The change in lifetime for a given change in charge carrier concentration is much larger for neutron than for γ irradiation. Hall effect measurements at 77°K following irradiation at near 300°K indicate a removal rate of 2.5×10^{-3} electron per incident photon in these measurements. Since the position of the various defect levels is expected to be essentially the same for either neutron or γ-ray exposures, the analysis for recombination above the middle of the gap leads to a hole capture cross section $\sigma_p=5\times10^{-16}$ cm^2. This value is nearly an order of magnitude less than the value obtained for neutron irradiation. Although the reason for the large difference is not clearly understood, this may be associated with the difference in the defect distribution for the two types of radiation. A large variation in local defect concentration is expected in the vicinity of the primary neutron collision while the defect distribution after γ-ray irradiation should be uniform. It seems reasonable, therefore, that the neutron produced effective capture cross sections would be larger because the region of recombination is more perturbed, allowing transitions to occur with greater ease.

Consideration of the recombination center above the middle of the forbidden energy gap (~0.23 ev below the conduction band) accounts reasonably well for the experimental results. However, the analysis given here does not rule out the alternative situation. If the low-lying state S_2 is associated with a vacancy then appreciable recombination may be taking place at this level. Annealing studies, measurements on p-type material, and studies of lifetime as a function of temperature are underway and are expected to throw more light on the problem of recombination in irradiated germanium.

In summary, the minority carrier lifetime in n-type germanium is extremely sensitive to irradiation by fast neutrons and γ rays. This fact is of great importance to those who desire to use semiconductor devices in the presence of radiation. The simple dependence of the recombination rate upon irradiation received permits a prediction of the expected decrease in lifetime in a known radiation field. For the same change in carrier concentration, the change in lifetime produced by γ irradiation is much smaller than that produced by fast neutrons.

JOURNAL OF APPLIED PHYSICS VOLUME 28, NUMBER 10 OCTOBER, 1957

Equipartition of Energy and Local Isotropy in Turbulent Flows

MAHINDER S. UBEROI
Department of Aeronautical Engineering, University of Michigan, Ann Arbor, Michigan
(Received March 25, 1957; revised manuscript received July 1, 1957)

Homogeneous turbulence in which $\langle v^2\rangle=\langle w^2\rangle\neq\langle u^2\rangle$ was produced experimentally, where $\langle u^2\rangle$, $\langle v^2\rangle$, and $\langle w^2\rangle$ are the mean-square turbulent velocities in x, y, and z direction, respectively. The decay of turbulence and the energy transfer between $\langle u^2\rangle$ and $(\langle v^2\rangle+\langle w^2\rangle)$ were measured, and it was found that the larger components ($\langle v^2\rangle$ and $\langle w^2\rangle$) are losing more energy due to viscosity than by transfer to the smaller component ($\langle u^2\rangle$). However, $\langle u^2\rangle$ is receiving enough energy by transfer to compensate for its decay and is in fact slowly increasing. The measurement of mean-square vorticity components shows that the turbulence is becoming locally isotropic at a faster rate than the equipartition of energy is taking place.

In another set of experiments it was found that when approximately isotropic turbulence is subjected to deformation, the three components of turbulent energy become widely different in magnitude and that the turbulence is not locally isotropic. This indicates that even at high Reynolds number the deformation in a shear flow may cause anisotropy. The data on the turbulent shear flow near a solid wall confirm this conjecture.

The connection of this investigation to turbulent flows in general is discussed. In particular, it follows that neither the turbulent energy nor the small-scale structure of turbulence rapidly settles down statistically to quasi-equilibrium.

