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Green's Distributions and the Cauchy Problem for the Multi-Mass Klein-Gordon Operator

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Explicit forms of the Green's functions (which are to be regarded as distributions in the sense of Schwartz) for the multi-mass Klein-Gordon operator in n-dimensional spaces are presented. The homogeneous Green's functions $G_N(x)$ and $G_N{}^1(x)$, defined in the usual way by independent paths of integration in the k_0 plane, are investigated in the neighborhood of the light cone. The parameter N indicates the total number of masses involved. The singularities on the light cone reflect the well-known difference between even- and odd-dimensional wave propagation. It is found that $G_N(x;\text{ odd } n)$ contains a finite jump on the light cone as well as a linear combination of derivatives up to order $\frac{1}{2}(n-2N-1)$ of $\delta(x^2)$; the singular part of $G_N{}^1(x;\text{ odd } n)$ consists of a logarithmic singularity $\ln(|x^2|)$ along with a polynomial in $(x^2)^{-1}$ of degree $\frac{1}{2}(n-2N-1)$. For even-dimensional spaces, the singular part of both Green's functions consists of a polynomial in $(x^2)^{-1/2}$ of degree $n-2N+1$ vanishing *outside* the light cone for G_N and vanishing *inside* the light cone for $G_N{}^1$. In all cases no singularities or finite jumps occur when the order $2N$ of the operator is greater than the number $n+1$ of space-time dimensions. The general solution of the Cauchy problem is given both for the data carrying surface $t = 0$ and for arbitrary spacelike data surfaces.

1. INTRODUCTION

MULTI-MASS equations have enjoyed a long history in field theory and often arise from attempts to eliminate the typical divergences occurring within the theory.[1] For example, the regularized propagators of Pauli and Villars,[2] obtained with the help of an introduction of discrete auxiliary masses, may easily be seen to satisfy equations of the form

$$F(\Box)\varphi(x) = \rho(x), \tag{1}$$

where

$$F(\Box) = \prod_i (\Box + \mu_i^2). \tag{2}$$

In the neighborhood of the light cone, no singularities or finite jump discontinuities appear in the propagators provided the total number of masses is at least three.

Equations of the type given in (1) have been investigated in detail by Pais and Uhlenbeck.[3] As these authors have shown, if $F(\Box)$ is a polynomial in $\Box$ with arbitrary constant coefficients, then (1) is an equation of hyperbolic type and a well-defined initial value of Cauchy problem exists for the homogeneous equation. The case of polynomials in $\Box$ has also been treated by Rzewuski,[4] who gives an invariant form of the general solution of the homogeneous equation, but leaves the corresponding Green's functions implicit in their Fourier representations.

The present paper is devoted to the Green's functions and the Cauchy problem for the multi-mass Klein-Gordon operator in multi-dimensional spaces. To fix the notation, we shall seek a solution of the Cauchy problem for the equation

$$(\Box + \mu_1^2)^{\lambda_1} \cdots (\Box + \mu_L^2)^{\lambda_L}\varphi(x) = 0, \tag{3}$$

where the d'Alembert operator is given by

$$\Box = \partial_0^2 - \partial_1^2 - \cdots - \partial_n^2. \tag{4}$$

The sum of the nonnegative integers $\lambda_1 \cdots \lambda_L$ will be denoted

$$N \equiv \lambda_1 + \cdots + \lambda_L \tag{5}$$

where $N = 1, 2, \cdots$ gives the total number of rest masses involved. The rest masses $\mu_1, \cdots, \mu_L$, will be taken to be distinct, real and positive (although many of the results can be extended to complex μ_ν); if one of them must be zero, the limit $\mu \to 0$ may be taken after the calculations are made. Clearly, the wave equation is of order $2N$; hence, the Cauchy data on the spacelike plane $t = 0$ consist of the time

[1] Multi-mass equations also arise naturally when one considers particles of higher spin; see, e. g., J. D. Harris, Phys. Rev. **112**, 2124 (1958).
[2] W. Pauli and F. Villars, Revs. Modern Phys. **21**, 434 (1949).
[3] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).
[4] J. Rzewuski, Acta Phys. Polon. **12**, 100 (1953).

derivatives $\partial_0^M \varphi(\mathbf{x}, 0)$ with $M = 0, 1, \cdots, 2N - 1$.

In a previous paper[5] we obtained the explicit form of the homogeneous Green's function $\Delta(x; m; \mu)$ associated with the multi-dimensional, iterated Klein-Gordon operator $(\Box + \mu^2)^m$. The Fourier representation of the Green's function was expressed, after some angular integrations, as a one-dimensional, infinite integral of the Sonine type. It was shown that although this integral is classically divergent when the order of the operator is less than the number of space dimensions, it can be treated rigorously under these conditions using the concepts of distribution analysis. The Green's function is then to be regarded as a (tempered) distribution in the sense of Schwartz.[6] A new distribution introduced for the purpose of giving the improper Sonine integral its generalized meaning was used to investigate the singularities of the Green's function on the light cone.

These results may be easily extended to include the Green's functions associated with the multi-mass operator $(\Box + \mu_1^2)^{\lambda_1} \cdots (\Box + \mu_L^2)^{\lambda_L}$ either by a partial fraction decomposition of the integrand of the Fourier representations,[7] or by the method given in Sec. 2. Explicit expressions for the complete set of Green's functions for the multi-mass operator will be presented; this includes an expression for $\Delta^1(x; m; \mu)$ which has not been previously given in (I). (The two distributions Δ and Δ^1 are linearly independent.)

The behavior of the Green's functions in the neighborhood of the light cone will be explicitly investigated in Sec. 3. Because of the well-known difference between wave propagation in spaces with an even, and spaces with an odd, number of dimensions, the two cases must be treated separately. Some very interesting results are obtained. For *odd-dimensional spaces*, the Green's function $G_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$, intimately related to $\Delta(x; m; \mu)$, contains a finite linear combination of derivatives of the Dirac delta function $\delta(x^2)$ as well as a finite jump discontinuity on the light cone. The highest derivative appearing is of order $\frac{1}{2}(n-2N-1)$. The singular part of $G_N^1(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$, closely connected with $\Delta^1(x; m; \mu)$, consists of a polynomial in $(1/x^2)$ of degree $\frac{1}{2}(n - 2N - 1)$ along with a logarithmic singularity $\ln(|x^2|)$. On the other hand, for even-dimensional spaces, the singular part of both Green's functions consists essentially of a polynomial in $1/(x^2)^{1/2}$ of degree $(n - 2N + 1)$; but the polynomial vanishes *outside* the light cone for G_N, and vanishes *inside* the light cone for G_N^1. In all cases, no singularities or finite jumps appear when the order of the operator is greater than the number of space-time dimensions (i.e., $2N > n + 1$). When this is true, the first $N - (n + 3)/2$ derivatives with respect to x^2 when n is odd, or first $N - 1 - n/2$ derivatives when n is even, of both $G_N(x)$ and $G_N^1(x)$, are also continuous on the light cone.

