The statistics of the organized vortical structure in turbulent mixing layers

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The statistics of the large scale vortex structure in turbulent mixing layers have been investigated theoretically. It is shown that similarity in the fully developed flow results in a common description of the Eulerian and Lagrangian statistics. In the Eulerian frame of reference, a conservation equation is derived and solved to show that the distribution of vortex circulation is lognormal. It is also shown that the standard deviation normalized by the mean value of the distribution depends only on the amalgamation mechanism. The value for pairing is in good agreement with experimental measurements. These results are used to calculate the life span and survival probabilities of the vortices in the Lagrangian frame of reference. These distributions are in good agreement with direct measurements of the life span probability and with space-time correlation measurements, respectively. Some implications of these results on the dynamics of the large scale vortices in the fully developed turbulent flow are discussed.

I. INTRODUCTION

An essential feature of the structure of turbulence in plane mixing layers is the presence of spanwise coherent large scale vortices. An important aspect of this structure is the observation at any downstream location of a broad distribution of scales associated with the large scale vortices. Measurements of this distribution have been reported by Brown and Roshko,[1] Winant and Browand,[2] Bernal,[3] Hernan and Jimenez,[4] and Koochesfahani *et al.*[5] The life span of the vortices also shows a broad distribution. Measurements of the life span probability were reported by Brown and Roshko and by Hernan and Jimenez. These distributions are in a sense the primary manifestation of the turbulent character of the flow. Another manifestation is the presence of smaller scale three-dimensional structures and motions, but these play only an indirect role in the distribution of large scales. The nature of these small scale motions was investigated by Bernal and Roshko.[6] Their results showed that three-dimensional smaller scale motions within the large scale vortices do not destroy their spanwise coherence. The large scale vortices and their dynamics remain by and large two dimensional. In this paper we study theoretically the distributions of scale and life span of the large scale vortices, and the relation of these distributions to the dynamics of the vortices in fully developed turbulent mixing layers.

A sequence of photographs from a high speed motion picture of a high Reynolds number constant density mixing layer is shown in Fig. 1. The motion picture was obtained by Bernal[3] in collaboration with Roshko[7] and Brown. The mixing layer between a nitrogen stream at 1000 cm/sec and a mixture of helium and argon of the same density as nitrogen at 380 cm/sec was visualized using the shadowgraph technique. On each photograph the nitrogen stream is on top and the helium/argon stream is at the bottom. The pictures illustrate typical evolutions of the large scale vortices. The vortices are convected downstream during lifetimes that terminate when each vortex interacts or amalgamates with its neighbors to form vortices of a larger scale. A form of interaction, the pairing interaction, in which two vortices rotate around each other until they amalgamate can be observed at the center of the first three photographs in Fig. 1. This form of interaction was first documented by Winant and Browand[2] at low and moderate Reynolds numbers. Other evolutions in pairing interactions have been found in numerical studies by Patnaik *et al.*[8] and Riley and Metcalfe.[9] In all cases of pairing, however, two vortices combine to form one. Another form of interaction involving the destruction of three vortices to form one, the tripling interaction, is also observed in the sequence in Fig. 1 just upstream of the pairing interaction. This form of interaction has also been observed at high Reynolds numbers by Dimotakis and Brown.[10] Moore and Saffman[11] suggested a different possible form of amalgamation in which a vortex is torn apart by its neighbors and assimilated into them. Thus in this form of interaction three vortices are destroyed to form two new vortices. Tearing processes were reported by Dimotakis and Brown and Hernan and Jimenez.[4] Hernan and Jimenez also found, using computer analysis of a shadowgraph motion picture, the same number of triplings and tearings in a sample of over 100 amalgamations. Their results show that pairing is the more frequent form of amalgamation. Each of the other forms contributed approximately ten percent to the total.

The parameters needed to characterize the large scale vortices are the scale of the vortices and their downstream location. The precise definition of these parameters used in this paper are based on the features observed in shadowgraphs like those in Fig. 1. On the shadowgraphs the large scale vortices are outlined by dark lines shown schematically in Fig. 2. The scale of the vortices is characterized by their wavelength λ, defined as the distance between adjacent intersections of the braids with the x axis. The circulation of a vortex is given in first approximation by $\Gamma = \Delta U \lambda$, where $\Delta U = U_1 - U_2$ is the velocity difference across the layer. Since the velocity difference is constant, λ also characterizes the circulation of the vortex. For this reason, throughout this paper λ will be referred to as the circulation of the vortex. Bernal[3] showed that within experimental uncertainty λ is constant during the vortex life span. This is consistent with

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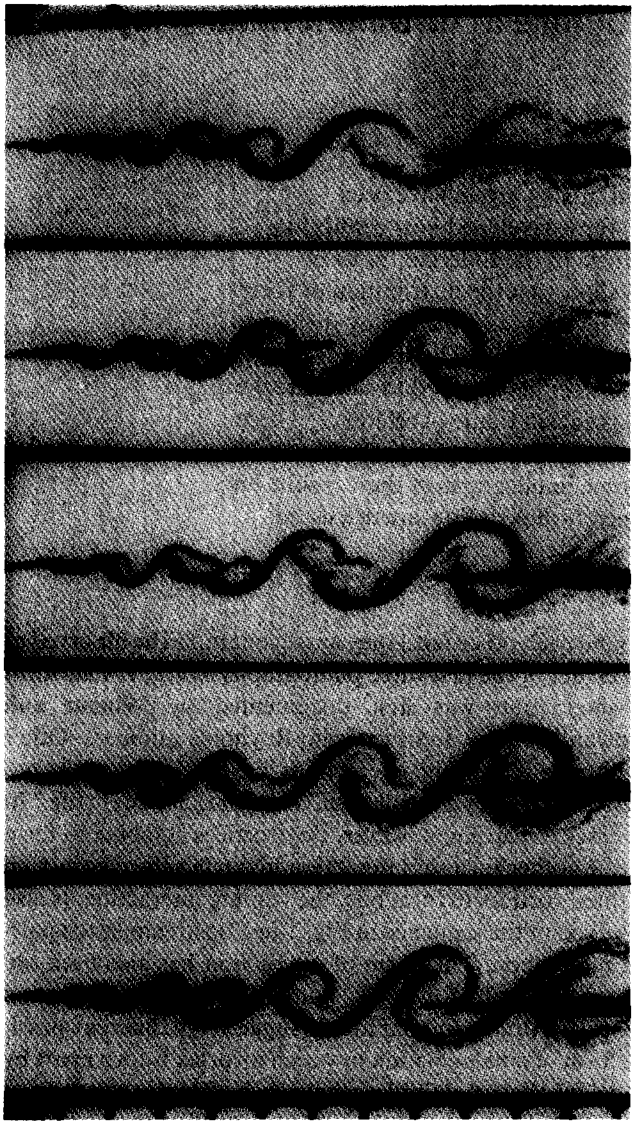

FIG. 1. Sequence of shadowgraphs from a high speed motion picture (3000 ppsec) of a constant density mixing layer: High speed stream on top at 1000 cm/sec; low speed stream at the bottom at 380 cm/sec; probe tip location $x = 10$ cm; test section pressure 8 atm; Reynolds number 8.5×10^5. Pictures number 1, 6, 9, 13, and 19 of a sequence are shown.

the observation that fluid entrained by a vortex during its life span is irrotational since it originates in a free stream. All the vorticity in the mixing layer is introduced at the trailing edge of the splitter plate. Thus during the evolution of a vortex its vorticity is simply redistributed while the total amount remains constant. Similarly, at an amalgamation the sum of the circulation of the vortices destroyed equals the sum of the circulation of the vortices formed. Therefore the sum of λ is conserved at an amalgamation.