I. INTRODUCTION

THE motion of an incompressible viscous fluid is governed by the Navier-Stokes equations. In principle these equations may be solved, although in practice this may prove to be a formidable task. In the case of turbulent motion we are not interested in detailed motion, but only in certain statistical averages of the motion. Accordingly, we take the appropriate averages of the equations. Irrespective of the particular averages and the mode of taking these averages, the averaged equations become indeterminate; that is, we have more unknowns than the equations relating them. We must complement these equations by one or more hypotheses relating the unknowns. In the case of homogeneous turbulence the proper average to consider is the energy spectrum or its Fourier transform, the correlation of velocities at two points. Using the equations of motion von Kármán and Howarth derived an equation governing this correlation. The equation involves a triple velocity correlation which is unknown. There are two types of hypotheses which are used to complement this averaged equation of motion.

The first type was put forward by Millionshtchikov.[1] From the equation of motion he derived a governing equation for triple correlation. The equation involves yet another unknown, the quadruple velocity correlation. The above two equations involve three unknowns and in order to make the system determinate Millionshtchikov postulated a relation between quadruple and double velocity correlations. The equations are now determinate and may be solved. There are some errors in Millionshtchikov's work. Recently Proudman and Reid[2] and Chandrasekhar[3] have done work along these lines. The present author[4] has measured most of the quadruple correlations and these satisfy Millionshtchikov's hypothesis. However, it is the differences among the quadruple correlations which enter in the final averaged equations of motion and these differences have not been measured with any accuracy which would justify the use of the hypothesis without any reservation.

The second type of hypothesis is based on a physical picture of turbulence. According to Kolmogoroff,[5] at sufficiently high Reynolds number the turbulence is made up of a hierarchy of eddies. It is assumed that there is a transfer of energy from larger to smaller eddies, in the same way as the transfer of energy from the mean to the turbulent motion in a shear flow. The large scale eddies dissipate little energy by viscosity and pass on most of it to eddies of next smaller scale, and so on to the smallest scale eddies which are responsible for most of the viscous dissipation. Various investigators[6] have given expressions for the energy transfer based on this cascade process. This amounts to postulating a relation between triple and double velocity correlations.

If this picture is correct and if we deform a fluid in isotropic turbulent motion then it should become rapidly isotropic once the deformation has ceased. It is also a natural consequence of the cascade process that small eddies in a shear flow are locally isotropic. We have conducted experiments which throw some light on the mechanism of turbulence and afford a check on some of these predictions.

Before discussing the experiments, it is appropriate to make a few remarks of general nature. The first type of approach to the turbulence problem is quite analogous to that used in nonequilibrium statistical mechanics.[7] We start out with a determinate system of of equations for N particles and by appropriately averaging these equations we get a set of equations. The first equation relates the probability of finding two particles with specified positions and momenta with the probability of finding three particles. The second equation relates the probability of finding three particles with that of finding four particles. It is clear that this is as far as we can go with the use of equations of motions and the probability theory. In order to solve an actual problem it is necessary to terminate the series of equations at some point and to make the truncated set of equations determinate by postulating a relation between the probability involving n particles with that involving $(n+1)$ particles. In the two cases of kinetic theories of gases and liquids, it has been possible to find hypotheses which are supposedly valid quite generally. On the other hand, we require one hypothesis for isotropic turbulence, another for shear flow near a wall, still another for free turbulent flows. Our understanding of the mechanics of turbulent flow is so meager that we cannot cover all types of turbulent flow with a single hypothesis. In the case of turbulence merely setting up equations involving correlations offers no difficulty and the procedure may be easily extended to compressible gases, conducting gases, etc. Hopf[8] has made an unsuccessful effort to obtain a determinate system by using the equation of motion and functional calculus. He derives a governing equation for the characteristic functional or the Fourier transform of the phase distribution which is completely equivalent to an infinite system of partial differential equations expressing n-velocity correlation in terms of $(n+1)$-velocity correlation. In effect Hopf's procedure is quite similar to that of Millionshtchikov and Chandrasekhar. It appears that even the most advanced methods of functional calculus and stochastic processes cannot make the averaged equations determinate. Actually, we should not expect the averaged equations to be determinate. We declare our ignorance of the detailed motion by averaging the equations and this lack of information has to be made up by one or more hypotheses about the properties of average motion. Most profitable research in turbulence will involve theoretical and experimental work on the mechanics of turbulence which will in the end lead to one or more hypotheses making the system of equations determinate.