The remainder of the paper is concerned with obtaining the general solution of Cauchy's problem for the homogeneous equation (3). In this enterprise, the set of Green's functions $G_\eta(x; r_1 \cdots r_L; \mu_1 \cdots \mu_1)$ with $0 \leq r_p \leq \lambda_p$, $\eta = r_1 + \cdots + r_L$, and $1 \leq \eta \leq N$ is particularly useful. The initial conditions satisfied by these functions are given in Sec. 4 and the general solution $\varphi(x)$ is obtained in Sec. 5. Finally, an invariant form of the general solution is presented in Sec. 6 and agrees with that given by Rzewuski.[4] There are two Appendices.

2. THE GREEN'S FUNCTIONS

All of the Green's functions for the multi-mass operator $(\Box + \mu_1^2)^{\lambda_1} \cdots (\Box + \mu_L^2)^{\lambda_L}$ may be obtained using the general Fourier representation

$$K(x) = (2\pi)^{-n-1} \times \int dk e^{-ikx} (\mu_1^2 - k^2)^{-\lambda_1} \cdots (\mu_L^2 - k^2)^{-\lambda_L}. \qquad (6)$$

As is well known, such an expression is not completely defined until the path of integration around the poles of the integrand has been specified. In the k_0 plane, open paths of integration that coincide with the real k_0 axis at $\pm\infty$ give rise to inhomogeneous Green's functions. Closed paths which encircle one or more of the poles lead to homogeneous Green's functions.

There are only $2L$ independent ways of encircling the $2L$ poles of the integrand. However, by reducing the multiplicity of the poles, $2N - 2L$ more independent solutions of the homogeneous equation may be obtained, giving a total of $2N$ independent homogeneous Green's functions. The general inhomogeneous Green's function may be written in the form of a sum of the particular solution of the equation

$$(\Box + \mu_L^2)^{\lambda_L} \cdots (\Box + \mu_L^2)^{\lambda_L} K(x) = \delta(x) \qquad (7)$$

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[5] J. J. Bowman and J. D. Harris, J. Math. Phys. **3**, 396 (1962), hereafter called (I).

[6] L. Schwartz, *Théorie des distributions I, II* (Hermann et Cie, Paris, 1950–51).

[7] Green's functions for multi-mass operators like $[\Box^m - (-\mu^2)^m]^l$ may be calculated directly without recourse to a partial fraction expansion. Such operators have been investigated by J. J. Bowman and J. D. Harris, J. Math. Phys. **3**, 1291 (1962).

and of the $2N$ independent solutions of the homogeneous equation with arbitrary coefficients. The multi-mass operator thus has a total of $2N + 1$ independent Green's functions.

The homogeneous Green's function $G_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$ is defined by choosing the path of integration in the k_0 plane to consist of a closed curve C encircling *all* of the poles in a clockwise fashion. The notation $G_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$ will often be abbreviated to $G_N(x)$ or simply G_N. The inhomogeneous Green's function $\bar{G}_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$, defined by taking the principal part of the k_0 integration over the singularities, is related to the homogeneous Green's function $G_N(x)$ by the formula

$$G_N(x) = 2\epsilon(x)\bar{G}_N(x), \tag{8}$$

where $\epsilon(x) = \pm 1$ for $t \gtrless 0$. A second homogeneous Green's function $G_N^1(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$ is characterized by a closed path of integration C^1 which encircles all of the poles on the positive real k_0 axis in a clockwise fashion and all of the poles on the negative real k_0 axis in a counterclockwise direction. Clearly the paths C^1 and C are not equivalent, hence G_N^1 and G_N are linearly independent distributions. We will give explicit expressions for $G_N(x)$ and $G_N^1(x)$.

In what follows, the notation $G_h(x)$ will be used to denote any homogeneous Green's function defined by a path of integration $\mathcal{C}$ which consists of a linear combination of the paths C and C^1. Using the theory of residues and the properties of Dirac δ functions, one easily obtains the identity

$$\Gamma(\lambda) \int_{\mathcal{C}} dk(\mu^2 - k^2)^{-\lambda} = 2\pi i \int_{-\infty}^{\infty} dkf(k)\ \delta^{(\lambda-1)}(k^2 - \mu^2), \tag{9}$$

where $f(k)$ is determined by the path $\mathcal{C}$; in particular,

$$f(k) = \begin{cases} \epsilon(k) & \text{for} \quad \mathcal{C} = C \\ 1 & \text{for} \quad \mathcal{C} = C^1. \end{cases} \tag{10}$$

Making use of (9), and taking into account the contribution of all the poles in (6), one may write $G_h(x)$ in the form

$$G_h(x) = \sum_{p=1}^{L} \frac{i(2\pi)^{-n}}{\Gamma(\lambda_p)} \int_{-\infty}^{\infty} dk e^{-ikx} f(k) \times \delta^{(\lambda_p-1)}(k^2 - \mu_p^2) \prod_{j=1}^{L}{}' (\mu_j^2 - k^2)^{-\lambda_j}, \tag{11}$$

where the prime on the product means that the factor for $j = p$ is to be omitted. Each summand can be expressed in terms of the operator

$$\frac{1}{\Gamma(\lambda_p)} \left(-\frac{d}{d\mu_p^2}\right)^{\lambda_p - 1}$$

acting on

$$i(2\pi)^{-n} \int_{-\infty}^{\infty} dk e^{-ikx} f(k)\ \delta(k^2 - \mu_p^2) \prod_{j=1}^{L}{}' (\mu_j^2 - k^2)^{-\lambda_j} = \Delta_h(x; \mu_p)\Phi(-\mu_p^2), \tag{12}$$

with Δ_h representing the corresponding Green's function for the Klein-Gordon operator $\square + \mu^2$, and

$$\Phi(-\mu_p^2) \equiv \prod_{j=1}^{L}{}' (\mu_j^2 - \mu_p^2)^{-\lambda_j}. \tag{13}$$

Finally we obtain

$$G_h(x) = \sum_{p=1}^{L} \sum_{m=1}^{\lambda_p} \frac{\Phi^{(\lambda_p - m)}(-\mu_p^2)}{(\lambda_p - m)!} \Delta_h(x; m; \mu_p) \tag{14}$$

by applying the Leibniz rule for differentiating a produce of two functions and using the result

$$\frac{1}{\Gamma(m)} \left(-\frac{d}{d\mu^2}\right)^{m-1} \Delta_h(x; \mu) = \Delta_h(x; m; \mu), \tag{15}$$

where $\Delta_h(x; m; \mu)$ is the homogeneous Green's function for the iterated Klein-Gordon operator $(\square + \mu^2)^m$. We note that (15) is easily obtained from the Fourier representation

$$\Delta_h(x; m; \mu) = (2\pi)^{-n-1} \int_{\mathcal{C}} dk e^{-ikx} (\mu^2 - k^2)^{-m}, \tag{16}$$

although a rigorous justification of the identity follows only from considerations of distribution theory [cf. (I)].

The result expressed in (14) may alternatively be obtained using the general partial fraction expansion (Appendix A)

$$(\mu_1^2 - k^2)^{-\lambda_1} \cdots (\mu_L^2 - k^2)^{-\lambda_L} = \sum_{p=1}^{L} \sum_{m=1}^{\lambda_p} \frac{\Phi^{(\lambda_p - m)}(-\mu_p^2)}{(\lambda_p - m)!} (\mu_p^2 - k^2)^{-m} \tag{17}$$

for the integrand of (6). Equation (14) follows immediately.