The downstream location of a vortex, x, is defined as the midpoint between adjacent braid/x-axis intersections as shown in Fig. 2.[3] Browand and Weidman[12] determined the vorticity distribution in the large scale vortices using phase averaged velocity measurements. Their results show that the maximum vorticity is located approximately at the midpoint between adjacent braids in a well-formed vortex. As the vortex approaches amalgamation this simple relation does not hold. The detailed measurements of the velocity field by Browand and Weidman were only obtained at two conditions and therefore cannot be used to follow the evolution of the vortices. Data on the evolution of the large scale vortices has been obtained primarily from flow visualization motion pictures. Brown and Roshko[1] and Hernan and Jimenez[4] used slightly different definitions for the location of the vortices in shadowgraph motion pictures. Their definitions approximately correspond to the midpoint between adjacent braids. Those measurements as well as Bernal's[3] showed that within experimental accuracy the vortices are convected with constant velocity U_c. The convection velocity U_c is approximately the average of the free-stream velocities in a constant density mixing layer. It also depends on the density ratio in the case of a nonuniform mixing layer.[13] Also shown in Fig. 2 are the definitions of the velocity and density ratios.

Similarity is a particularly useful concept in the study of the fully developed turbulent mixing layer. Similarity requires a linear growth with downstream distance of any mean length scale associated with the large scale vortices. This similarity scaling is a consequence of the inviscid character of the large scale vortex dynamics. It is not inconsistent with conservation of circulation during the vortex life span provided that the distribution of circulation is self-similar. In earlier investigations[1,2] similarity was shown for the mean velocity profiles in regions where the vortices were observed. Similarity of the mean and rms values of circulation of the scale vortices was documented by Bernal.[3] These results showed that after an initial region of nonlinear growth, the mean properties of the vortices increase linearly with downstream distance. The extent of the transition region depends on the initial conditions at the trailing edge of the partition. The results based on the large scale properties are in good agreement with the velocity field results obtained by Hussain and Zedan.[14]

Takaki and Kovasznay[15] studied the probability distribution of large scale vortex spacing. They consider the time evolution of a mixing layer uniform in space. They derived a conservation equation for the distribution of spacings in which the amalgamation process results in the formation and destruction of spacings. Their formulation of the amalgamation process included only amalgamations by pairing, which were described in terms of a second statistic, the rate of merging. The rate of merging could not be obtained from

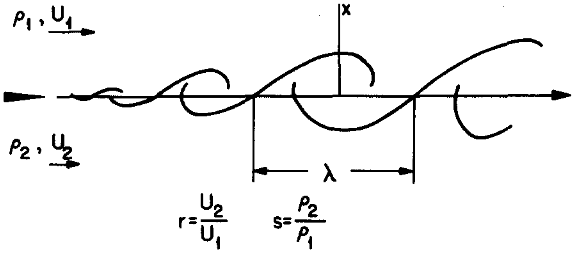

FIG. 2. Schematic diagram of the plane mixing layer showing the definition of the large scale vortex circulation and location.

the dynamics of the flow and had to be modeled in order to solve for the distribution of spacings.

In considering possible alternative approaches to this problem, Kolmogorov's[16,17] statistical theory of breakage is of considerable interest. Kolmogorov formulated the theory for the distribution of particle size in a process in which the size of the particles changes only in a sequence of grinding events. He showed that if the sizes of the particles before and after the grinding events are uncorrelated, then after a sufficient number of grinding events the size distribution tends to the lognormal distribution. The evolution of the large scale vortices in the mixing layer can be viewed as a reverse breakage process in which the scale increases only in discrete events, the amalgamations. Thus it could be argued that the distribution of circulation of vortices of the same generation approaches a lognormal distribution after a sufficient number of amalgamations. Vortices of the same generation are those that have been formed by the same number of amalgamations. It follows that a large number of amalgamations may be required to reach the self-similar state. However, Kolmogorov theory does not apply in a strict sense because it fails to account for the randomization of vortex life span, an important part of the mixing layer dynamics in the fully developed flow.

In this investigation we study the probability distribution of vortex circulation in plane mixing layers. In Sec. II the main assumptions of the analysis are formulated. It is shown that similarity results in a common description of the Eulerian and Lagrangian descriptions of the statistics of the large scale vortices. In Sec. III a conservation equation is derived and solved for the distribution of vortex circulation in the Eulerian frame of reference. The parameters of the solution are related to known features of the evolution of the vortices. In Sec. IV, the results obtained in the Eulerian frame of reference are used to derive the Lagrangian statistics of the vortices. The life span probability and the survival probability are calculated and compared with experimental measurements. These results and their implications on the dynamics of the vortices in the mixing layer are discussed in Sec. V.

II. FORMULATION

Our objective is to determine the probability of finding a vortex with circulation less than λ at a location less than x. From dimensional considerations, in the fully developed turbulent flow, we expect this probability to depend only on the similarity variable $\xi = \lambda/x$,

$$P(x_s \leqslant x; \lambda_s \leqslant \lambda) = P(\xi).$$

The similarity scaling implies that in the x-λ plane $P(x,\lambda)$ is constant along straight lines passing through the origin. Thus $P(x,\lambda)$ is a singular distribution in the sense discussed by Cramer[18] and the corresponding two-dimensional density function does not exist. Only the one-dimensional probability density function,

$$p(\xi) = \frac{dP(\xi)}{d\xi},$$

has physical significance. For an observer located at a fixed downstream position x, in the Eulerian frame of reference, $p(\xi)$ describes the probability of finding a vortex with circulation $(\lambda, \lambda + d\lambda)$. On the other hand, for an observer following vortices with circulation λ, in the Lagrangian frame of reference, $p(\xi)$ describes the probability of finding the vortex at a position $(x, x + dx)$. Therefore similarity results in the rather important simplification that a single function $p(\xi)$ is sufficient to describe the Eulerian and Lagrangian statistics of the large scale vortices.

The statistics of the amalgamation process is fully characterized by the probability that a vortex of circulation less than λ is formed at a position less than x_1 and destroyed at a position less than x_2. Using the similarity scaling it can be written as a joint probability distribution $W(\xi_1,\xi_2)$ with $\xi_1 = \lambda/x_1$ and $\xi_2 = \lambda/x_2$. The corresponding probability density is given by the usual relation

$$w(\xi_1,\xi_2) = \frac{\partial^2 W(\xi_1,\xi_2)}{\partial\xi_1\,\partial\xi_2}.$$

An important property of this function is that $w(\xi_1,\xi_2) = 0$ for $\xi_1 \leqslant \xi_2$, which results from the fact that vortices are always destroyed downstream of where they are formed.

Of particular interest are the marginal probability densities $p_F(\xi)$ and $p_D(\xi)$ defined in the usual way:

$$p_F(\xi) = \int_0^\infty w(\xi,\xi_2)\,d\xi_2,$$

$$p_D(\xi) = \int_0^\infty w(\xi_1,\xi)\,d\xi_1.$$

These marginal distributions describe the probability of formation and destruction, respectively, of a vortex of circulation λ at a location x. Both $p_F(\xi)$ and $p_D(\xi)$ have dual physical interpretations analogous to $p(\xi)$ in the Eulerian and Lagrangian frames of reference.