II. EQUIPARTITION OF TURBULENT ENERGY

Approximately homogeneous and isotropic turbulence was produced by placing a square mesh grid in a uniform flow. This turbulence is passed through a 4:1 axisymmetric contraction followed by a duct of constant cross section. Schematic diagram of the arrangement is shown in Fig. 1. Measurements show that after the contraction the turbulent field is statistically homogeneous and $\langle v^2\rangle=\langle w^2\rangle>\langle u^2\rangle$, where u is the turbulent velocity component along the mean flow or the contraction axis and v and w are the components perpendicular to it. Angular parentheses $\langle\ \rangle$ denote an average. The degree

[1] M. Millionshtchikov, Compt. rend. acad. Sci., U.S.S.R. 32, 615 (1941).
[2] I. Proudman and W. H. Reid, Phil. Trans. Roy. Soc. (London) A247, 163 (1954).
[3] S. Chandrasekhar, Proc. Roy. Soc. (London) A299, 1 (1955).
[4] M. S. Uberoi, J. Aeronaut. Sci. 20, 197 (1953); Natl. Advisory Comm. Aeronaut. Tech. Notes No. 3116 (1954).
[5] A. N. Kolmogoroff, Compt. rend. acad. Sci., U.S.S.R. 30, 301 (1941); 32, 16 (1941).
[6] W. Heisenberg, Z. Physik 124, 628 (1948); A. Obukhoff, Compt. rend. acad. Sci., U.S.S.R. 32, 19 (1941).
[7] M. Born, Proc. Roy. Soc. (London) A188, 10 (1946).
[8] E. Hopf, J. Rational Mech. Anal. 1, 87 (1952).

of anistropy or the ratio $\langle v^2\rangle/\langle u^2\rangle$ depends on the amount of contraction. The work on the effect of contraction on free-stream turbulence is discussed elsewhere.[9] Here we are not interested in this effect and we can regard the grid and the contraction as a method of producing homogeneous anisotropic turbulence. An insight of the mechanism of turbulent motion can be gained by the study of partition of turbulent energy among the three velocity components. Measurements of turbulent velocity fluctuations in the uniform section after the contraction are shown in Fig. 2. $\langle v^2\rangle$, the larger component, is losing energy by transfer to $\langle u^2\rangle$ and viscosity. $\langle u^2\rangle$, the smaller component, is gaining an equal amount of energy from $\langle v^2\rangle$ and $\langle w^2\rangle$ and is losing some due to viscosity. In the beginning the gain of $\langle u^2\rangle$ is enough to compensate for its decay and in fact it is slowly increasing. After a while the transfer cannot keep up with decay and $\langle u^2\rangle$ decreases with time. In the absence of transfer from $\langle v^2\rangle$ and $\langle w^2\rangle$, $\langle u^2\rangle$ would decrease rapidly with time. Since there is no production of energy the total energy $\langle u^2\rangle+2\langle v^2\rangle$ is decreasing due to viscosity alone. Approximately one third of the energy

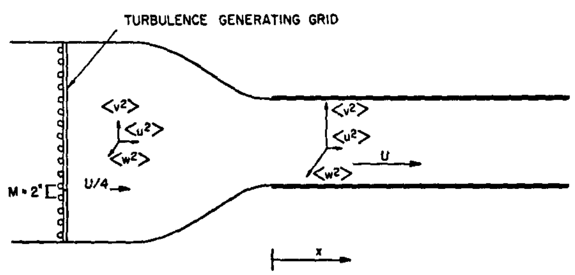

FIG. 1. Schematic diagram of the experimental arrangement. The grid Reynolds number $(MU/4)/\nu \simeq 10{,}000$.

is left by the time $\langle u^2\rangle$ and $\langle v^2\rangle$ become nearly equal. The time required for equipartition is of the same order as the decay time; the latter is, of course, the natural time scale for our problem.