Equation (14) yields an expansion of $G_N(x)$ in terms of the N independent Δ solutions of the homogeneous equation (3); similarly, an expansion of $G_N^1(x)$ in terms of the N independent Δ^1 solutions is obtained. Clearly the distributions $G_\eta(x; r_1 \cdots r_L; \mu_1 \cdots \mu_L)$ and $G_\eta^1(x; r_1 \cdots r_1; \mu_1 \cdots \mu_L)$ with $0 \leq r_p \leq \lambda_p$, $\eta = r_1 + \cdots = r_L$, and $1 \leq \eta \leq N$ are also solutions of the homogeneous equation. There is a large degeneracy with respect to the parameter η and not all of these solutions are linearly independent. Out of the combined set $\{G_\eta, G_\eta^1\}$ one may choose $2N$ independent homogeneous solutions; in particular, the functions $\Delta(x; m; \mu_p)$ and $\Delta^1(x; m; \mu_p)$ form a complete set.

Explicit forms of the distributions Δ and Δ^1 may be calculated (see Appendix B) using the method set forth in (I); the results are

$$\Delta(x; m; \mu) = \epsilon(x) \operatorname{Re} \mathbf{K}(x; m; \mu), \quad (18)$$

$$\Delta^1(x; m; \mu) = i \operatorname{Im} \mathbf{K}(x; m; \mu), \quad (19)$$

where

$$\mathbf{K}(x; m; \mu) = \frac{(2\pi)^{(1-n)/2}}{\Gamma(m)2^m}\left(\frac{\sqrt{x^2}}{\mu}\right)^{m-(n+1)/2} \times H^{(2)}_{m-(n+1)/2}(\mu\sqrt{x^2}). \quad (20)$$

Here $H_\nu^{(2)}$ is the Hankel function of the second kind.[8] We stress the fact that $\Delta(x; m; \mu)$ and $\Delta^1(x; m; \mu)$ are in general to be considered as distributions. The $\Delta(x; m; \mu)$ vanish outside the light cone, whereas the $\Delta^1(x; m; \mu)$ do not.

The general homogeneous Green's function for the multi-mass operator $(\square + \mu_1^2)^{\lambda_1} \cdots (\square + \mu_L^2)^{\lambda_L}$ is now obviously

$$G_{\text{hom}}(x) = \sum_{p=1}^{L}\sum_{m=1}^{\lambda_p}[a_{pm}\,\Delta(x; m; \mu_p) + b_{pm}\,\Delta^1(x; m; \mu_p)], \quad (21)$$

where a_{pm} and b_{pm} are some constants which may involve the μ_p. The general inhomogeneous Green's function for that operator is therefore

$$G_{\text{inhom}}(x) = \bar{G}_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L) + G_{\text{hom}}(x), \quad (22)$$

where $G_{\text{hom}}(x)$ is a linear combination of the N functions $\Delta(x; m; \mu_p)$ vanishing outside, and the N functions $\Delta^1(x; m; \mu_p)$ not vanishing outside the light cone.

3. SINGULARITIES OF THE GREEN'S FUNCTIONS

Using the results of the preceding section, we may write the distributions $G_N(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$ and $G_N^1(x; \lambda_1 \cdots \lambda_L; \mu_1 \cdots \mu_L)$ in the following manner:

$$\{G_N, G_N^1\} = \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{-d}{d\mu_p^2}\right)^{\lambda_p-1}\Phi(-\mu_p^2) \times \{\Delta(x; \mu_p), \Delta^1(x; \mu_p)\}, \quad (23)$$

where the Green's functions for the Klein-Gordon operator have the explicit forms

$$\Delta(x; \mu) = \frac{\epsilon(x)\theta(x^2)}{2}\left(\frac{\mu}{2\pi}\right)^{(n-1)/2}(\sqrt{x^2})^{(1-n)/2} \times J_{(1-n)/2}(\mu\sqrt{x^2}), \quad (24)$$

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed.

$$\Delta^1(x; \mu) = -i\frac{\theta(x^2)}{2}\left(\frac{\mu}{2\pi}\right)^{(n-1)/2}(\sqrt{x^2})^{(1-n)/2} \times Y_{(1-n)/2}(\mu\sqrt{x^2}) + i\frac{\theta(-x^2)}{2}\frac{2}{\pi}\left(\frac{\mu}{2\pi}\right)^{(n-1)/2}(\sqrt{-x^2})^{(1-n)/2} \times K_{(1-n)/2}(\mu\sqrt{-x^2}). \quad (25)$$

In the above, J_ν is the Bessel function of the first kind, Y_ν the Neumann function, K_ν the modified Bessel function of the third kind, and $\theta(y) = 1$ for $y > 0$, $\theta(y) = 0$ for $y < 0$.

The cases of even- and odd-dimensional spaces must be considered separately in order to investigate the nature of the singularities near the light cone. The singular part of $\Delta(x; \mu)$ has already been determined in (I); for small x^2 we find

$$\Delta(x; \mu) \simeq \frac{\epsilon(x)}{2}\left(\frac{-\mu^2}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n-1)/2}\frac{(-\mu^2/4)^{-m}\theta^{(m)}(x^2)}{\Gamma(n/2 - m + \frac{1}{2})}\right\} \quad (n \text{ odd}) \quad (26)$$

$$\Delta(x; \mu) \simeq \frac{\epsilon(x)\theta(x^2)}{2}\left(\frac{-\mu^2}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(1/2)n-1}\frac{(-\mu^2/4)^{-m-1/2}(x^2)^{-m-1/2}}{\Gamma(n/2 - m)\Gamma(\frac{1}{2} - m)}\right\} \quad (n \text{ even}) \quad (27)$$

where the terms that vanish for $x^2 \to 0$ have been omitted. Since $\theta'(x^2) = \delta(x^2)$, one finds that $\Delta(x; \mu$: odd $n)$ contains δ-function singularities as well as a finite jump discontinuity on the light cone; the singular part of $\Delta(x; \mu$: even $n)$ consists of a polynomial in $(x^2)^{-1/2}$ vanishing outside the light cone, and no finite discontinuity is present.

