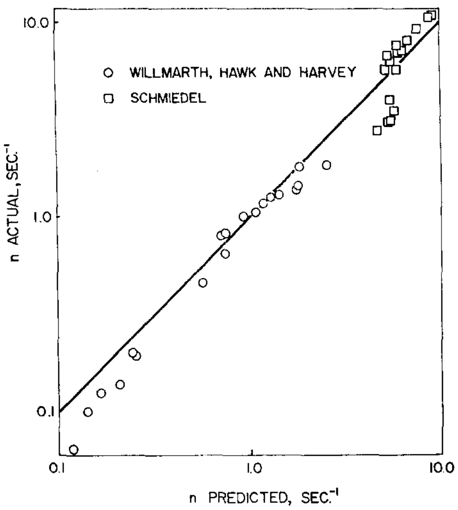

FIG. 1. Pitching frequencies.

obtained of the torque T tending to restore the disk to the normal orientation.

The angular force balance on the disk may then be written. In the absence of any other consideration,

$$I\ d^2\alpha/d\theta^2 = -T,$$

where I is the moment of inertia of the disk; that is

$$(\pi/64)(\rho_p D^4 t)\ d^2\alpha/d\theta^2 = -(0.44)(C_D/8\pi)(\rho U^2 D^3)\alpha. \quad (3)$$

The pitching motion is thus simple harmonic and it has a frequency of

$$n = 0.169U[\rho C_D/(\pi\rho_p tD)]^{\frac{1}{2}}. \quad (4)$$

ρ and ρ_p are the densities of the fluid and particle, respectively; t and D are the thickness and diameter of the disk.

Equation (4) has been applied to the data of Schmiedel, using a value of 1.65 for C_D, which is the approximate average of the values he reported. Willmarth's drag coefficient data are too scattered to be useful. The commonly accepted value of C_D for the Reynolds number range in which Willmarth's data lie is 1.2, and this value is used in the present calculation. The frequencies obtained from Eq. (4) are compared in Fig. 1 with the observed frequencies.

Some weaknesses in the present treatment should be pointed out. It takes no account of the lateral oscillation of the disk. This motion, which has the same frequency as the pitching, undoubtedly plays a part in determining that frequency. It may also help set the amplitude of the pitching motion. Viscous forces are neglected in the present model and, as a result, the pressure force normal to disks of low Re is probably overestimated. This consideration may explain why five of Schmiedel's data points lie significantly below the predicted curve. Finally, in setting up the angular force balance, the fact is neglected that the fluid close to the disk follows its oscillations and therefore contributes an "added moment of inertia." This consideration assumes importance for particles of low I. It is likely responsible for the departure from the curve of Willmarth, Hawk, and Harvey's low-frequency data all of which were obtained from particles with low moments of inertia.

[1] J. Schmiedel, Physik. Z. **29**, 593 (1928).
[2] W. W. Willmarth, N. E. Hawk, and R. L. Harvey, Phys. Fluids **7**, 197 (1964).
[3] E. K. Marchildon, A. Clamen and W. H. Gauvin, Can. J. Chem. Eng. **42**, 178 (1964).
[4] A. Fage and F. C. Johansen, Proc. Roy. Soc. (London) **A116**, 170 (1927).
[5] Lord Rayleigh, Phil. Mag. (5th Ser.) **2**, 430 (1876).

Reply to the Comments by E. K. Marchildon, A. Clamen, and W. H. Gauvin

WILLIAM W. WILLMARTH*
Department of Aeronautical and Astronautical Engineering, University of Michigan, Ann Arbor, Michigan.

(Received 3 July 1964)

MARCHILDON, Clamen, and Gauvin give an expression for the frequency of oscillation of freely falling disks, Eq. (4). Their Eq. (4), with $C_D = 1.2$, may be written in the dimensionless form $nd/U = 0.023(I^*)^{-\frac{1}{2}}$. I would like to compare this expression with disk oscillation results taken from Figs. 12 and 13 of our paper,[1] Schmiedel's results,[2] and a more complete set of our data.[3] The comparison is shown in Fig. 1 and includes 29 additional data points that did not appear in our abbreviated Table III[1] or in Marchildon, Clamen, and Gauvin's figure.

It is evident from Fig. 1 and from Figs. 12 and 13 of Ref. (1) that nd/U is not proportional to $(I^*)^{-\frac{1}{2}}$. It has already been demonstrated that nd/U depends in a complicated manner upon both Reynolds number and I^* (see discussion and Fig. 11 of Ref. 1).

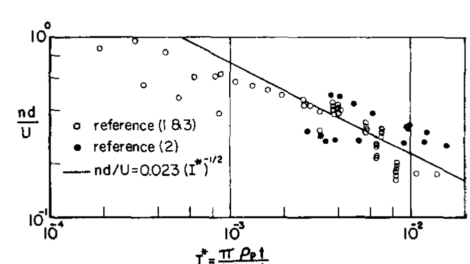

FIG 1. Comparison between measurements and Marchildon, Clamen, and Gauvin's Eq. (4).

