

$$J_1 = \frac{(c_0')^2}{m_\eta(a_1 - a_2)}\left[m_\eta[H_1'(1) - 2H_1(1)] - \frac{2}{n}\gamma H_2(-n) - \gamma H_2'(-n) - n^2K_2 + \frac{m_\eta - n^2}{2(1+n)}\left(H_1(1) - H_1'(1) - \frac{\gamma}{n^2}H_2(-n) - \frac{\gamma}{n}H_2'(-n) - 2nK_2\right)\right], \quad (34)$$

and

$$J_2 = \frac{(c_0')^2}{m_\eta(a_1 - a_2)}\left(\frac{n^2 - m_\eta}{2(1+n)}K_1 - \frac{n(n + m_\eta)}{1+n}K_3\right). \quad (35)$$

V. RESULTS AND DISCUSSION

The $(c_1)_i$ given by Eq. (33) is the criterion of stability. It consists of two parts: RJ_1 due to the viscosity and ΔMJ_2 due to the elasticity of the liquids. For $\Delta M = 0$, Eq. (33) becomes

$$(c_1)_i = RJ_1(\gamma, n, m_\eta),$$

which reproduces Yih's[8] result for Newtonian fluids.

As pointed out by Yih, we can verify directly from Eq. (34) that

$$\frac{m_\eta}{n^2}J_1(n, m_\eta) = J_1\left(\frac{1}{n}, \frac{1}{m_\eta}\right)$$

when $\gamma = 1$. This is still true in this investigation. In addition, one from Eq. (35) can also verify that

$$\frac{1}{m_\lambda}J_2(n, m_\eta, m_\lambda) = J_2\left(\frac{1}{n}, \frac{1}{m_\eta}, \frac{1}{m_\lambda}\right), \quad (36)$$

which is to say that the part of $(c_1)_i$ due to the elasticity also remains the same when the two superposed layers of fluid are interchanged, as it should from a physical point of view. The factor $1/m_\lambda$ on the left-hand side of Eq. (36) arose as long as the definition of ΔM is retained before and after the interchange.

To see whether the elasticity of the liquids has any stabilizing influence, numerical calculations have been carried out. The results are illustrated by curves shown in Figs. 2–4. Since $\lambda_1 > \lambda_2$, ΔM is positive.

Keeping Eq. (33) in mind, we see that the elasticity in liquid causes the flow stability if $J_2 < 0$ and instability if $J_2 > 0$.

For $n < 0.5$, the elasticity in liquid stabilizes the flow if $m_\eta < 1$, no matter what values m_λ may have, and destabilizes the flow if $m_\eta > 1$. The stabilizing effect is increased with increasing m_λ, but the value of m_λ seems to have only a slight influence on the growth rate of the disturbance for cases in which the elasticity destabilizes the flow. The variation of J_2 with m_λ and m_η for the special case, $n = 0.1$, is shown in Fig. 2.

For n increasing from 0.5 toward 1, the range of value of m_η for which the elasticity of the fluids stabilizes the flow becomes narrower for $m_\eta < 1$, and wider for $m_\eta > 1$. Figure 3 illustrates the variation of J_2 with m_λ and m_η for the special case, $n = 1$. Whether stabilizing or destabilizing, the damping rate or growth rate of the disturbance attributable to the action of elasticity of the liquids varies only slightly with m_λ when $m_\lambda < 1$, but greatly with m_λ when $m_\lambda > 1$.

The values of J_2 for $n > 1$ can be calculated from the values of J_2 for $n < 1$ which are plotted in figures by using the correlation (36).

Figure 4 shows the variation of J_2 with m_λ and n for $m_\eta = 0.4$. From this we see that nonuniformity in elastic property of the liquids stabilizes the flow when $n < 1$, and destabilizes the flow when $n \geq 1$ for this particular value of m_η.

As one can see from Figs. 2–4, J_2 vanishes whenever m_η approaches 1. J_2 is other than zero when $m_\eta \neq 1$. This result reveals that the stabilizing or destabilizing effect of the elasticity in liquid is absent in the absence of viscosity stratification, and is only brought about when the viscosity varies.

