Fluid sources for Bianchi I and III space-times

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Four analytic solutions to the Einstein field equations are presented. The solutions are parametrized to have either Bianchi I or Bianchi III symmetry. The associated fluid parameters are given and some of them are discussed in detail.

I. INTRODUCTION

Space-times admitting a three-parameter group of automorphisms are important in discussing cosmological models. The case where the group is simply transitive over the three-dimensional, constant-time subspace is particularly useful for two reasons. First, Bianchi[1] has shown there are only nine distinct sets of structure constants for groups of this type so that the algebra may be easily used to classify homogeneous space-times. The second reason for the importance of the Bianchi spaces is the simplicity of the field equations. The relative ease of solution has made these space-times useful in constructing models of spatially homogeneous cosmologies. There is a large literature concerning specific Bianchi spaces which contain fluids with specified equations of state. A partial list is given in the bibliography.[2-17]

In this paper we present and discuss some solutions which can belong to either Bianchi type I or III. These solutions have a special interest as they allow one to investigate fluid behavior across two Bianchi types. The solutions we find are also locally rotationally symmetric and fit into type II of the classification scheme given by Stewart and Ellis.[18]

II. FIELD EQUATIONS AND THEIR SOLUTIONS

The metric we consider is

$$ds^2 = -dt^2 + \gamma_1(t)dx^2 + \gamma_2(t)e^{-2ax}dy^2 + \gamma_3(t)dz^2, \tag{2.1}$$

where $a=0$ gives Bianchi I and $a=1$ gives Bianchi III space-times. The field equations are $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$.

The Einstein tensors $G_{\mu\nu}$ for our metric are

$$G_{00} = \frac{\dot\gamma_3\dot\gamma_1}{4\gamma_1\gamma_3} + \frac{\dot\gamma_3\dot\gamma_2}{4\gamma_3\gamma_2} + \frac{\dot\gamma_1\dot\gamma_2}{4\gamma_1\gamma_2} - \frac{a^2}{\gamma_1}, \tag{2.2}$$

$$G^1_1 = -\frac{\ddot\gamma_2}{2\gamma_2} - \frac{\ddot\gamma_3}{2\gamma_3} - \frac{\dot\gamma_3\dot\gamma_2}{4\gamma_3\gamma_2} + \frac{\dot\gamma_2^2}{4\gamma_2^2} + \frac{\dot\gamma_3^2}{4\gamma_3^2}, \tag{2.3}$$

$$G^2_2 = -\frac{\ddot\gamma_1}{2\gamma_1} - \frac{\ddot\gamma_3}{2\gamma_3} - \frac{\dot\gamma_3\dot\gamma_1}{4\gamma_1\gamma_3} + \frac{\dot\gamma_1^2}{4\gamma_1^2} + \frac{\dot\gamma_3^2}{4\gamma_3^2}, \tag{2.4}$$

$$G^3_3 = -\frac{\ddot\gamma_1}{2\gamma_1} - \frac{\ddot\gamma_2}{2\gamma_2} - \frac{\dot\gamma_1\dot\gamma_2}{4\gamma_1\gamma_2} + \frac{\dot\gamma_2^2}{4\gamma_1^2} + \frac{\dot\gamma_2^2}{4\gamma_2^2} + \frac{a^2}{\gamma_1}, \tag{2.5}$$

$$R_{01} = +\frac{a}{2}\frac{\dot\gamma_1}{\gamma_1} - \frac{a}{2}\frac{\dot\gamma_2}{\gamma_2}. \tag{2.6}$$

To find analytic solutions with Bianchi I and III symmetry we consider locally rotationally symmetric metrics, with $\gamma_1 = \gamma_2$. This metric falls into type II of the classification scheme given by Stewart and Ellis.[18] For this choice of γ, G^1_1 is identically equal to G^2_2, and the isotropic pressure requirement for perfect fluid sources gives

$$\frac{\ddot\gamma_3}{2\gamma_3} - \frac{\ddot\gamma_1}{2\gamma_1} + \frac{\dot\gamma_3\dot\gamma_1}{4\gamma_1\gamma_3} - \frac{\dot\gamma_3^2}{4\gamma_3^2} + \frac{a^2}{\gamma_1} = 0. \tag{2.7}$$