The singular part of $\Delta^1(x; \mu)$ may easily be obtained using well-known formulas[8] of the Bessel functions. In the neighborhood of the light cone, one finds, for odd dimensional spaces

$$\Delta^1(x; \mu) \simeq -\frac{i}{\pi}\left(\frac{-\mu^2}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n-1)/2}\frac{(-\mu^2/4)^{-m}}{\Gamma(n/2 - m + \frac{1}{2})}\left(\frac{d}{dx^2}\right)^m \ln(|x^2|)^{1/2}\right\} + \frac{i}{\pi}\left(\frac{-\mu^2}{4\pi}\right)^{(n-1)/2}\frac{h(n; \mu)}{\Gamma(n/2 + \frac{1}{2})} \quad (n \text{ odd}) \quad (28)$$

where

$$h(n; \mu) \equiv \tfrac{1}{2}[1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + 2/(n-1)] - \ln(\gamma\mu/2), \quad (29)$$

with $\ln\gamma$ $(=0.5772\cdots)$ representing the Euler-

Mascheroni constant. For even-dimensional spaces we have

$$\Delta^1(x;\mu) \simeq \frac{\theta(-x^2)}{2}\left(\frac{-\mu^2}{4\pi}\right)^{(1/2)(n-1)} \times \left\{\sum_{m=0}^{(1/2)n-1} \frac{(-\mu^2/4)^{-m-1/2}(x^2)^{-m-1/2}}{\Gamma(n/2-m)\Gamma(\frac{1}{2}-m)}\right\} - \tfrac{1}{2}\left(\frac{-\mu^2}{4\pi}\right)^{(1/2)(n-1)} \frac{1}{\Gamma(n/2+\frac{1}{2})} \quad (n \text{ even}), \tag{30}$$

where the factor of i is contained implicitly. In both (28) and (30), the terms that vanish for $x^2 \to 0$ have been omitted. Again, the remarkable difference between even- and odd-dimensional spaces is reflected in the nature of the singularities on the light cone: the singular part of $\Delta^1(x;\mu$: even $n)$ consists of a polynomial in $(x^2)^{-1/2}$ vanishing *inside* the light cone, whereas $\Delta^1(x;\mu$: odd $n)$ contains logarithmic and multiple pole singularities.

When one substitutes (24) and (25) into (23) to get explicit expressions for $G_N(x)$ and $G_N^1(x)$, one finds that sums of the form

$$S(l) = \sum_{p=1}^{L} \frac{1}{\Gamma(\lambda_p)}\left(\frac{-d}{d\mu_p^2}\right)^{\lambda_p-1}(-\mu_p^2)^l\Phi(-\mu_p^2) \tag{31}$$

must be evaluated. Such sums are easily determined explicitly (when l is an integer) using the algebra of partial fractions (see Appendix A); in particular we have

$$S(l) = \begin{cases} 0 & \text{for } l = 0, 1, \cdots, N-2; \\ 1 & \text{for } l = N-1. \end{cases} \tag{32}$$

Carrying out the indicated procedure, one easily finds the following explicit results:

For both even and odd values of n,

$$G_N(x) = \frac{\epsilon(x)}{2\pi^{(n-1)/2}} \sum_{m=0}^{\infty} \frac{S(N+m-1)}{4^{N+m-1}\Gamma(N+m)} \times \mathfrak{H}_{N+(1-n)/2+m}(x^2), \tag{33}$$

where [cf. I, Eqs. (14) to (22)]

$$\mathfrak{H}_\beta(y) = \theta(y)y^{\beta-1}/\Gamma(\beta) \qquad (\beta \neq 0, -1, -2, \cdots), \tag{34a}$$

$$\mathfrak{H}_{-\beta}(y) = \delta^{(\beta)}(y) \qquad (\beta = 0, 1, 2, \cdots). \tag{34b}$$

For even n,

$$G_N^1(x) = \tfrac{1}{2}\left(\frac{1}{\pi}\right)^{(n-1)/2} \times \left\{\theta(-x^2)\sum_{m=0}^{\infty} \frac{S(m)(x^2)^{m+(1-n)/2}}{4^m\Gamma(m+1)\Gamma[(3-n)/2+m]} - \sum_{m=0}^{\infty} \frac{S[m+(n-1)/2](x^2)^m}{4^{m+(n-1)/2}\Gamma(m+1)\Gamma[(n+1)/2+m]}\right\}. \tag{35}$$

For odd n,

$$G_N^1(x) = \frac{i}{\pi}\left(\frac{1}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{\infty} \frac{(x^2)^m}{4^m\Gamma(m+1)\Gamma((n+1)/2+m)} \times \left[S((n-1)/2+m)\left[\frac{1}{2}\left(1+\cdots+\frac{1}{m}\right) + \frac{1}{2}\left(1+\cdots+\frac{1}{(n-1)/2+m}\right) - \ln\frac{\gamma\,|x^2|^{1/2}}{2}\right] - \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(-\frac{d}{d\mu_p^2}\right)^{\lambda_p-1}(-\mu_p^2)^{(n-1)/2+m}\Phi(-\mu_p^2)\ln\mu_p\right] - \sum_{m=1}^{(n-1)/2} \frac{S((n-1)/2-m)}{4^{-m}\Gamma((n+1)/2-m)}\left(\frac{d}{dx^2}\right)^m \ln|x^2|^{1/2}\right\}. \tag{36}$$

where the sum $1 + \cdots + 1/m$ is understood to be zero when $m = 0$.

From Eq. (32) and the immediately preceding Eqs. (33) to (36), it easily follows that on the light cone $x^2 = 0$,

(i) There are no singularities in $G_N(x)$, and $G_N^1(x)$ contains no multiple pole singularities, when $2N > n$, that is, when the order of the wave equation is greater than the number of space dimensions. However, for odd n and $2N = n + 1$, a finite jump discontinuity occurs in $G_N(x)$ and a logarithmic singularity occurs in $G_N^1(x)$.

(ii) Both $G_N(x)$ and $G_N^1(x)$ are continuous together with their first $N - (n+3)/2$ derivatives if n is odd, or with their first $N - 1 - n/2$ derivatives if n is even, provided that $2N > n + 1$, that is provided the order of the wave equation is greater than the number of space-time dimensions.

We observe that the inequality $2N > n + 1$ determines the minimum number of masses required for the regularization of the n-dimensional propagators according to the Pauli-Villars[2] procedure.

When $2N \leq n$, one may write, by means of (A6), expressions which exhibit the singularities of $G_N(x)$ and $G_N^1(x)$ explicitly. For odd-dimensional spaces we find

$$G_N^{(\text{sing})}(x) = \frac{\epsilon(x)}{2}\left(\frac{1}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n+1)/2-N} \frac{S(n/2-m-\frac{1}{2})}{4^{-m}\Gamma(n/2-m+\frac{1}{2})}\,\theta^{(m)}(x^2)\right\}, \tag{37}$$

$$G_N^{1(\text{sing})}(x) = -\frac{i}{\pi}\left(\frac{1}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n+1)/2-N} \frac{S(n/2-m-\frac{1}{2})}{4^{-m}\Gamma(n/2-m+\frac{1}{2})} \times \left(\frac{d}{dx^2}\right)^m \ln(|x^2|)^{1/2}\right\}. \tag{38}$$

It is clear that $G_N(x;\text{ odd } n)$ contains a finite jump $\theta(x^2)$ on the light cone in addition to a linear combination of derivatives up to order $\frac{1}{2}(n-2N-1)$ of $\delta(x^2)$, whereas the singular part of $G_N^1(x;\text{ odd } n)$ consists of a logarithmic singularity $\ln(|x^2|)$ along with a polynomial in $(x^2)^{-1}$ of degree $\frac{1}{2}(n-2N-1)$.