It will be assumed in the analysis that the location of the vortices at formation and destruction are statistically independent. It is not immediately apparent that the vortices in the mixing layer will satisfy this assumption. It proves to be a good assumption because of the agreement found between theoretical results and experimental measurements. In Sec. V the implications of this assumption on the dynamics of the vortices in the fully developed turbulent flow are discussed. It follows from this assumption that $w(\xi_1,\xi_2) = p_F(\xi_1)p_D(\xi_2)$. In this case the requirement that $w(\xi_1,\xi_2) = 0$ for $\xi_1 \leqslant \xi_2$ gives

$$\begin{aligned} &p_F(\xi) = 0, && p_D(\xi) \geqslant 0, && \xi > \xi_C, \\ &p_F(\xi) \geqslant 0, && p_D(\xi) = 0, && \xi < \xi_C, \\ &p_F(\xi) = 0, && p_D(\xi) = 0, && \xi = \xi_C, \end{aligned} \tag{1}$$

where ξ_c is a constant. Thus under the assumption of statistical independence of the formation and destruction processes, not only do $p_F(\xi)$ and $p_D(\xi)$ fully characterize the statistics of the amalgamation process but also these functions are nonoverlapping, i.e., they are different from zero in different regions of the ξ axis.

III. EULERIAN STATISTICS

In the Eulerian frame of reference a conservation equation is derived relating $p(\xi)$, $p_F(\xi)$, and $p_D(\xi)$. If the total number of vortices convected per unit time through a location x is $n_T(x)$, the number of vortices of circulation $(\lambda,\lambda+d\lambda)$ convected through that location is

$$n(\lambda,x)d\lambda = n_T(x)p(\xi)d\xi = n_T(x)\xi p(\xi)\frac{d\lambda}{\lambda}. \quad (2)$$

The function $n_T(x)$ can be obtained by considering the total length convected per unit time, which by definition equals the convection velocity U_c. Thus

$$n_T(x) = \frac{U_c}{\bar{\xi}x}, \quad \bar{\xi} = \int_0^\infty \xi p(\xi)d\xi.$$

Here, $n_T(x)$ is readily interpreted as the mean passage frequency of the large scale vortices at x.

Similarly, if the total number of vortices formed and destroyed by amalgamation per unit length and time is $n_F(x)$ and $n_D(x)$, respectively, then the number of vortices with circulation $(\lambda,\lambda+d\lambda)$ formed and destroyed, respectively, per unit length and time are

$$n_F(x)\xi p_F(\xi)\frac{d\lambda}{\lambda}, \quad n_D(x)\xi p_D(\xi)\frac{d\lambda}{\lambda}.$$

Thus the overall effect of the amalgamation process is

$$n_a(\lambda,x)d\lambda = [n_F(x)p_F(\xi) - n_D(x)p_D(\xi)]\xi\frac{d\lambda}{\lambda}, \quad (3)$$

where $n_a(\lambda,x)$ is positive if more vortices with circulation λ are formed than destroyed at x, and negative otherwise.

A conservation equation for $p(\xi)$ is obtained by stating that λ is constant during the life span of the vortices. Therefore the change with downstream distance in the number of vortices of circulation λ convected per unit time through a location x equals the change due to amalgamations, $n_a(\lambda,x)d\lambda$. Thus

$$\frac{\partial n(\lambda,x)}{\partial x}d\lambda = n_a(\lambda,x)d\lambda.$$

Substituting relations (2) and (3) we find

$$\xi p(\xi) + \xi\frac{d[\xi p(\xi)]}{d\xi} = k_D\xi p_D(\xi) - k_F\xi p_F(\xi), \quad (4)$$

and

$$n_F(x) = -k_F\frac{dn_T(x)}{dx}, \quad n_D(x) = -k_D\frac{dn_T(x)}{dx},$$

where k_F and k_D are positive constants which represent the number of vortices formed and destroyed, respectively, per vortex lost during amalgamation. Since $dn_T(x)/dx = n_F(x) - n_D(x)$, then $k_D - k_F = 1$.

It may seem that Eq. (4) is of limited value because our understanding of the amalgamation process does not allow the determination of the functions $p_F(\xi)$ and $p_D(\xi)$. However, an exact solution for $p(\xi)$ can be obtained without explicit reference to these functions. To obtain this solution it is convenient to introduce the function $g(\ln\xi)$ defined as

$$\xi p(\xi)\frac{dg(\ln\xi)}{d\xi} = k_D p_D(\xi) - k_F p_F(\xi),$$

and the new variables $\alpha = \ln\xi$ and $P(\alpha) = \ln[\xi p(\xi)]$. Equation (4) in these variables reads

$$1 + \frac{dP(\alpha)}{d\alpha} = \frac{dg(\alpha)}{d\alpha},$$

which can be integrated to give

$$P(\alpha) = g(\alpha) - \alpha + \ln C.$$

The constant of integration C is obtained from the normalization condition on $p(\xi)$. Therefore the general form of $p(\xi)$ consistent with similarity is

$$p(\xi) = C\exp[g(\ln\xi)]/\xi^2. \quad (5)$$

An asymptotic expansion for this solution is derived in the Appendix. It is argued that although ξ changes by a factor of the order of 2 during the amalgamations, $\ln\xi$ changes only by a small additive constant. Thus the variance of $\ln\xi$ is expected to be small. This justifies the use of an expansion for $p(\xi)$ with small parameter the variance of $\ln\xi$. This expansion gives to leading order a lognormal distribution for $p(\xi)$, namely,

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma\xi}\exp\left\{-\frac{1}{2\sigma^2}\left[\ln\left(\frac{\xi}{\bar{\xi}}\right) + \frac{\sigma^2}{2}\right]^2\right\}. \quad (6)$$

In this expression explicit reference to the function $g(\alpha)$ was dropped in favor of the following parameters: $\bar{\xi}$, the mean value of the distribution, and σ^2, the variance of $\ln\xi$. The parameter σ characterizes the width of the distribution. The properties of the lognormal distribution can be found in the monograph by Aitchison and Brown.[17]

Comparison of this approximate solution for $p(\xi)$ with experimental measurements is presented in Fig. 3 in terms of $\lambda/\bar{\lambda} = \xi/\bar{\xi}$ for $\sigma = 0.276$. The lognormal distribution is in good agreement with the experimental results. The accuracy of the approximation can be determined from the moments of measured distributions. Using the experimental results of Bernal[3] shown in Fig. 3 we find $\overline{\ln\xi} = -1.4806$, $\sigma^2 = 0.0656$, and the third-order moment about the mean $\mu_3 = -0.0027$. Clearly the magnitude of μ_3 is smaller than σ^4, which confirms the validity of the proposed asymptotic expansion for $p(\xi)$ and justifies the use of the lognormal distribution in the rest of the analysis. The error incurred by this approximation is of the same order as μ_3, i.e., less than 0.3%.

The significance of this result is that the problem of finding $p(\xi)$ has been reduced to the evaluation of two parameters, $\bar{\xi}$ and σ, which may depend only on a few specific properties of the large scale vortex dynamics. It should be emphasized, however, that there is no fundamental reason to assume the higher-order terms in the expansion for $p(\xi)$ to be identically zero. If higher-order terms are retained, additional constants are introduced that must be determined from the large scale vortex dynamics. The validity of the approximation must then be confirmed after those constants have been determined. Although as shown above these correction terms are expected to be very small.