The method we shall use to generate solutions to (2.7) is to assume a solution for $\gamma_1(t)$ [or $\gamma_3(t)$] and then solve for $\gamma_3(t)$ [or $\gamma_1(t)$]. At this stage the only criteria we use is the integrability of the field equations. We then will evaluate some of the physical parameters and discuss them.[19]

Solution 1: Assume $\gamma_3(t) = C_1 e^{\tilde t}$. Then

$$\gamma_1(t) = -(1/C_0)[A_1 e^{\tilde t} + A_2 e^{-(1/2)\tilde t} + 8a^2], \tag{2.8}$$

with[20] $\tilde t = \sqrt{C_0/2}\,t$. The pressure and density are

$$\begin{aligned} 8\pi p(t) = {} & -\frac{C_0}{8} + \frac{3}{8}\frac{A_1 e^{\tilde t}}{\gamma_1} + \frac{1}{8\gamma_1^2 C_0}\left(A_1 e^{\tilde t} - \frac{1}{2}A_2 e^{-(1/2)\tilde t}\right)^2 + \Lambda, \\ 8\pi\epsilon(t) = {} & -\frac{a^2}{\gamma_1} - \frac{1}{4\gamma_1}\left(A_1 e^{\tilde t} - \frac{1}{2}A_2 e^{-(1/2)\tilde t}\right) + \frac{(A_1 e^{\tilde t} - \frac{1}{2}B e^{-(1/2)\tilde t})^2}{8\gamma_1^2 C_0} - \Lambda. \end{aligned} \tag{2.9}$$

We have included the cosmological constant in Eq. (2.9) for generality. Its inclusion does not alter (2.7). As $t \to 0$

$$\begin{aligned} 8\pi p(0) = {} & -\frac{C_0}{8}\frac{(4A_1 + 8A_2 + 8a^2)}{(A_1 + A_2 + 8a^2)} + \frac{C_0}{8}\frac{(A_1 - A_2/2)^2}{(A_1 + A_2 + 8a^2)^2} + \Lambda, \\ 8\pi\epsilon(0) = {} & \frac{C_0}{8}\frac{(2A_1 - A_2 + 8a^2)}{(A_1 + A_2 + 8a^2)} + \frac{C_0}{8}\frac{(A_1 - A_2/2)^2}{(A_1 + A_2 + 8a^2)^2} - \Lambda. \end{aligned} \tag{2.10}$$

At $t \to \infty$, using $\gamma_1(t)$ we find

$$8\pi p(\infty) = \Lambda - \tfrac{3}{8}C_0, \quad 8\pi\epsilon(\infty) = \tfrac{3}{8}C_0 - \Lambda. \tag{2.11}$$

Choosing $\Lambda = \frac{3}{8}C_0$, one can generate in a Bianchi III space-time, a variety of initial perfect fluids which expand at large times to a very dilute, essentially empty space solution.

For example, choosing $A_2 = 0$ and $A_1 < 0$, we have, with $\Lambda = \frac{3}{8}C_0$,

$$\begin{aligned} 8\pi p(0) &= C_0 a^2(-|A_1| + 16a^2)/(-|A_1| + 8a^2)^2, \\ 8\pi\epsilon(0) &= C_0 a^2(3|A_1| - 16a^2)/(-|A_1| + 8a^2)^2. \end{aligned} \tag{2.12}$$

Physical solutions correspond to $8 < |A_1|/a^2 \leqslant 16$.

The Bianchi I solution is a vacuum. In Bianchi III the solution is a fluid which can have an equation of state of the form $p = \alpha\epsilon$ with $0 \leqslant \alpha < 1$ at $t = 0$.