For spaces with an even number of dimensions, one obtains

$$G_N^{(\text{sing})}(x) = \frac{\epsilon(x)\theta(x^2)}{2}\left(\frac{1}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n-N)/2} \frac{S(n/2-m-1)(x^2)^{-m-1/2}}{4^{-m-1/2}\Gamma(n/2-m)\Gamma(\frac{1}{2}-m)}\right\}, \quad (39)$$

$$G_N^{1(\text{sing})}(x) = \frac{\theta(-x^2)}{2}\left(\frac{1}{4\pi}\right)^{(n-1)/2} \times \left\{\sum_{m=0}^{(n-N)/2} \frac{S(n/2-m-1)(x^2)^{-m-1/2}}{4^{-m-1/2}\Gamma(n/2-m)\Gamma(\frac{1}{2}-m)}\right\}. \quad (40)$$

Here the singular part of the two Green's functions consists of a polynomial in $(x^2)^{-1/2}$ of degree $n-2N+1$, with $G_N^{(\text{sing})}(x;\text{ even } n)$ vanishing *outside*, and $G_N^{1(\text{sing})}(x;\text{ even } n)$ vanishing *inside*, the light cone.

In the last four equations, the sum $S(l)$ is given by

$$S(N+q-1) = \sum_{\zeta_1+\cdots+\zeta_L=q} \frac{(\lambda_1)_{\zeta_1}\cdots(\lambda_L)_{\zeta_L}}{\zeta_1!\cdots\zeta_L!} \times(-\mu_1^2)^{\zeta_1}\cdots(-\mu_L^2)^{\zeta_L}, \quad (41)$$

where $q=0,1,\cdots$ and $(\lambda_p)_{\zeta_p}=\lambda_p(\lambda_p+1)\cdots(\lambda_p+\zeta_p-1)$.

Finally, it is clear that $G_{\text{hom}}(x)$ in (21) is regular on the light cone, provided

$$\sum_{p=1}^{L}\sum_{m=1}^{\lambda_p}\frac{C_{pm}}{\Gamma(m)}\left(\frac{-d}{d\mu_p^2}\right)^{m-1}(-\mu_p^2)^l = 0 \quad (l=0,1,\cdots,\nu) \quad (42)$$

where $C_{pm}=a_{pm}+b_{pm}$ and

$$\nu = \begin{Bmatrix}\frac{1}{2}(n-1)\\ \frac{1}{2}n-1\end{Bmatrix} \text{ for } \begin{Bmatrix}\text{odd } n\\ \text{even } n\end{Bmatrix}. \quad (43)$$

In the case of a multi-mass operator with *distinct* rest masses ($\lambda_p\equiv 1$), the singularities and finite jumps cancel if

$$\sum_{p=1}^{L} C_p(\mu_p^2)^l = 0 \qquad (l=0,1,\cdots,\nu). \quad (44)$$

These equations represent the multi-dimensional analog of the well-known regularization conditions of Pauli and Villars,[2] guaranteeing the absence of singularities on the light cone.

If one considers (42) with the condition $\nu=N-2$ instead of the condition (43), the homogeneous system (42) contains $N-1$ equations in the N parameters C_{pm}. From (31) and (32), after applying Leibniz's rule, it follows that

$$C_{pm} = \text{const}\times\frac{1}{\Gamma(\lambda_p-m+1)}\left(-\frac{d}{d\mu_p^2}\right)^{\lambda_p-m}\Phi(-\mu_p^2) \quad (45)$$

is a solution of this system. Since the solution of such a system is determined within a multiplicative constant, it is the only solution, apart from the trivial one

$$C_{pm}=0. \quad (46)$$

Substituting these solutions in (21) and remembering (15) and (23), it is easily seen that the regularization condition (42) with $\nu=N-2$ is satisfied only for the homogeneous functions

$$G_{\text{hom}} = AG_N + BG_N^1 \quad (47)$$

and

$$G_{\text{hom}} = \sum_{p=1}^{L}\sum_{m=1}^{\lambda_p} d_{pm}[\Delta(x;m;\mu_p)-\Delta^1(x;m;\mu_p)], \quad (48)$$

where A, B, and the d_{pm} are arbitrary constants. Since, as one can now see, Eq. (42) with $\nu=N-2$ is equivalent to the vanishing on the light cone of the first $N-(n+3)/2$ (if n is odd) or $N-n/2-1$ (if n is even) derivatives of G_{hom} with respect to x^2, it follows that no Green's function vanishing outside the light cone can be more regular than G_N, and that G_N is determined uniquely within a multiplicative constant by these regularity and causality conditions.

4. INITIAL CONDITIONS OF THE GREEN'S FUNCTIONS

It will become evident that the set of homogeneous Green's functions $G_\eta(x;r_1\cdots r_L;\mu_1\cdots\mu_L)$ with $0\le r_p\le\lambda_p$, $\eta=r_1+\cdots+r_L$, and $1\le\eta\le N$ is particularly useful for constructing the solution of the Cauchy problem. For this reason, we shall briefly investigate the behavior of these functions and their time derivatives at $t=0$. The initial behavior of the G_η^1 functions will not be considered, aside from the obvious statement that for $m=0,1,2,\cdots$ the equations

$$\partial_0^{2m+1}G_\eta^1(\mathbf{x},0)=0, \quad (49)$$

$$\partial_0^{2m}G_\eta(\mathbf{x},0)=0 \quad (50)$$

follow immediately from (B1) and (B2). Because of (50), we need only calculate the time derivatives of odd order for $G_\eta(x)$.

There are of course many ways of performing the calculation, the shortest of which seems to be the following. We use a result given in (I),

$$\partial_0^{2m+1}\,\Delta(\mathbf{x}, 0; \mu) = (\nabla^2 - \mu^2)^m\,\delta(\mathbf{x}), \tag{51}$$

which, when substituted into (23), gives

$$\partial_0^{2m+1}G_N(\mathbf{x}, 0) = \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}(b_p)^m\Phi(b_p)\,\delta(\mathbf{x}), \tag{52}$$

where

$$b_p = \nabla^2 - \mu_p^2 \quad\text{and}\quad \Phi(b_p) = \prod_j{}'\,(b_p - b_j)^{-\lambda_j}.$$

The sum is evaluated explicitly in Appendix A; using those results, we obtain the initial conditions

$$\partial_0^m G_N(\mathbf{x}, 0) = \begin{cases} 0 & \text{for}\quad m = 0, 1, \cdots, 2N-2; \\ \delta(\mathbf{x}) & \text{for}\quad m = 2N-1; \\ 0 & \text{for all even}\quad m; \end{cases} \tag{53}$$

along with

$$\partial_0^{2N+2m-1}G_N(\mathbf{x}, 0; \lambda_1, \cdots, \lambda_L; \mu_1, \cdots, \mu_L) = \sum_{\zeta_1+\cdots+\zeta_L=m}\frac{(\lambda_1)_{\zeta_1}\cdots(\lambda_L)_{\zeta_L}}{\zeta_1!\cdots\zeta_L!} \times (\nabla^2 - \mu_1^2)^{\zeta_1}\cdots(\nabla^2 - \mu_L^2)^{\zeta_L}\,\delta(\mathbf{x}), \tag{54}$$

where $(\lambda_p)_{\zeta_p} = \lambda_p(\lambda_p + 1)\cdots(\lambda_p + \zeta_p - 1)$ is the Pochhammer symbol. Of course similar initial conditions are obtained for the $G_\eta(x; r_1 \cdots r_L; \mu_1 \cdots \mu_L)$.