The relationship between the large scale dynamics and the parameter σ can be obtained from the statistics of the amalgamation process. The lognormal density function for $p(\xi)$ can be used to obtain the statistics of the amalgamation process. If $p(\xi)$ given by Eq. (6) is used in Eq. (4), it is found that

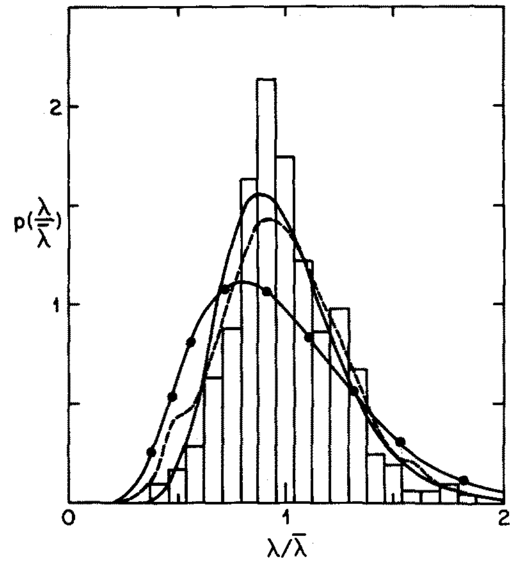

FIG. 3. Probability density of large scale vortex circulation: —, present theory; histogram, measurements of Bernal[3] ($r = 0.38$, $s = 1$); – –, measurements of Brown and Roshko[1] ($r = 0.38$, $s = 7$); —•—, theory of Takaki and Kovasznay.[15]

$$k_D p_D(\xi) - k_F p_F(\xi) = [(\sigma^2/2 - \ln \xi/\bar{\xi})/\sigma^2] p(\xi), \tag{7}$$

where $p(\xi)$ is the lognormal density function given by Eq. (6). As indicated above, in Sec. II, if the locations of formation and destruction of a vortex are statistically independent then p_F and p_D are nonoverlapping. Then p_F and p_D are uniquely determined by Eq. (7), namely,

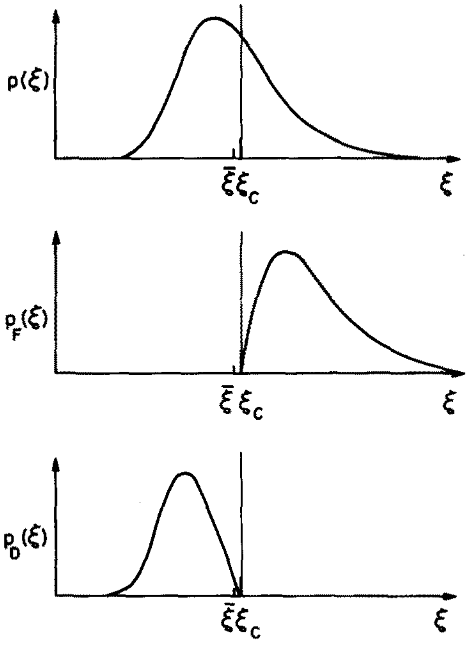

FIG. 4. Schematic diagram of the functions $p(\xi)$, $p_F(\xi)$, and $p_D(\xi)$ in the Eulerian frame of reference.

$$\begin{aligned} &p_F(\xi) = [(\ln \xi/\xi_c)/k_F\sigma^2] p(\xi), \quad p_D(\xi) = 0, \\ &\qquad \infty > \xi \geqslant \xi_C; \\ &p_F(\xi) = 0, \quad p_D(\xi) = [(-\ln \xi/\xi_C)/k_D\sigma^2] p(\xi), \\ &\qquad \xi_C > \xi > 0. \end{aligned} \tag{8}$$

where $\xi_C = \bar{\xi} \exp(\sigma^2/2)$.

The functions $p(\xi)$, $p_D(\xi)$, and $p_F(\xi)$ are sketched in Fig. 4. The basic feature described by these equations is that at a location x amalgamations destroy vortices with circulation λ such that $\lambda/x < \xi_C$, while the vortices formed by the amalgamation have circulation λ such that $\lambda/x > \xi_C$. Thus the picture emanating from this analysis is that if the fully developed region vortices are formed with $\xi \geqslant \xi_C$, then, as they are convected downstream, the value of ξ decreases until it reaches a value $\xi < \xi_C$, where they amalgamate to form new vortices with $\xi > \xi_C$. These new vortices follow the same evolution, i.e., they amalgamate farther downstream with $\xi < \xi_C$ to form even larger scale vortices also with $\xi > \xi_C$, and so on. This picture is consistent with the observed evolution of the large scale vortices.

The value of σ is determined from the normalization condition on p_F and p_D, which results in the equations

$$k_F = I(\sigma) - 1, \quad k_D = I(\sigma), \tag{9}$$

where

$$I(\sigma) = \frac{\exp(-\sigma^2/2)}{\sqrt{2\pi}\sigma} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sigma} \exp\left(\frac{-t^2}{2}\right) dt. \tag{10}$$

The function $I(\sigma)$ is plotted in Fig. 5. As noted earlier, $k_D - k_F = 1$ and therefore the two equations for σ are consistent.

The constants k_F and k_D were defined as the number of vortices formed and destroyed, respectively, per vortex lost in the amalgamation process. It is immediately apparent that the values of k_F and k_D depend on the amalgamation mechanism: for tearing $k_F = 2$, $k_D = 3$; for pairing $k_F = 1$, $k_D = 2$; and for tripling $k_F = \frac{1}{2}$, $k_D = \frac{3}{2}$. In general we expect a combination of these amalgamation mechanisms. If ν_{te} is

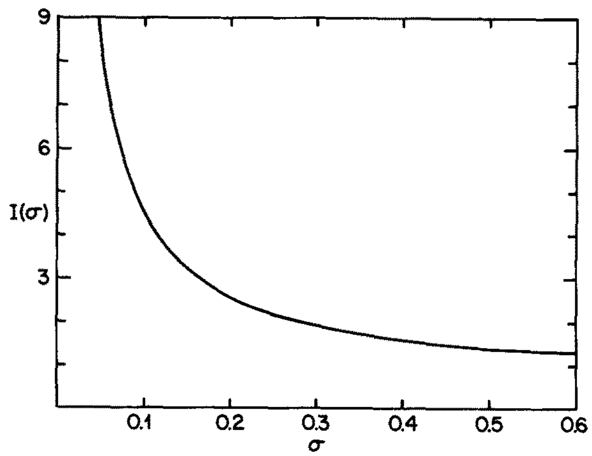

FIG. 5. Plot of the function $I(\sigma)$ used to determine the value of σ for various amalgamation mechanisms.

the frequency of tearing processes and ν_{tr} is the frequency of tripling processes then the values of k_F and k_D are given by the equations

$$k_F = \frac{1 + \nu_{te}}{1 + \nu_{tr}}, \quad k_D = \frac{2 + \nu_{tr} + \nu_{te}}{1 + \nu_{tr}}.$$

Note that if the frequencies of tearing and tripling are equal then $k_F = 1$ and $k_D = 2$, the values for pairing.

Therefore the width of the distribution $p(\xi)$ characterized by the parameter σ is determined by the type of amalgamation mechanism. The value of σ for several amalgamation mechanisms is given in Table I. Amalgamations by tearing, pairing, and tripling give increasing values of σ, respectively, and therefore broader probability distributions. In order to obtain the value of σ we note that amalgamations by pairing dominate the evolution of the mixing layer. Further, amalgamations by tearing and tripling tend to balance each other in their effect on σ. Hernan and Jimenez[4] found experimentally the same number of tearings and triplings. Thus the value for pairing $\sigma = 0.276$ should be a good estimate of this parameter in a fully developed turbulent mixing layer. The density and velocity ratio can only influence σ through their effect on the relative frequency of the various forms of amalgamations. These effects are expected to be small and therefore this value of σ is expected to be only weakly dependent on the velocity and density ratios.