The equations governing the decay and transfer may be easily derived from the equations of motion.[9] These are

$$\frac{d\langle u^2\rangle}{dt}=\frac{2}{\rho}\left\langle p\frac{\partial u}{\partial x}\right\rangle-6\nu\left\langle\left(\frac{\partial u}{\partial s}\right)^2\right\rangle \qquad (1)$$

(gain) (decay)

$$\frac{d\langle v^2\rangle}{dt}=-\frac{1}{\rho}\left\langle p\frac{\partial u}{\partial x}\right\rangle-6\nu\left\langle\left(\frac{\partial v}{\partial s}\right)^2\right\rangle, \qquad (2)$$

(transfer) (decay)

where $\partial/\partial s$ is the gradient in the direction making equal angles with all three axes, p, ρ, and ν are the pressure, density, and kinematic viscosity, respectively. $\langle u^2\rangle$, $\langle v^2\rangle$, and $\langle(\partial u/\partial s)^2\rangle$ were measured at various distances downstream from the contraction and other

[9] M. S. Uberoi, J. Aeronaut. Sci. 23, 754 (1956).

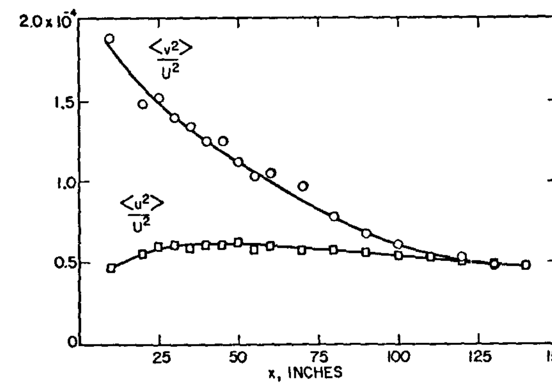

FIG. 2. Velocity fluctuations in axisymmetric turbulence.

quantities were calculated from Eqs. (1) and (2). The distance downstream from the grid is proportional to time, $x=Ut$, since the mean velocity, U, is uniform. The measured values for the ratios

$$\frac{1}{3\rho}\left\langle p\frac{\partial u}{\partial x}\right\rangle\Big/\nu\left\langle\left(\frac{\partial u}{\partial s}\right)^2\right\rangle \quad\text{and}\quad \frac{1}{6\rho}\left\langle p\frac{\partial u}{\partial x}\right\rangle\Big/\nu\left\langle\left(\frac{\partial v}{\partial s}\right)^2\right\rangle$$

are shown in Fig. 3. Both of these ratios are less than unity except in early stages; that is, energy transfer is a small part of viscous loss. Townsend[10] has noted the weak tendency to isotropy although he did not measure the rate of energy transfer.

III. LOCAL ISOTROPY IN HOMOGENEOUS TURBULENCE

It is of interest to see if the small-scale motion is becoming isotropic at a faster rate than the rate of equipartition of energy. The velocity derivatives are mainly determined by small-scale motion and for local isotropy the ratios of their mean squares have a definite value, as in the case of isotropic turbulence. The value for the most commonly measured ratio

$$\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle\Big/\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle$$

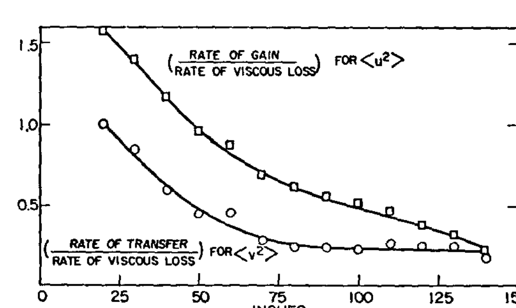

FIG. 3. Energy transfer in axisymmetric turbulence.