Numerical calculation also shows that the values of J_1 and J_2 can be of opposite sign for the case $\gamma = 1$ with the depth ratio n larger than about 0.5. J_1 can be positive while J_2 is negative or vice versa depending on the values of n, m_η, and m_λ. This indicates that the effect of the elasticity in the liquid can be either stabilizing or destabilizing. Computations for J_1 are also carried out for the cases of $\gamma = 1.1$, 1.2, and 1.4 each for four different values of the Froude number $F = 0.1$, 0.5, 1, and 2. The results demonstrate that J_1 decreases with increasing the value of γ. This feature is expected since the effect of gravity is to stabilize horizontal flows when there is a negative density gradient in the direction of the vertical. For the cases of $F = 0.1$ ($\gamma \geq 1.1$), and $F = 0.5$ ($\gamma \geq 1.2$), the destabilizing effect of viscosity variation in Newtonian fluid is completely overshadowed by the

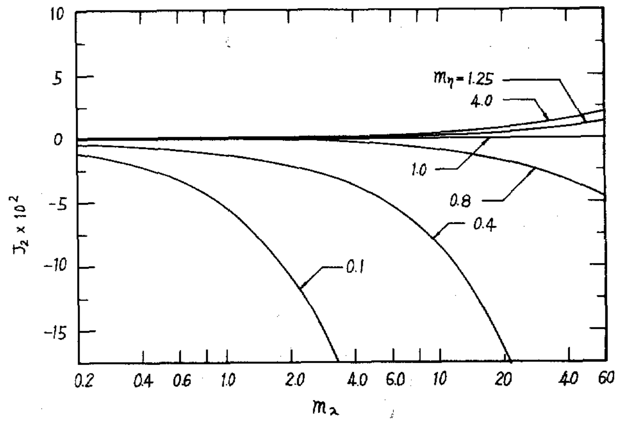

FIG. 2. Variation of J_2 with the elasticity ratio m_λ for various values of the viscosity ratio m_η for the case of depth ratio $n = 0.1$.

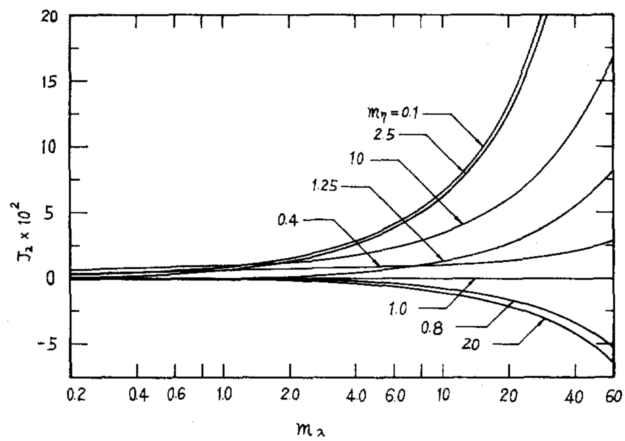

FIG. 3. Variation of J_2 with the elasticity ratio m_λ for various values of the viscosity ratio m_η for the case of depth ratio $n = 1$.

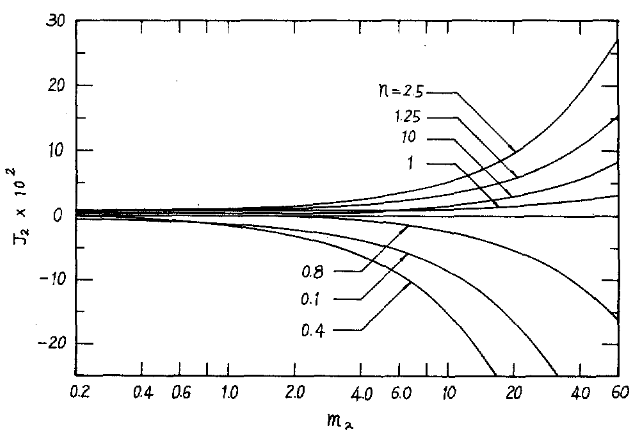

FIG. 4. Variation of J_2 with the elasticity ratio m_λ for various values of the depth ratio, n, for the case of viscosity ratio $m_\eta = 0.4$.

stabilizing effect of gravity. In these cases, the positive value of J_2 indicates that the elasticity of the liquid destabilizes the flow while the negative value of J_2 indicates that the elasticity behavior in liquid increases the degree of stability of the flow. For $F \geq 1$, the destabilizing effect of viscosity variation in Newtonian fluid is not completely overshadowed by the stabilizing effect of gravity, at least for $\gamma \leq 1.4$. The oppositeness in sign of J_1 and J_2 is still there for certain values of n, m_η, and m_λ. In order to shorten the paper, the numerical results for J_1 are not presented. For the case of $\gamma = 1$, the value of J_1 can be found in Yih's[8] paper.

From the foregoing it can be concluded that the elasticity of the liquids destabilizes the flow for certain values of n, m_η, and m_λ, but stabilizes it for other values of these variables. The detailed results are shown in the figures mentioned before.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to Professor Chia-Shun Yih for his guidance and encouragement during the period of this research.

The financial support from the National Science Foundation and the Army Research Office (Durham) is highly appreciated.