The main conclusion of this study is that the probability distribution of large scale vortices in the self-similar region of a turbulent mixing layer is a lognormal distribution with the standard deviation normalized by the mean value equal to 0.28 ($\sigma = 0.276$) independent of velocity and density ratio. Comparison with other experimental and theoretical results is presented in Fig. 3 in terms of $\lambda/\bar{\lambda}$. The measurements of Brown and Roshko[1] in a mixing layer between nitrogen and helium streams and Bernal[3] in a constant density mixing layer are in good agreement with the result of the present analysis. The theoretical result of Takaki and Kovasznay[15] is also shown in the figure. They found a broader distribution with the mode at a smaller value of $\lambda/\bar{\lambda}$. A more quantitative comparison can be made in terms of the standard deviation of $\lambda/\bar{\lambda}$. Experimental measurements of the standard deviation are in the range 0.24–0.33,[1–4] in agreement with the theoretical result. These experimental results include measurements at several velocity and density ratios. Takaki and Kovasznay found a standard deviation of 0.39, somewhat larger than the measurements or the value found in the present theory. Koochesfahani *et al.*[5] measured a value for the standard deviation of 0.031. They also found a normalized mean value of the circulation somewhat higher than that of other investigators. These anomalous results might be attributable to the technique used to measure the vortex circulation, which may have biased the sample toward vortices early in their evolution where the velocity signature is expected to be cleaner.

TABLE I. The calculated averaged values of large scale vortex parameters for different forms of amalgamation.

	Tearing	Pairing	Tripling
k_D	3	2	$\frac{3}{2}$
σ	0.162	0.276	0.436
$\overline{\bar{\zeta}_F/\bar{\zeta}}$	0.798	0.660	0.484
$\overline{\bar{\zeta}_D/\bar{\zeta}}$	1.198	1.330	1.495
$\overline{L/x}$	0.518	1.074	2.306
$\overline{L/(x+L/2)}$	0.398	0.659	0.977

IV. LAGRANGIAN STATISTICS

The results obtained in the Eulerian frame of reference provide important new insight into the Lagrangian statistics. It is convenient to discuss the basic features of the Lagrangian statistics in terms of the similarity variable $\zeta = x/\lambda$. The probability of finding a vortex of circulation λ at a position $(x, x + dx)$ is given by $p(\xi)$, which after a change of variables reads

$$p_\zeta(\zeta) = \frac{1}{\sqrt{2\pi}\sigma\zeta} \exp\left\{ -\frac{1}{2\sigma^2}\left[\ln\left(\frac{\zeta}{\bar{\zeta}}\right) + \frac{\sigma^2}{2}\right]^2\right\},$$

with $\bar{\zeta} = \exp(\sigma^2)/\bar{\xi}$. Its maximum value is found at $\zeta_M = \bar{\zeta}\exp(-3\sigma^2/2)$. The probabilities of formation $p_F(\xi)$ and destruction $p_D(\xi)$ are given by Eqs. (8), which after a change of variables read

$$p_F(\zeta) = [(-\ln \zeta/\zeta_M)/k_F\sigma^2]p(\zeta), \quad p_D(\zeta) = 0, \quad 0 < \zeta < \zeta_M;$$

$$p_F(\zeta) = 0, \quad p_D(\zeta) = [(\ln \zeta/\zeta_M)/k_D\sigma^2]p(\zeta), \quad \zeta_M < \zeta < \infty.$$

In Fig. 6 $p(\zeta)$, $p_F(\zeta)$, and $p_D(\zeta)$ are sketched. These equations show that vortices with circulation λ are formed at $x < \zeta_M\lambda$ and destroyed at $x > \zeta_M\lambda$. If we focus on vortices of a given circulation λ the probability of finding vortices with

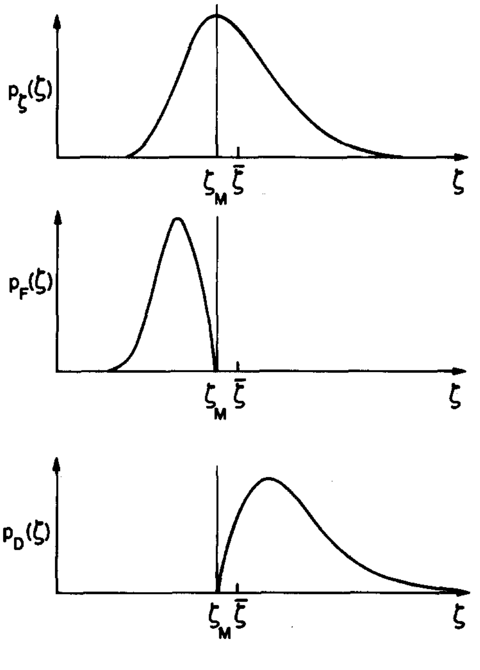

FIG. 6. Schematic diagram of the functions $p_\zeta(\zeta)$, $p_F(\zeta)$, and $p_D(\zeta)$ in the Lagrangian frame of reference.

circulation λ close to the origin of the mixing layer is zero. As we move downstream vortices of that circulation are formed by amalgamation, $p_F(\zeta) > 0$, and therefore $p_\xi(\zeta)$ increases. At $x = \lambda\zeta_M$ the probabilities of formation and destruction are both zero and $p_\xi(\zeta)$ has a maximum. Downstream of that point vortices of circulation λ are destroyed by amalgamation, $p_D(\zeta) > 0$, and therefore $p_\xi(\zeta)$ decreases.

An important property of the Lagrangian evolution of the vortices is their life span. If the life span of the vortices is normalized by their circulation the mean value can be calculated using the statistical independence of $p_F(\zeta)$ and $p_D(\zeta)$, which gives $\overline{L/\lambda} = \overline{\zeta}_D - \overline{\zeta}_F$, where $\overline{\zeta}_F$ and $\overline{\zeta}_D$ are the normalized mean formation and destruction locations, respectively. The values of $\overline{\zeta}_F$ and $\overline{\zeta}_D$ are obtained from the above equations as

$$\overline{\zeta}_F = 2\overline{\zeta}\,[I(2\sigma) - 1]/[I(\sigma) - 1], \quad \overline{\zeta}_D = 2\overline{\zeta}I(2\sigma)/I(\sigma),$$

where the function $I(\sigma)$ is defined by Eq. (10). The values of $\overline{\zeta}_F/\overline{\zeta}$ and $\overline{\zeta}_D/\overline{\zeta}$ depend only on σ and are given in Table I for the three amalgamation mechanisms. It follows that in this normalization the mean value of L/λ depends on both $\overline{\zeta}$ and σ.

The statistics of the vortex life span takes a simpler form when normalized by the formation location. The life span probability density, $p_L(L/x)$, is defined as the probability that a vortex formed at x will be destroyed at $x + L$. The contribution to this probability by vortices of circulation λ is $w[\xi,\xi/(1 + L/x)]$ with $\xi = \lambda/x$. The sum of all contributions gives the life span probability density, which after using the statistical independence of p_F and p_D reads

$$p_L\left(\frac{L}{x}\right) = \frac{1}{(1 + L/x)^2}\int_0^\infty \xi p_F(\xi) p_D\left(\frac{\xi}{1 + L/x}\right) d\xi.$$

Using p_F and p_D from Eqs. (8) it is found that

$$p_L(L/x) = \frac{\exp(-\Lambda^2/4)}{2\pi\, k_F k_D \sigma^3} \times [M(a,b) - M(b,a)]\,\frac{1}{1 + L/x},$$

where $\Lambda = 1/\sigma \ln(1 + L/x)$, $a = \sigma - \Lambda/2$, $b = \sigma + \Lambda/2$, and

$$M(x,y) = \left(xy + \frac{1}{2}\right)\int_0^x \exp(-t^2)dt + y\exp(-x^2).$$

Note that $p_L(0) = 0$ consistent with the fact that the life span of a vortex must be greater than zero.