[10] A. A. Townsend, Quart. J. Mech. Appl. Math. 7, 104 (1954).

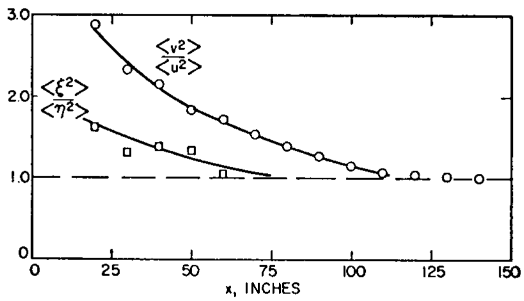

FIG. 4. Local isotropy in axisymmetric turbulence.

is 2. Determination of this single ratio cannot provide conclusive evidence for local isotropy. Local isotropy requires that vorticity be isotropic even though the over-all velocity field is anisotropic. In the present case of homogeneous turbulence, the decay of total energy is directly proportional to mean-square vorticity,

$$\frac{d}{dt}(\langle u^2\rangle+2\langle v^2\rangle)=2\nu(\langle\xi^2\rangle+2\langle\eta^2\rangle), \quad (3)$$

where $\langle\xi^2\rangle$ and $\langle\eta^2\rangle$ are the mean-square vorticities in x and y directions, respectively. Furthermore, in the special case of homogeneous axisymmetric turbulence it may be shown that (see Appendix)

$$\langle\eta^2\rangle=\frac{3}{2}\left\langle\left(\frac{\partial v}{\partial s}\right)^2\right\rangle+\frac{1}{2}\left[\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle+2\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle\right]. \quad (4)$$

These equations are useful since they express the vorticity in terms of easily and accurately measurable quantities. All mean square derivatives appearing in the last two equations have been measured and the ratios $\langle\xi^2\rangle/\langle\eta^2\rangle$ and $\langle v^2\rangle/\langle u^2\rangle$ are shown in Fig. 4. The results show that in the beginning the turbulence is not locally isotropic but becomes so at a somewhat faster rate than the rate of equipartition of energy. Townsend's conjecture that even highly anisotropic turbulence is locally isotropic is not borne out.

IV. LOCAL ISOTROPY IN SHEAR FLOWS

If we picture the energy transfer from large-scale to small-scale motion as a cascade process, then at high Reynolds number there are enough of these cascades so that the small-scale motion is not influenced by the anisotropy of the over-all flow. Turbulence should be locally isotropic in a shear flow or in the presence of rate of deformation of the fluid. This prediction was checked by measuring the ratio

$$\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle\Big/\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle$$

on the axis of the contraction used to produce anisotropic turbulence (see Fig. 1). This ratio has a value of two for local isotropy. Measurements of this ratio are shown in Fig. 5. Deviations from the value two show that the turbulence is locally anisotropic in the presence of rate of deformation of the fluid. This conjecture has also been put forward by Townsend.[10] In view of this, one has serious doubt that the turbulence is locally isotropic in shear flows. The crucial test for local isotropy is provided by measuring the mean-square vorticity in the direction of the principal axes of the rate of deformation. For local anisotropy the vorticity should be highest in the direction in which the fluid is being stretched and lowest in the direction of contraction. These directions are at $\pm 45°$ to the mean flow for parallel or nearly parallel flows (channel, boundary layer, etc.). Measurement of the ratio of mean-square vorticities in the above two directions were made in the boundary layer on the wall of a fourteen-inch square duct. The location at which measurements were made is not far enough downstream from the duct entrance for the flow to become independent of x, but is of no consequence here. Measurements are shown in Fig. 6 where $\langle\omega_1^2\rangle$ and $\langle\omega_2^2\rangle$ are the mean-square vorticities in the directions of the principal axes of the rate of deformation. Near the wall the turbulence is locally anisotropic; the mean-square vorticity is larger in the direction of rate of elongation and smaller in the direction of rate of contraction. Towards the center the rate of deformation decreases and the ratio tends to unity, that is, the turbulence becomes locally isotropic. Six hot-wire anemometers, suitably grouped together, had to be used for these measurements so the absolute accuracy is rather low, but the trend can be trusted. The ratio of vorticities has a value of 1.3 near the wall, but this value is of no significance. It is greater than unity near the wall and tends to unity in the center where the shear disappears. One might say that the turbulence is at least locally isotropic in the