5. GENERAL SOLUTION OF THE CAUCHY PROBLEM

The solution of

$$(\Box + \mu_1^2)^{\lambda_1}\cdots(\Box + \mu_L^2)^{\lambda_L}\varphi(x) = 0$$

taking specified values for $\varphi(x)$, $\partial_0\varphi(x)$, $\cdots$, $\partial_0^{2N-1}\varphi(x)$ on the spacelike plane $t = 0$, may be written in the form

$$\varphi(x) = \int_{t'=0} d\mathbf{x}' \sum_{m=0}^{N-1}\sum_{\eta=m+1}^{N} A_{m\eta}\,\partial_0^{2\eta-2m-2}G_\eta(x - x') \times (\overleftarrow{\partial_0'} - \overrightarrow{\partial_0'})\,\partial_0'^{2m}\varphi(x'), \tag{55}$$

where the arrows indicate the direction in which the differentiation is to be carried out. For each $m = 0$, $1, \cdots, N-1$, the $N - m$ constants $A_{m\eta}$ are to be determined from the $N - m$ equations

$$\sum_{\eta=m+1}^{N} A_{m\eta}\,\partial_0^{2\eta+2j-1}G_\eta(\mathbf{x}, 0) = \delta_{0j}\,\delta(\mathbf{x}), \tag{56}$$

with $j = 0, 1, \cdots, N - m - 1$.

We first show that (55), subject to the conditions (56), does indeed represent the solution of Cauchy's problem. Consider

$$[\partial_0^{2M}\varphi(x)]_{t=0} = \int_{t'=t=0} d\mathbf{x}' \sum_{m=0}^{M}\sum_{\eta=m+1}^{N} A_{m\eta}\,\partial_0^{2\eta+2(M-m)-1} \times G_\eta(x - x')\,\partial_0'^{2m}\varphi(x'), \tag{57}$$

where the upper limit to the sum over M is determined by (53). Clearly the integral in (55) corresponding to $\overrightarrow{\partial_0'}$ makes no contribution because of (50). Hence, summing over η by virtue of (56), one obtains the desired result

$$\partial_0^{2M}\varphi(\mathbf{x}, 0) = \int d\mathbf{x}' \sum_{m=0}^{M} \delta_{mM}\,\delta(\mathbf{x} - \mathbf{x}')\,\partial_0'^{2m}\varphi(\mathbf{x}', 0). \tag{58}$$

An exactly analogous proof may be used for $\partial_0^{2M+1}\varphi(\mathbf{x}, 0)$; thus (55) is the required solution.

The problem now is to find the $A_{m\eta}$ explicitly. Evidently the existence of a solution to (56) depends on what G_η functions are employed; one must choose an independent set. In the present case, we can guarantee that a solution exists by using *all* of the $G_\eta(x; r_1 \cdots r_L; \mu_1 \cdots \mu_L)$. Because of (54), the conditions (56) may be reduced to algebraic equations for the $A_{m\eta}$; namely,

$$\sum_{\eta=m+1}^{N} A_{m\eta} \sum_{\zeta_1+\cdots+\zeta_L=j}\frac{(r_1)_{\zeta_1}\cdots(r_L)_{\zeta_L}}{\zeta_1!\cdots\zeta_L!} \times (b_1)^{\zeta_1}\cdots(b_L)^{\zeta_L} = \delta_{0j}, \tag{59}$$

where the sum over $\eta = r_1 + \cdots + r_L$ is understood as a sum over all the r_p consistent with $0 \le r_p \le \lambda_p$ and $m + 1 \le r_1 + \cdots + r_L \le N$. Since we expect $A_{m\eta} = A_{m\eta}(r_1 \cdots r_L; \lambda_1 \cdots \lambda_L)$ to be independent of the rest masses, we shall seek a solution of

$$\sum_{\eta=m+1}^{N} A_{m\eta}(r_1)_{\zeta_1}\cdots(r)_{\zeta_L} = \delta_{0j} \tag{60}$$

with $j = \zeta_1 + \cdots + \zeta_L$. Such a solution will, of course, be a solution of (59).

We assert that

$$A_{m\eta} = (-)^{\eta-m-1}\binom{\eta-1}{m}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L} \tag{61}$$

satisfies (60) in addition to satisfying the requirements $0 \le r_p \le \lambda_p$. To prove this, consider the function

$$F(x, b) = -\frac{(-)^m}{m!}\sum_{r_1,\cdots,r_L}(-)^\eta\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}b^{-\eta}x^{\eta-1} = -\frac{(-)^m}{m!}\frac{1}{x}\frac{(b - x)^N}{b^N}. \tag{62}$$

Differentiating with respect to x, one finds

$$(\partial^m/\partial x^m)F(1, b) = -1 + \sum_{\eta=m+1}^{N} (-)^{\eta-m-1} \times \binom{\eta-1}{m}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}b^{-\eta} \tag{63}$$

where the -1 comes from the $r_1 + \cdots + r_L = 0$ term. Differentiating with respect to b, one may write

$$\frac{\partial^j}{\partial b^j}\frac{\partial^m}{\partial x^m}F(1, 1) + \delta_{0j} = \sum_{\zeta_1+\cdots+\zeta_L=j} \frac{(-)^j j!}{\zeta_1!\cdots\zeta_L!}\sum_{\eta=m+1}^{N}(-)^{\eta-m-1} \times \binom{\eta-1}{m}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}(r_1)_{\zeta_1}\cdots(r_L)_{\zeta_L}, \tag{64}$$

where we have used the multinomial differentiation rule

$$\frac{d^N}{dx^N}u_1u_2\cdots u_t = \sum_{r_1+\cdots+r_t=N}\frac{N!}{r_1!\cdots r_t!}u_1^{(r_1)}\cdots u_t^{(r_t)} \tag{65}$$

which follows by induction from Leibniz's rule.

A direct calculation from the second equality in (62) gives

$$\frac{\partial^j}{\partial b^j}\frac{\partial^m}{\partial x^m}F(1, 1) = 0 \tag{66}$$

for $m = 0, 1, \cdots, N-1$ and $j = 0, 1, \cdots, N - m - 1$; therefore,

$$\sum_{\zeta_1+\cdots+\zeta_L=j}\frac{1}{\zeta_1!\cdots\zeta_L!}\sum_{\eta=m+1}^{N}(-)^{\eta-m-1} \times \binom{\eta-1}{m}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}(r_1)_{\zeta_1}\cdots(r_L)_{\zeta_L} = \delta_{0j}. \tag{67}$$

However, because of the symmetry of the summand, we clearly must have

$$\sum_{\eta=m+1}^{N}(-)^{\eta-m-1}\binom{\eta-1}{m}\binom{\lambda_1}{r_1}\cdots \times \binom{\lambda_L}{r_L}(r_1)_{\zeta_1}\cdots(r_L)_{\zeta_L} = \delta_{0j}, \tag{68}$$

so our assertion is proved.[9]