The life span probability is by definition the probability that a vortex formed at x will have a life span greater than L. Thus

$$P_L\left(\frac{L}{x}\right) = \int_{L/x}^\infty p_L\left(\frac{L}{x}\right) d\left(\frac{L}{x}\right),$$

which after integration reads

$$P_L\left(\frac{L}{x}\right) = \frac{1}{2\pi k_F k_D}\int_\sigma^\infty \frac{\beta - \sigma}{\sigma} K(\beta - \Lambda) \times \exp\left(-\frac{\beta^2}{2}\right) d\beta,$$

where

$$K(x) = \int_{-\infty}^x \exp\left(\frac{-t^2}{2}\right)dt + \frac{1}{\sigma}\exp\left(\frac{-x^2}{2}\right), \quad -\infty < x < \sigma,$$

$$K(x) = K(\sigma) = \sqrt{2\pi}I(\sigma), \quad \sigma < x, \tag{11}$$

and $\Lambda = 1/\sigma \ln(1 + L/x)$. In constrast with the mean life span normalized by λ, P_L depends only on σ. The function $P_L(L/x)$ is plotted in Fig. 7 for the values of σ corresponding to tearing, pairing, and tripling amalgamations. As σ is increased the weight of the life span distribution shifts toward higher values of L/x, i.e., the vortices have a longer life span. This trend is shown clearly by the mean value of L/x given in Table I for the three amalgamation mechanisms. Also of interest is the average value of the life span normalized by the mean life span location, i.e., the mean value of $L/(x + L/2)$, given in Table I.

Comparison of the life span probability $P_L(L/x)$ for $\sigma = 0.276$ with the experimental measurements of Hernan and Jimenez[4] is presented in Fig. 8. Clearly the theoretical calculation agrees well with the measurements. The measured mean value is $\overline{L/x} = 0.97$ in good agreement with the theoretical result $\overline{L/x} = 1.074$. Brown and Roshko[1] reported an average value of the life span normalized with the mean life span location of 0.43, which is significantly lower than the calculated value 0.659 for $\sigma = 0.276$. It must be noted that the number of vortices in the experimental samples was small.

An independent characterization of the life span of the vortices is obtained from space-time correlation measurements between two velocity probes deployed in the downstream direction. Roshko[7] argued that the decay with increased distance between the probes of the maximum correlation coefficient is a manifestation of the finite life span of the large scale vortices. He proposed an exponential decay

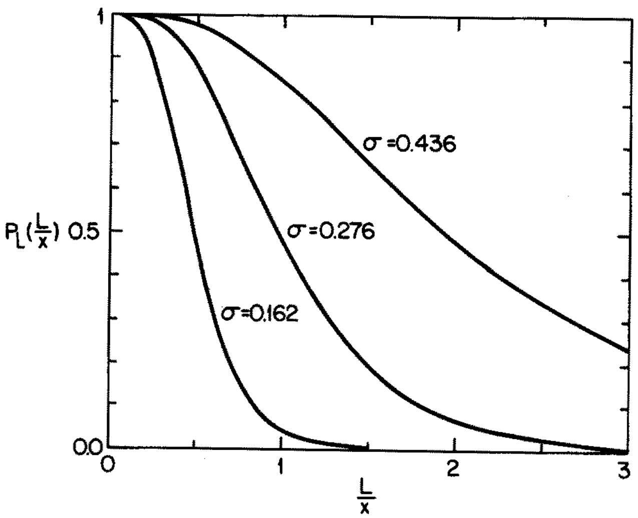

FIG. 7. Life span probability distribution $P_L(L/x)$ for values of σ corresponding to tearing ($\sigma = 0.162$), pairing ($\sigma = 0.276$), and tripling ($\sigma = 0.436$) amalgamations.

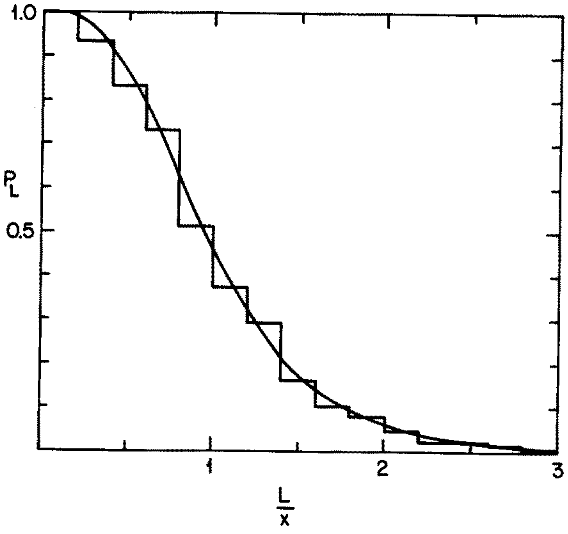

FIG. 8. Comparison of the calculated life span distribution with experimental measurements: —, present theory for pairing amalgamations; histogram, measured distribution of Hernan and Jimenez.[4]

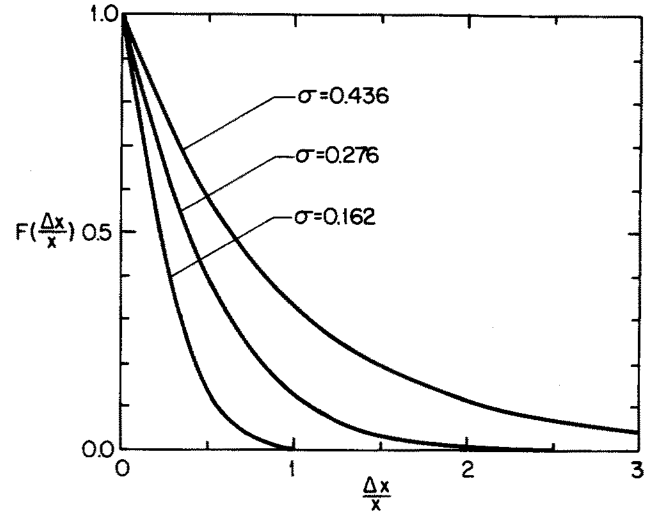

FIG. 9. Survival probability distribution $F(\Delta x/x)$ for values of σ corresponding to tearing ($\sigma = 0.162$), pairing ($\sigma = 0.276$), and tripling ($\sigma = 0.436$).

of characteristic length scale equal to the average value of the life span normalized by the mean position of the vortices. In order to relate the large scale statistics to the space-time correlation measurements the following three assumptions are made: (a) the velocity signature of the large scale vortices does not change during their life span (this assumption is consistent with the observation that the circulation of the vortices is constant during their life span); (b) the velocity signature of the large scale vortices before and after an amalgamation are uncorrelated; and (c) the contribution to the correlation coefficient by smaller scale motions is negligible. Under these assumptions the maximum value of the correlation coefficient for fixed probe spacing equals the probability that a vortex convected by the first probe is destroyed downstream of the second probe. We call this probability the survival probability of the vortices, $F(\Delta x/x)$, where Δx is the distance between the two probes and x is the location of the first probe.