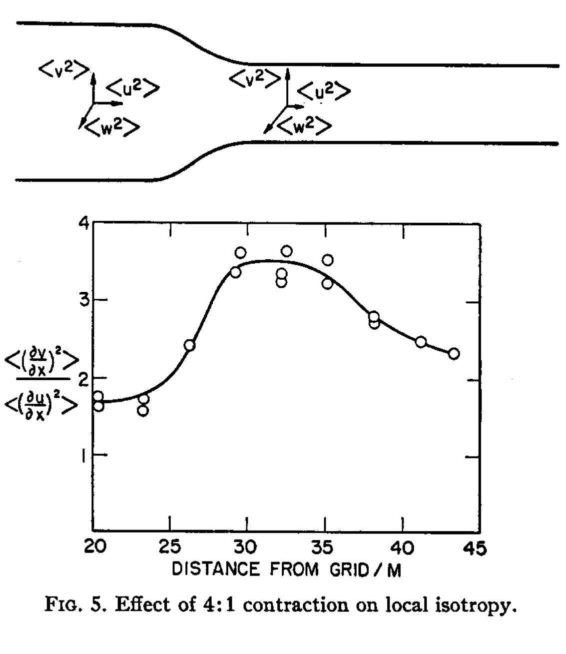

FIG. 5. Effect of 4:1 contraction on local isotropy.

center of the channel. But in a shear flow the region of high rate of deformation is of prime importance, since it is here that maximum production and dissipation of energy is taking place. Recently Sandborn and Braun[11] also concluded that there is no evidence for local isotropy in the turbulent boundary layer.

The rate of deformation due to mean motion increases the vorticity in a preferred direction while the deformation due to turbulence increases the vorticity more or less equally in all directions. The degree of anisotropy produced is determined by the ratio

$$\frac{\partial U}{\partial y}\Big/\left[\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle\right]^{\frac{1}{2}},$$

where

$$\frac{\partial U}{\partial y} \quad \text{and} \quad \left[\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle\right]^{\frac{1}{2}}$$

are the representative velocity gradients due to mean motion and turbulence, respectively. According to the general argument based on the cascade process it is claimed that all turbulent flows become locally isotropic at sufficiently high Reynolds number, that is the above ratio tends to zero. However, it can be shown that in the important region of high-energy production this ratio in some cases tends to a constant value independent of Reynolds number while in others it tends to zero with increasing Reynolds number. We illustrate this by considering two typical flows.

In the case of boundary layer and channel flows most of the production and dissipation of energy take place near the wall. In this region all the properties of turbulence depend only on the local shear which is approximately equal to the shear at the wall. The production and the dissipation of energy are nearly in balance, that is,

$$\underset{\text{(production)}}{-\langle uv\rangle\frac{\partial U}{\partial y}} = \nu\Bigg[2\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle+2\left\langle\left(\frac{\partial v}{\partial y}\right)^2\right\rangle +2\left\langle\left(\frac{\partial w}{\partial z}\right)^2\right\rangle+\left\langle\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2\right\rangle +\left\langle\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\rangle\Bigg]_{\text{(dissipation)}}. \quad (5)$$

All terms in the dissipation function are of the same order of magnitude and we may replace the function by a representative velocity gradient squared, say $c\langle(\partial u/\partial y)^2\rangle$ where c is a constant. The Reynolds stress

[11] V. A. Sandborn and W. H. Braun, Natl. Advisory Comm. Aeronautic. Tech. Notes No. 3761 (1956).