Solution 2: Assume $\gamma_3(t) = C_1 \cos^2 \tilde{t}$. Then one finds

$$\gamma_1(t) = (1/C_0)(\bar{\gamma}_1(t) + 2a^2), \quad C_0 > 0, \tag{2.13}$$

with $\tilde{t} = \sqrt{(C_0/2)}\,(t - C_2)$ and

$$\bar{\gamma}_1(t) = \cos^2(\tilde{t})\left[A_1 - A_2 \tan\tilde{t}\sec\tilde{t} - A_2 \ln\left|\frac{1 + \sin\tilde{t}}{\cos\tilde{t}}\right|\right].$$

The pressure and density are

$$\begin{aligned} 8\pi p(t) &= C_0 - \frac{a^2}{\gamma_1} - \frac{C_0 \tan^2\tilde{t}}{2} + \frac{(2a^2\tan\tilde{t} - A_2\sec\tilde{t})^2}{2C_0\gamma_1^2} + \Lambda, \\ 8\pi\epsilon(t) &= \frac{3}{2}C_0\tan^2\tilde{t} - \frac{4a^2\tan^2\tilde{t}}{\gamma_1} + \frac{2A_2\sec\tilde{t}\tan\tilde{t}}{\gamma_1} - \frac{a^2}{\gamma_1} \\ &\quad + (1/2C_0\gamma_1^2)[2a^2\tan\tilde{t} - A_2\sec\tilde{t}\,]^2 - \Lambda. \end{aligned} \tag{2.14}$$

This solution could be valid over a range of time.

Solution 3:

(a) Assume $\gamma_3(t) = \tilde{t}^{\,2}$, where $\tilde{t} = C_1 t + C_2$, then one obtains

$$\gamma_1(t) = A_1\tilde{t}^{\,2} + a^2\tilde{t}^{\,2}(\ln(A_2\tilde{t}) - \tfrac{1}{2})/C_1^2. \tag{2.15}$$

The pressure and density distributions are

$$\begin{aligned} 8\pi p(t) &= -\frac{C_1^2}{\tilde{t}^{\,2}} - \frac{a^2}{\gamma_1} + \frac{\tilde{t}^{\,2}a^4}{4\gamma_1^2 C_1^2} + \Lambda, \\ 8\pi\epsilon(t) &= \frac{3C_1^2}{\tilde{t}^{\,2}} + \frac{a^2}{\gamma_1} + \frac{\tilde{t}^{\,2}a^4}{4C_1^2\gamma_1^2} - \Lambda. \end{aligned} \tag{2.16}$$

As $t \to \infty$, $p(\infty) = \Lambda$ and $\epsilon(\infty) = -\Lambda$, so $\Lambda = 0$ would give a large time an essentially empty universe. As t approaches zero we have

$$\begin{aligned} 8\pi p(0) &= -\frac{C_1^2}{C_2^2} + a^2\frac{a^2C_2^2}{4\gamma_1^2C_1^2} - \frac{1}{\gamma_1}, \\ 8\pi\epsilon(0) &= \frac{3C_1^2}{C_2^2} + a^2\frac{a^2C_2^2}{4\gamma_1^2C_1^2} - \frac{1}{\gamma_1}. \end{aligned} \tag{2.17}$$

For Bianchi I, a positive pressure is not possible with $\Lambda = 0$. For Bianchi III, a proper choice of constants will produce a range of physical solutions.

For example, take C_1 and C_2 as input, $a^2 = 1$. Choose $A_2 = 1/C_2$. This choice sets a limit on A_1

$$A_1 > 1/2C_1^2, \quad C_1 \neq 0,$$

or (2.18)

$$A_1 = n/2C_1^2, \quad n > 1.$$

This is required to keep $\gamma_1 > 0$ and maintain the signature. Here, n parametrizes A_1. The pressure and density are

$$\begin{aligned} \frac{8\pi p(t)\tilde{t}^{\,2}}{C_1^2} &= -1 - \frac{2}{n-1} + \frac{1}{(n-1)^2}, \\ \frac{8\pi\epsilon(t)\tilde{t}^{\,2}}{C_1^2} &= 3 + \frac{2}{n-1} + \frac{1}{(n-1)^2}. \end{aligned} \tag{2.19}$$