In order to calculate the survival probability we introduce first the survival probability density, $f(\Delta x/x)$, defined as the probability that a vortex found at x will be destroyed at a distance Δx from x. The contribution to $f(\Delta x/x)$ by vortices of circulation λ is the product of the probability of finding a vortex of that circulation at x, $p(\xi)$, times the conditional probability that a vortex of circulation λ will be destroyed at $x + \Delta x$, $p_D[\xi/(1+\Delta x/x)]/W(\infty,\xi)$. Thus $f(\Delta x/x)$ is given by the equation

$$f\left(\frac{\Delta x}{x}\right) = \frac{1}{(1+\Delta x/x)^2} \times \int_0^\infty \frac{\xi p(\xi) p_D[\xi/(1+\Delta x/x)]}{\int_0^\xi p_D(t)dt}\, d\xi.$$

The survival probability distribution is then

$$F\left(\frac{\Delta x}{x}\right) = \int_{\Delta x/x}^\infty f\left(\frac{\Delta x}{x}\right) d\left(\frac{\Delta x}{x}\right),$$

which after integration gives

$$F\left(\frac{\Delta x}{x}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{K(\beta - \Lambda)}{K(\beta)} \exp\left(\frac{-\beta^2}{2}\right) d\beta,$$

where $K(x)$ is given by Eq. (11) and $\Lambda = 1/\sigma \ln(1 + \Delta x/x)$. We find then that the survival probability, $F(\Delta x/x)$, depends only on σ. It is independent of the velocity and density ratio. The function $F(\Delta x/x)$ is plotted in Fig. 9 for several values of σ. The mean value of the distribution shifts toward higher values of $\Delta x/x$ as σ is increased, consistent with Roshko's argument. An important difference between the life span probability and the survival probability is that the former has zero slope at the origin while the latter has finite negative slope. This is because the probability of a vortex having zero life span is zero but some vortices are destroyed by amalgamation at any downstream location and therefore $f(0) > 0$. The slope of $F(\Delta x/x)$ at the origin is proportional to the number of vortices destroyed at x.

Comparison of this result with experimental measurements is presented in Fig. 10. The solid line in the figure is the survival probability for $\sigma = 0.276$. The symbols are maximum values of the correlation coefficient for several probe spacings, Δx, normalized by the location of the upstream probe, x. These data were calculated from the reported measurements of Davies *et al.*[19] The experimental measurements are in good agreement with the theoretical result. It should be noted that these measurements were obtained in the mixing layer at the edge of the potential core of a circular jet and therefore the velocity ratio was zero. Also shown in Fig. 10 is the exponential curve proposed by Roshko.[7]

V. DISCUSSION

An essential aspect of the dynamics of the fully developed flow uncovered by this analysis is the nonoverlapping feature of the amalgamation statistics, i.e., only vortices with $\xi < \xi_C$ are destroyed and only vortices with $\xi > \xi_C$ are

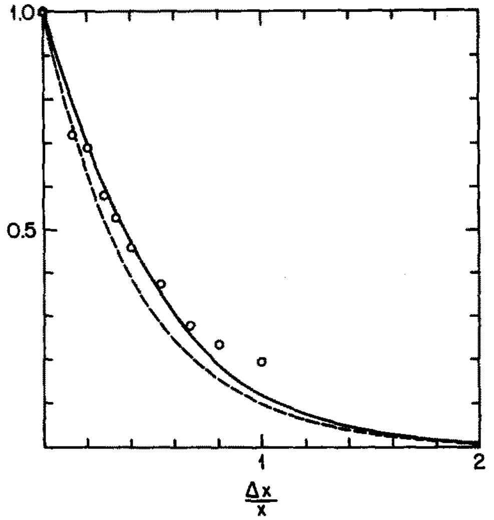

FIG. 10. Comparison of the calculated survival probability distribution with experimental measurements: —, present theory for pairing amalgamation; – –, proposed distribution of Roshko[7]; O, cross correlation measurements of Davies *et al.*[19]

formed by amalgamation in the fully developed turbulent flow. Thus vortices with $\lambda < \lambda_C = \xi_C x$ are "dynamically unstable" and vortices with $\lambda > \lambda_C$ are "dynamically stable." This feature of the dynamics of the large scale vortices in the fully developed flow is a manifestation of the underlying stability characteristics of the flow. The cutoff value λ_C varies linearly with downstream distance. These features are analogous to the stability characteristics of the laminar mixing layer. In the laminar layer, disturbances with small wavelength compared to the local thickness of the layer tend to decay while disturbances with wavelength large compared to the local thickness of the layer tend to grow. The cutoff point in the large scale vortex dynamics corresponds to the neutral stability point in the laminar stability problem. Given the lack of detailed quantitative understanding of the dynamics of the large scale vortices, this analogy can be used to estimate the value of λ_C from linear stability theory,[20–22] in the spirit of the model proposed by Ho.[23,24] For typical mixing layer velocity profiles the inviscid analyses of Monkewitz and Huerre,[21] for the constant density layer, and of Maslowe and Kelly,[22] for the nonuniform density layer, give values of λ_C/δ_ω in the range 3.14–3.43 at the neutral stability point. Based on these results the calculated values of $\bar{\lambda}/\delta_\omega$ for $\sigma = 0.276$ are in the range 3.0–3.3 in good agreement with experimental results.[1]

An interesting result of this investigation is the finding that the distribution of large scale vortex circulation in the fully developed turbulent mixing layer is lognormal. This result is quite general since it was based only on self-similarity, conservation of circulation during the life span of the vortices, and the small change of $\ln \xi$ in the amalgamations. It is independent of details of the amalgamation process. Takaki and Kovasznay[15] also considered the statistics of the large scale vortices under these same basic assumptions but used a different model of the amalgamation process. Hence the result of their analysis is expected to be a lognormal distribution even though they used a different approximate form of the distribution in their calculations. If the variance and third moment about the mean of $\ln \xi$ are evaluated for their distribution, it is found that $(\sigma^2)_{TK} = 0.153$ and $(\mu_3)_{TK} = -0.0013$, respectively. The very small value of the third moment confirms our expectations. Clearly Takaki and Kovasznay's distribution is well approximated by a lognormal distribution.

A second observation in comparing these results with the results of Takaki and Kovasznay is the large difference between the variance of $\ln \xi$, σ^2. In their analysis the width of the probability distribution is determined by the rate of merging, a function of vortex spacing. They assumed a rate of merging inversely proportional to the square of the distance between the vortices. This functional dependence was justified on the basis of the behavior of vortices with small spacing. However, they also found, by solving the inverse problem, some discrepancies between the assumed form of the merging function and the merging function calculated from the experimental measurements of Brown and Roshko.[1] In the present analysis the value of σ was obtained based on the assumption of statistical independence of the amalgamation locations, which leads to the nonoverlapping feature of $p_F(\xi)$ and $p_D(\xi)$. It is apparent that the Takaki–Kovasznay model is not consistent with statistical independence of the amalgamation locations. It follows that for the assumed form of the merging function $p_F(\xi)$ and $p_D(\xi)$ will overlap. This more general case was treated by Bernal.[3] The main difference in the analysis is that a non-negative function appears on the solution for $p_F(\xi)$ and $p_D(\xi)$. This function results in a positive constant added to the right-hand side of Eq. (9). Therefore, if $p_F(\xi)$ and $p_D(\xi)$ are not statistically independent, the value of σ for a given form of amalgamation must be larger than the value obtained for statistical independence. Thus the value of σ for the Takaki–Kovasznay distribution must be greater than the value obtained in the present investigation, as indeed it is the case. We conclude then that the reason for the discrepancy in the values of σ is the merging function assumed by Takaki and Kovasznay, which does not accurately model the nonoverlapping feature of the amalgamation statistics.