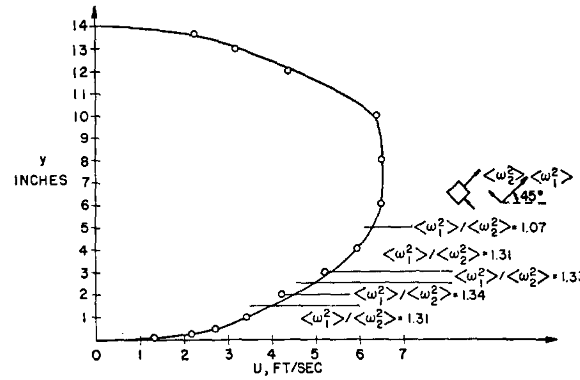

FIG. 6. Effect of rate of deformation on local isotropy in a shear flow.

may be replaced by τ_0, the shear at the wall. Thus

$$\frac{\tau_0}{\rho}\cdot\frac{\partial U}{\partial y} = \nu c\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle. \quad (6)$$

It is an experimental fact that $U/(\tau_0/\rho)^{\frac{1}{2}} = F(y^*)$ where $y^* = y(\tau_0/\rho\nu^2)^{\frac{1}{2}}$ is the nondimensional distance from the wall and F is a universal function independent of Reynolds number. It follows that

$$(\tau_0/\rho\nu)F' = \frac{\partial U}{\partial y}. \quad (7)$$

Substituting for τ_0 in Eq. (6) we get

$$\frac{\partial U}{\partial y}\Big/\left[\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle\right]^{\frac{1}{2}} = [cF'(y^*)]^{\frac{1}{2}}. \quad (8)$$

Since the region of high production and dissipation of energy depends on y^* and not on y, therefore we want to study the ratio

$$\frac{\partial U}{\partial y}\Big/\left[\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle\right]^{\frac{1}{2}}$$

at a fixed value of y^* instead of y. The above equation shows that the ratio has a constant value independent of Reynolds number. It has been experimentally shown that the turbulence is locally anisotropic and the above analysis shows that it will be so at all Reynolds numbers.

The situation is quite different for free turbulence, say a jet issuing into open air. Consider the energy equation

$$-\langle uv\rangle\frac{\partial U}{\partial y} = \text{dissipation}+\text{diffusion}+\text{convection}, \quad (9)$$

where y is the distance from the jet axis. The value of these terms depends on the location but at a fixed position their ratios do not change with Reynolds number and we may take $\langle(\partial u/\partial y)^2\rangle$ as representative of all three terms on the right side of the above equation, that is

$$-\langle uv\rangle\frac{\partial U}{\partial y}=\nu c_1\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle, \qquad (10)$$

where c_1 is a constant. It is an experimental fact that $-\langle uv\rangle=c_2U^2$ and $\partial U/\partial y=c_3U/D$ or

$$-\langle uv\rangle=\frac{c_2}{c_3}UD\frac{\partial U}{\partial y},$$

where D is the half-width of the jet and c_2 and c_3 are functions of position but are independent of Reynolds number. Substituting this result in the above equation, we have

$$\frac{\partial U}{\partial y}\Big/\left[\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle\right]^{\frac{1}{2}}=\left(\frac{c_1c_3\nu}{c_2UD}\right)^{\frac{1}{2}}=\left(\frac{c_1c_3}{c_2\,\mathrm{Re}}\right)^{\frac{1}{2}}. \qquad (11)$$

The ratio tends to zero as $UD/\nu=\mathrm{Re}$, the jet Reynolds number, increases to infinity.