I would also like to mention that when I^* is small ($I^* < 10^{-3}$) and Reynolds number relatively large ($\mathrm{Re} > 10^3$) the amplitude of the lateral oscillation of freely falling disks becomes large. Appreciable lift is developed as the disk "flies" through the fluid during large lateral oscillations. In this case the average upward force (which we have called drag) is large. Our drag measurements, which vary from $C_D = 0.99$ to $C_D = 5.76$, show the development of considerable lift during large amplitude lateral oscillations. When lift is being produced there will be a change in the flow field and torque acting on the disk. This effect, in addition to the viscous and apparent mass effects discussed by Marchildon, Clamen, and Gauvin, will place additional limitations on the validity of their analysis leading to Eq. (4).

*On leave of absence at the Joint Institute for Laboratory Astrophysics, University of Colorado, Boulder, Colorado.

[1] W. W. Willmarth, N. E. Hawk, and R. L. Harvey, Phys. Fluids 7, 197 (1964).
[2] J. Schmiedel, Physik Z. 29, 594 (1928).
[3] W. W. Willmarth, N. E. Hawk, and R. L. Harvey, Aerospace Research Laboratory Report ARL 64-19 (1964).

Comment on "Free Molecule Density Field for Orifice Flow"

Roddam Narasimha
Department of Aeronautical Engineering
Indian Institute of Science, Bangalore, India
(Received 6 May 1964)

In Ref. 1, Gustafson and Kiel have expressed the density field in free molecule flow through a circular orifice as an integral which they have evaluated numerically, noting that when ρ (which is the polar distance to any field point from the centre of the orifice in terms of the radius of the orifice) is unity, the result is a complete elliptic integral of the first kind. They also quote Howard[2] as giving the result for the density along the centreline of the orifice.

The purpose of the present note is to point out that the author has previously given[3] a closed expression for the density at any point in the field, i.e., for *all* ρ, in terms of already-tabulated elliptic integrals. Further details will be found in Ref. 3, which includes some curves of the density distribution and a discussion of other flow quantities as well.

[1] W. A. Gustafson and R. E. Kiel, Phys. Fluids 7, 472 (1964).
[2] W. M. Howard, Phys. Fluids 4, 521 (1961).
[3] R. Narasimha, J. Fluid Mech. 10, 371 (1961).

Reply to the Comments by R. Narasimha

W. A. Gustafson and R. E. Kiel
Purdue University, Lafayette, Indiana
(Received 8 June 1964)

We wish to thank R. Narasimha[1] for pointing out his work on orifice flow, and we regret that a reference to his paper was omitted from our note.[2]

The primary purpose of Ref. 2 was to treat the free molecular flow through a rectangular and a circular orifice as an initial value problem of the collisionless Boltzmann equation. One can then proceed in a purely mathematical way to obtain the distribution function in terms of its initial value and certain discontinuous functions involving the characteristics of the partial differential equation. The density field is then determined by an integration in velocity space. In the papers of both Narasimha[3] and Howard[4] the determination of the density has been reduced to a geometrical calculation in physical space of the solid angle subtended by the orifice opening at an arbitrary point. Hence, there is some difference in the point of view between these papers and Ref. 2.

Narasimha[1] has erroneously stated that the symbol $\rho \equiv r/R$ in Ref. 2 represents the nondimensional polar distance to any point in the field from the center of the orifice. Our formulation of the circular orifice problem was based on cylindrical coordinates in both velocity and physical space. Hence, ρ represents the radial distance normal to the axis of symmetry of the orifice (z axis). It should also be noted that the angle θ, which in Ref. 2 is the variable of integration in the final expression for the density, is an angle in velocity space and after the integration we obtain $n = n(\rho, Z)$. Thus, the notation (ρ, θ) or (r, θ) in Ref. 2 refers to completely different quantities than do (r, θ) in Narasimha's paper.[3]

A paper by Naito[5] has recently come to our attention which treats in considerable detail the computation of the solid angle subtended by a circular aperture. His results are given in terms of elliptic integrals. Narasimha[3] has expressed the solid angle in terms of Heuman's lambda function and the complete elliptic integral of the first kind. Since the lambda function[6] can be expressed in terms of elliptic integrals, one can show with a little manipulation that Narasimha's expression for the solid angle is identical to that given by Naito's first formula. One might expect that the results of Ref. 2 could also be related to those of Ref. 5, however, this has not yet been accomplished.

[1] R. Narasimha, Phys. Fluids 7, 2020 (1964).
[2] W. A. Gustafson and R. E. Kiel, Phys. Fluids 7, 472 (1964).
[3] R. Narasimha, J. Fluid Mech. 10, 371 (1961).
[4] W. M. Howard, Phys. Fluids 4, 521 (1961).
[5] M. Naito, J. Phys. Soc. Japan 12, 1122 (1957).
[6] C. Heuman, J. Math. Phys. 20, 127 (1941).