The lognormality of the probability distribution reduces the problem of the large scale vortex statistics to the determination of two parameters: the mean value of the distribution, $\bar{\xi}$, and the variance of $\ln \xi$, σ^2. Current understanding of the large scale vortex dynamics in the fully developed turbulent flow is not adequate to determine either one of these parameters. The present theory can be used to relate these parameters to measurable quantities. In the Eulerian frame of reference $\bar{\xi}$ determines the average passage frequency of the large scale vortices, $n_T(x)$. In the Lagrangian frame of reference $\bar{\xi}$ determines the mean value of the vortex life span normalized by its circulation. The results presented earlier (Sec. IV) give $\overline{L/\lambda} = 0.723\ (\bar{\xi})^{-1}$ for $\sigma = 0.276$. This second physical interpretation can be used to obtain the relation between $\bar{\xi}$ and the velocity and density ratios. It is argued that the vortex life time scales as $\lambda / \Delta U$. Thus the life span is proportional to $\lambda U_C/\Delta U$ and, consequently, $\bar{\xi}$ is proportional to $\Delta U/U_C$. This scaling should be valid for arbitrary values of the velocity and density ratios.[1-4,25] These measurements give values of $\bar{\xi} U_C/\Delta U$ in the range 0.27–0.31. It is interesting to compare these results on the vortex life span in the fully developed flow with measurements of the life span of the vortices in the initial region of a forced mixing layer obtained by Ho and Huang.[26] They measure for a second generation vortex, i.e., after the first pairing, $L/\lambda = 4$ with $\Delta U/U_C = 0.62$. This value of L/λ is equal to the mean value calculated for the fully developed flow at the same flow conditions.

The second parameter of the lognormal distribution is the variance of $\ln \xi$, σ^2. In the Eulerian frame of reference σ determines the width of the distribution of vortex circulation. In the Lagrangian frame of reference σ determines the life span, normalized by the formation location, and the survival probability distributions. As σ is increased the width of the distribution of vortex circulation increases and the mean life span and the mean value of the survival probability also increase. The parameter σ was also related to the amalgamation mechanism. The value for pairing was found to be in good agreement with experimental measurements of the vortex circulation distribution, the life span distribution, and the survival probability distribution. The dominance of pairing in the fully developed flow is another feature common to the laminar stability problem. In the laminar layer it is related to the fact that the wavelength of the most amplified disturbances is approximately twice the wavelength of neutral disturbances. The experimental evidence available suggests that σ does not depend on the velocity or density ratios.

VI. CONCLUDING REMARKS

One initial motivation for this work was the realization that the measured probability distribution of large scale vortices is well approximated by a lognormal distribution. This observation led to an early attempt to use Kolmogorov's breakage theory[16] and to the idea that the fully developed flow can only be reached after a large number of amalgamations. It now appears that the lognormality of the distribution is a consequence of the underlying stability characteristics of the flow. Several aspects of the dynamics of the fully developed flow have been shown to be analogous to the instability characteristics of the laminar layer. This analogy provides very good quantitative estimates of several aspects of the evolution of the vortices in the fully developed turbulent flow. These instability characteristics combine with randomization of the vortex size and life span to produce a rapid transition to the self-similar lognormal distribution. Experimental evidence[3] suggests that only three or four amalgamations may be needed to obtain the lognormal distribution. This self-similar distribution is a necessary condition for the fully developed turbulent flow. In addition, the development of small scale three-dimensional motions will also be required. These, however, play a small role in the statistical distribution of large scale vortices. Better understanding of the nonlinear dynamics of the large scale vortices is needed to determine the parameters of the large scale vortex distribution. The theory described here does provide a self-consistent description of the large scale statistics in the Eulerian and Lagrangian frames of reference that is in good agreement with experimental results.

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APPENDIX: ASYMPTOTIC EXPANSION OF $p(\xi)$

The following derivation of an approximate form for $p(\xi)$ is motivated by the fact that in a typical amalgamation ξ changes by a factor of 2. Hence $\alpha = \ln \xi$ changes by an additive constant $\ln 2 = 0.693$. This suggests that the probability distribution when written in terms of α, $p_\alpha(\alpha)$, will have small variance. Thus an asymptotic expansion of $p_\alpha(\alpha)$ for small variance may provide a valuable approximation for $p(\xi)$.

The function $p_\alpha(\alpha)$ is given by Eq. (5), which after a change of variables reads

$$p_\alpha(\alpha) = C \exp[g(\alpha) - \alpha].$$

The asymptotic behavior of $p_\alpha(\alpha)$ is derived using the characteristic function $\Phi(\tau)$ defined in the usual way,

$$\Phi(\tau) = \int_{-\infty}^{\infty} \exp(i\tau\alpha) p_\alpha(\alpha) d\alpha$$

$$= C \int_{-\infty}^{\infty} \exp[i\tau\alpha + g(\alpha) - \alpha] d\alpha.$$

The asymptotic expansion for $\Phi(\tau)$ is found from the behavior of $g(\alpha) - \alpha$ near its maximum, α_0. The value of α_0 is determined from the equation $g'(\alpha_0) = 1$. An expansion of $g(\alpha) - \alpha$ in powers of $\alpha - \alpha_0$ gives, to second order,

$$\Phi(\tau) = \sqrt{2\pi}\sigma C \exp[g(\alpha_0) - \alpha_0] \exp(i\tau\alpha_0 - \sigma^2\tau^2/2)$$
$$\times [1 + \sigma^4 g^{\mathrm{iv}}(\alpha_0)/8 + i\sigma^4 g^{\mathrm{iii}}(\alpha_0)\tau/2 + O(\sigma^6)],$$

where $\sigma^2 = -1/g''(\alpha_0)$ is the variance of the distribution and $g^{\mathrm{iii}}(\alpha_0)$ and $g^{\mathrm{iv}}(\alpha_0)$ are the third and fourth derivatives of $g(\alpha)$ at $\alpha = \alpha_0$, respectively. This asymptotic expansion is valid for small variance, i.e., $\sigma^2 \ll 1$.

The normalization condition on $p_\alpha(\alpha)$ gives $\Phi(0) = 1$, which determines C as

$$C = \frac{1}{\sqrt{2\pi}\sigma} \frac{\exp[\alpha_0 - g(\alpha_0)]}{1 + \sigma^4 g^{\mathrm{iv}}(\alpha_0)/8} + O(\sigma^6),$$

and therefore

$$\Phi(\tau) = \exp(i\tau\alpha_0 - \sigma^2\tau^2/2)$$
$$\times [1 + i\sigma^4 g^{\mathrm{iii}}(\alpha_0)\tau/2 + O(\sigma^6)].$$

To leading order $p_\alpha(\alpha)$ is given by the exponential term that is the characteristic function of a normal distribution with mean (α_0) and variance σ^2. It follows that $p(\xi)$ is lognormally distributed with mean $\bar{\xi} = \exp(\alpha_0 + \sigma^2/2)$ and variance $(\bar{\xi})^2[\exp(\sigma^2) - 1]$.

The accuracy of this approximation can be determined from the moments of the distribution $p_\alpha(\alpha)$. Of particular interest is the third moment about the mean since this moment is zero for a normal distribution. These moments are obtained from the characteristic function, which give to second order

$$\overline{\alpha} = \alpha_0 + \sigma^4 g^{\text{iii}}(\alpha_0)/2,$$
$$\mu_2 = \sigma^2,$$
$$\mu_3 = 0.$$

Therefore retaining the second-order term in the expansion for $p_\alpha(\alpha)$ results in a small correction to the mean. The second and third moment about the mean equal the values for a normal distribution to order σ^6.

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