Returning to the original solution (55) of the Cauchy problem, using (61) and summing over m, we find

$$\varphi(x) = \int_{t'=0} d\mathbf{x}' \sum_{\eta=1}^{N}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}G_\eta \times (x - x'; r_1, \cdots, r_L; \mu_1, \cdots, \mu_L) \times (\vec{\partial}_0' - \overleftarrow{\partial}_0')(\vec{\partial}_0'^2 - \overleftarrow{\partial}_0'^2)^{\eta-1}\varphi(x'). \tag{69}$$

All possible G_η solutions are here involved, although only the N functions $\Delta(x; m; \mu_p)$ are ultimately present. Upon introducing the differential operator

$$X \equiv (\vec{\partial}_0 - \overleftarrow{\partial}_0)\sum_{\eta=1}^{N}\binom{\lambda_1}{r_1}\cdots \times \binom{\lambda_L}{r_L}(\vec{\partial}_0^2 - \overleftarrow{\partial}_0^2)^{\eta-1}\prod_{p=1}^{L}(\overleftarrow{\Box} + \mu_p^2)^{\lambda_p - r_p} \tag{70}$$

we can write the Cauchy solution in the neat form

$$\varphi(x) = \int_{t'=0} d\mathbf{x}' G_N(x - x')X'\varphi(x'). \tag{71}$$

We note that X may be written

$$(\vec{\partial}_0^2 - \overleftarrow{\partial}_0^2)X = (\vec{\partial}_0 - \overleftarrow{\partial}_0)\left[\prod_{p=1}^{L}(\overleftarrow{\Box} + \mu_p^2 + \vec{\partial}_0^2 - \overleftarrow{\partial}_0^2)^{\lambda_p} - \prod_{p=1}^{L}(\overleftarrow{\Box} + \mu_p^2)^{\lambda_p}\right]. \tag{72}$$

6. INVARIANT FORM OF THE SOLUTION

All of the Green's functions we have discussed are invariant within the proper Lorentz group, so that an invariant form of the field $\varphi(x)$ is easily obtained. As is well known, the derivative of $\Delta(x; \mu)$ normal to an arbitrary spacelike surface $\sigma(x)$ with normal $n_\beta(x)$ is given by[10]

$$n_\beta(x)\,\partial_\beta\,\Delta(x; \mu) = \delta_\sigma(x)\ (x^2 < 0) \tag{73}$$

where $\delta_\sigma(x)$ is the invariant surface δ function with the properties

$$\delta_\sigma(x) = 0\ (x \not\subset \sigma), \tag{74}$$

$$\int d\sigma(x)\,\delta_\sigma(x) = 1.$$

The corresponding behavior of $G_N(x)$ on spacelike surfaces may be obtained as follows.

Because Δ satisfies the Klein-Gordon equation, it follows that

$$\Box^m\,\Delta(x; \mu) = (-\mu^2)^m\,\Delta(x; \mu); \tag{75}$$

consequently, for $x^2 < 0$ we have

$$n_\beta(x)\,\partial_\beta\,\Box^m\,G_N(x) = S(m)\,\delta_0(x), \tag{76}$$

[9] A rigorous proof showing that (61) satisfies Eq. (59) directly is not hard to construct, but seems longer than this demonstration.

[10] See, e. g., J. Rzewuski, *Field Theory* (Hafner Publishing Company, New York, 1958).

where $S(m)$ is the sum given by (31). Utilizing the explicit expression for the sum, one immediately finds, for spacelike x,

$$n_\beta(x)\,\partial_\beta\,\Box^m\,G_N(x) = \begin{cases} 0 & \text{for}\quad m = 0, 1, \cdots, N-2; \\ \delta_\sigma(x) & \text{for}\quad m = N-1, \end{cases} \tag{77}$$

and

$$n_\beta(x)\,\partial_\beta\,\Box^{N+m-1}G_N(x) = \sum_{\zeta_1+\cdots\zeta_L=m} \frac{(\lambda_1)_{\zeta_1}\cdots(\lambda_L)_{\zeta_L}}{\zeta_1!\cdots\zeta_L!}(-\mu_1^2)^{\zeta_1}\cdots \times(-\mu_L^2)^{\zeta_L}\,\delta_\sigma(x). \tag{78}$$

Furthermore,

$$\Box^m\,G_N(x) = 0(x^2 < 0) \tag{79}$$

follows directly from the fact that G_N is an odd invariant function vanishing outside the light cone.

One may now write an invariant analog of (55) with constants $A_{m\eta}$ that satisfy the same algebraic equations. The result gives the general solution in the invariant form[11]

$$\varphi(x) = \int d\sigma_\beta(x')G_N(x-x')X'_\beta\varphi(x'), \tag{80}$$

where

$$X_\beta = (\vec{\partial}_\beta - \overleftarrow{\partial}_\beta)\sum_{\eta=1}^{N}\binom{\lambda_1}{r_1}\cdots\binom{\lambda_L}{r_L}\vec{\Box} - \overleftarrow{\Box})^{\eta-1} \times\prod_{p=1}^{L}(\overleftarrow{\Box}+\mu_p^2)^{\lambda_p-r_p}. \tag{81}$$

Equation (72) is replaced by

$$(\vec{\Box}-\overleftarrow{\Box})X_\beta = (\vec{\partial}_\beta - \overleftarrow{\partial}_\beta)\times\left[\prod_{p=1}^{L}(\vec{\Box}+\mu_p^2)^{\lambda_p} - \prod_{p=1}^{L}(\overleftarrow{\Box}+\mu_p^2)^{\lambda_p}\right]. \tag{82}$$

Using (82), the integral in (80) is easily seen to be independent of the data carrying surface $\sigma(x)$; we have

$$\begin{aligned}\delta\varphi(x)/\delta\sigma &= \partial'_\beta G_N(x-x')X'_\beta\varphi(x') \\ &= G_N(x-x')[\textstyle\prod(\vec{\Box}'+\mu_p^2)^{\lambda_p} \\ &\quad - \textstyle\prod(\overleftarrow{\Box}'+\mu_p^2)^{\lambda_p}]\varphi(x') = 0,\end{aligned} \tag{83}$$

since both $G_N(x)$ and $\varphi(x)$ are solutions of the homogeneous equation.

It is well to remark that these results can be extended to the case of complex μ_p. In general, for a polynomial $F(z)$ with real or complex coefficients, the homogeneous equation

$$F(\Box)\varphi(x) = 0 \tag{84}$$

has a complete solution in the form

$$\varphi(x) = \int d\sigma_\beta(x')\,\Delta_N(x-x')X'_\beta\varphi(x'), \tag{85}$$

where

$$\Delta_N(x) = (2\pi)^{-n-1}\int_C dke^{-ikx}/F(-k^2) \tag{86}$$

and

$$(\vec{\Box}-\overleftarrow{\Box})X_\beta = (\vec{\partial}_\beta-\overleftarrow{\partial}_\beta)[F(\vec{\Box})-F(\overleftarrow{\Box})]. \tag{87}$$

APPENDIX A. PARTIAL FRACTIONS

Partial fraction decompositions for quotients of polynomials have long been known (although the explicit formulas are difficult to find in textbooks[12]). We shall list here the pertinent results for quotients of the form $z^q\prod(z-b_j)^{-\lambda_j}$ assuming the b_j to be distinct and nonzero, and setting $N \equiv \lambda_1 + \cdots + \lambda_L$.