V. CONCLUDING REMARKS

The study of homogeneous and anisotropic turbulence shows that neither the small-scale nor the large-scale motion become isotropic in a time appreciably smaller than the time required for the decay of total energy. In homogeneous as well as shear flow turbulence the energy containing eddies are more or less permanent and their influence is felt directly by the small eddies. This implies that hierarchy of eddies or the number of cascades is quite limited. It is true that the entire motion may be divided into two scales of motion; the large, energy containing, eddies and small eddies which dissipate the energy by direct action of viscosity. There are not many intermediate sizes of eddies which are required if (a) the small-scale motion is to be locally isotropic or independent of large-scale motion; (b) there is to exist at sufficiently high Reynolds number an "inertial" subrange of eddies which is affected neither by viscosity nor by the large-scale motion. The increase in Reynolds number is expected to increase the range of intermediate sizes of eddies provided the size of large-scale eddies is kept fixed. However, it can happen that with increase in Reynolds number both the large and the small eddies decrease in size, their ratio remaining the same with no increase in the range of intermediate size eddies. This is the case for the boundary layer. In other words, whether a particular turbulent flow becomes locally isotropic or not at large or infinite Reynolds number depends on the ratio of the rate of deformation due to mean motion to that due to turbulence. If the ratio is small, that is, the rate of deformation of the fluid by the mean motion is high, then turbulence is not locally isotropic.

VI. ACKNOWLEDGMENTS

The author is indebted to Professor A. M. Kuethe for helpful suggestions and to Mr. C. Van Atta for assistance in the experimental work. This work was supported by the Office of Naval Research and the main results were presented at the Ninth International Congress of Applied Mechanics on September 5–13, 1956, Brussels, Belgium.

VII. APPENDIX

The mean-squared vorticities in y and z directions are numerically equal; therefore,

$$\begin{aligned}2\langle\eta^2\rangle&=\left\langle\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)^2\right\rangle\\&=\left\langle\left(\frac{\partial u}{\partial z}\right)^2\right\rangle+\left\langle\left(\frac{\partial w}{\partial x}\right)^2\right\rangle-2\left\langle\frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial x}\right\rangle\\&\quad+\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle-2\left\langle\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right\rangle.\end{aligned}$$

In a homogeneous turbulence

$$\left\langle\frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial x}\right\rangle=\left\langle\frac{\partial u}{\partial x}\cdot\frac{\partial w}{\partial z}\right\rangle \quad\text{and}\quad \left\langle\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right\rangle=\left\langle\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right\rangle.$$

Furthermore for an incompressible fluid

$$\left\langle\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)^2\right\rangle=\left\langle\left(-\frac{\partial v}{\partial y}\right)^2\right\rangle$$

or

$$2\left\langle\left(\frac{\partial u}{\partial x}\cdot\frac{\partial w}{\partial z}\right)\right\rangle=\left\langle\left(\frac{\partial v}{\partial y}\right)^2\right\rangle-\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle-\left\langle\left(\frac{\partial w}{\partial z}\right)^2\right\rangle.$$

Similarly

$$2\left\langle\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right)\right\rangle=\left\langle\left(\frac{\partial w}{\partial z}\right)^2\right\rangle-\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle-\left\langle\left(\frac{\partial v}{\partial y}\right)^2\right\rangle.$$

Substituting these results in the expression for $\langle\eta^2\rangle$ we have

$$\begin{aligned}2\langle\eta^2\rangle&=\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial u}{\partial y}\right)^2\right\rangle+\left\langle\left(\frac{\partial u}{\partial z}\right)^2\right\rangle\\&\quad+\left\langle\left(\frac{\partial w}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle\\&=3\left\langle\left(\frac{\partial u}{\partial s}\right)^2\right\rangle+\left[2\left\langle\left(\frac{\partial v}{\partial x}\right)^2\right\rangle+\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle\right]\end{aligned}$$

for axisymmetric turbulence where $\partial/\partial s$ is the gradient in a direction making equal angles with all three axes.