When $q = 0, 1, \cdots N-1$ the following expansion holds:

$$\frac{z^q}{\prod_{j=1}^{L}(z-b_j)^{\lambda_j}} = \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}\times\left[\frac{(b_p)^q\prod_{j=1}^{L}{}'(b_p-b_j)^{-\lambda_j}}{z-b_p}\right] = \sum_{p=1}^{L}\sum_{m=1}^{\lambda_p}\frac{(z-b_p)^{-m}}{(\lambda_p-m)!}\times\left[\left(\frac{d}{db_p}\right)^{\lambda_p-m}(b_p)^q\prod_{j=1}^{L}{}'(b_p-b_j)^{-\lambda_j}\right], \tag{A1}$$

where the second equality is obtained using Leibniz's rule. The factor for $j = p$ is to be left out of the primed products. The important result

$$\sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}\left[(b_p)^l\prod_{j=1}^{L}{}'(b_p-b_j)^{-\lambda_j}\right] = 0 \qquad (l = 1, 2, \cdots, N-2) \tag{A2}$$

is obtained immediately upon setting $z = 0$ and $q = l+1$.

The partial fraction theorem also gives

$$\frac{z^{N+q}}{\prod_{j=1}^{L}(z-b_j)^{\lambda_j}} = E_q(z) + \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}\times\left[\frac{(b_p)^{N+q}\prod_{j=1}^{L}{}'(b_p-b_j)^{-\lambda_j}}{z-b_p}\right], \tag{A3}$$

[11] Compare with reference 4.

[12] A complete treatment of partial fractions is given by J. A. Serret, *Cours d'algèbre supréieure* (Gauthier-Villars, Paris, 1885), Tome I.

where $q = 0, 1, \cdots$, and $E_q(z)$ is a polynomial of degree q. An explicit form for $E_q(z)$ is easily obtained as follows: Since

$$E_q(z) = \left[\frac{z^{N+q}}{(z-b_1)^{\lambda_1}\cdots(z-b_L)^{\lambda_L}}\right]_{\text{entire part}}, \quad \text{(A4)}$$

expand the denominator using the binomial theorem to find

$$E_q(z) = [z^q(1-b_1/z)^{-\lambda_1}\cdots(1-b_L/z)^{-\lambda_L}]_{\text{entire part}}$$
$$= \sum_{\zeta_1+\cdots+\zeta_L\le q}\binom{\lambda_1-1+\zeta_1}{\lambda_1-1}\cdots\binom{\lambda_1-1+\zeta_L}{\lambda_L-1}$$
$$\times (b_1)^{\zeta_1}\cdots(b_L)^{\zeta_L}z^{q-\zeta_1-\cdots-\zeta_L}. \quad \text{(A5)}$$

In particular, then

$$E_q(0) = \sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}\left[\frac{(b_p)^{N+q-1}}{\prod_{j=1}^{L}{}'(b_p-b_j)^{\lambda_j}}\right]$$
$$= \sum_{\zeta_1+\cdots+\zeta_L=q}\frac{(\lambda_1)_{\zeta_1}\cdots(\lambda_L)_{\zeta_L}}{\zeta_1!\cdots\zeta_L!}(b_1)^{\zeta_1}\cdots(b_L)^{\zeta_L}$$
$$= \frac{1}{q!}\frac{d^q}{d\zeta^q}[(1-b_1\zeta)^{-\lambda_1}\cdots(1-b_L\zeta)^{-\lambda_L}]_{\zeta=0}, \quad \text{(A6)}$$

where the third equality is obtained using the multinomial differentiation rule (65). The important formula

$$\sum_{p=1}^{L}\frac{1}{\Gamma(\lambda_p)}\left(\frac{d}{db_p}\right)^{\lambda_p-1}\left[(b_p)^{N-1}\prod_{j=1}^{L}{}'(b_p-b_j)^{-\lambda_j}\right] = 1 \quad \text{(A7)}$$

is obtained because $E_0(0) = 1$.

APPENDIX B. CALCULATION OF Δ AND Δ^1

We first calculate the Green's functions for the Klein-Gordon operator following the procedure of (I). Upon performing the k_0 integration over the paths C and C^1, one finds the well-known integral representations

$$\Delta(x;\mu) = \frac{\epsilon(x)}{(2\pi)^n}\int_{-\infty}^{\infty}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\left(\frac{\sin\omega\,|t|}{\omega}\right) \quad \text{(B1)}$$

$$\Delta^1(x;\mu) = \frac{i}{(2\pi)^n}\int_{-\infty}^{\infty}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\left(\frac{\cos\omega t}{\omega}\right), \quad \text{(B2)}$$

where $\omega = +(\mathbf{k}^2+\mu^2)^{1/2}$. Since

$$(1/\omega)(\sin\omega t + i\cos\omega t) = (\pi t/2\omega)^{1/2}H_{1/2}^{(2)}(\omega t), \quad \text{(B3)}$$

we introduce (for $t > 0$) the function

$$\mathbf{K}(x;\mu) \equiv (2\pi)^{-n}\int_{-\infty}^{\infty}d\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}}\left(\frac{\pi t}{2\omega}\right)^{1/2}H_{1/2}^{(2)}(\omega t), \quad \text{(B4)}$$

and note that

$$\Delta(x;\mu) = \epsilon(x)\,\mathrm{Re}\,\mathbf{K}(x;\mu), \quad \text{(B5)}$$

$$\Delta^1(x;\mu) = i\,\mathrm{Im}\,\mathbf{K}(x;\mu). \quad \text{(B6)}$$

Integrating (B4) over the angles, one obtains

$$\mathbf{K}(x;\mu) = \frac{1}{2}\left(\frac{1}{2\pi}\right)^{(n-1)/2}$$
$$\times\int_0^{\infty}\frac{J_{(n-1)/2}(kR)}{R^{(n-1)/2}}\frac{H_{1/2}^{(2)}(\omega t)}{t^{-1/2}}\frac{k^{n/2}\,dk}{\omega^{1/2}}, \quad \text{(B7)}$$

where the integral over k is an integral of the Sonine-Gegenbauer type [Bateman,[13] Sec. 7.14.2 (48)]; thus, finally

$$\mathbf{K}(x;\mu) = \frac{1}{2}\left(\frac{\mu}{2\pi}\right)^{(n-1)/2}(\sqrt{x^2})^{(1-n)/2}H_{(1-n)/2}^{(2)}(\mu\sqrt{x^2}). \quad \text{(B8)}$$

In (B8), $(x^2)^{1/2}$ is defined as $-i(-x^2)^{1/2}$ when $x^2 < 0$ and Bateman's Sec. 7.2.2 (16) has also been used.

Just as in (I), the Sonine-Gergenbauer integral must be defined as a distribution for general orders of the Bessel functions in the integrand. Equation (20) now follows by an application of (15) and the well-known formula

$$(-d/z\,dz)^m\{z^{-\nu}H_\nu^{(2)}(z)\} = z^{-\nu-m}H_{\nu+m}^{(2)}(z). \quad \text{(B9)}$$

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[13] H. Bateman, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vols. I, II, III.