Positive pressure requires $n^2 < 2$ so range of n is

$$1 < n^2 < 2.$$

The equation of state of this fluid can be written as $p = \alpha\epsilon$, with

$$\frac{1}{\alpha} = \frac{3(n-1)^2 + 2(n-1) + 1}{2 - n^2}. \tag{2.20}$$

The positive pressure condition allows any $\alpha > 0$. A radiation equation of state $\alpha = \frac{1}{3}$ corresponds to $n = 1.22$, and $\alpha = 1$ is forbidden by the signature requirement $n > 1$. The allowed equation of states have any $\alpha < 1$.

(b) There are two additional solutions which we provide for completeness.

(1): Given

$$\gamma_3(t) = C_2, \tag{2.21}$$

one finds $\gamma_1(t) = a^2t^{\,2} + A_1 t + A_2$. The pressure and density are

$$\begin{aligned} 8\pi p &= -\frac{a^2}{\gamma_1} + \frac{(2a^2t + A_1)^2}{4\gamma_1^2} + \Lambda, \\ 8\pi\epsilon &= -\frac{a^2}{\gamma_1} + \frac{(2a^2t + A_1)^2}{4\gamma_1^2} - \Lambda. \end{aligned} \tag{2.22}$$

For zero Λ, these describe an empty universe at large time.

(2): Given

$$\gamma_1 = C_0(C_1t + C_2)^m \text{ and } a = 0, \tag{2.23}$$

one finds $\gamma_3 = (C_1t + C_2)^n$, with

$$2n(n-1) - 2m(m-1) + nm - n^2 = 0. \tag{2.24}$$

The fluid parameters are

$$\begin{aligned} 8\pi p &= \frac{C_1^2}{(C_1t + C_2)^2}\left[\frac{m^2}{4} - m(m-1)\right], \\ 8\pi\epsilon &= \frac{C_1^2}{(C_1t + C_2)^2}\left(\frac{nm}{2} + \frac{m^2}{4}\right). \end{aligned} \tag{2.25}$$

These fluid parameters go to zero at large time. Vacuum corresponds to $m = 0$ and to $m = \frac{4}{3}$, $n = -\frac{2}{3}$, the latter a Kasner solution.[21] This fluid can have an equation of state $p = \alpha\epsilon$, with

$$\alpha = (4 - 3m)/(m + 2n). \tag{2.26}$$

Equation (2.24) has two possible solutions, $n = 2(1 - m)$ and $n = m$. This allows an easy classification of possible fluids.

(*i*) $\alpha \neq 1$:

$$m = n = \tfrac{4}{3}(1+\alpha),$$

$$8\pi p = 8\pi\alpha\epsilon = 4C_1^2\alpha/3(1+\alpha)^2(C_1 t + C_2)^2. \qquad (2.27)$$

For example, a radiation fluid corresponds to $m = n = 1$.

(ii) $\alpha = 1$: All m with $n = 2(1-m)$. There is an infinite set of solutions for the stiff fluid. We find

$$8\pi p = 8\pi\epsilon = C_1^2(4m - 3m^2)/4(C_1 t + C_2)^2. \qquad (2.28)$$

Only those solutions with $m^2 < \tfrac{4}{3} m$ will correspond to physical fluids. An infinite sequence of hard universes has been noted before by Jacobs[22] and Gröbner and Hofreiter.[23] We believe our results are a simple case of their solutions. A recent method[24] using metrically valid but unphysical fluid solutions to generate physical fluids makes these simple solutions valuable.

Solution 4: Given $\gamma_1(t) = a^2t^2 + C_0 t + C_1$, one obtains

$$\gamma_3(t) = A_0 + \frac{A_1}{a}\ln|2a(a^2t^2 + C_0t + C_1)^{1/2} + 2a^2t + C_0|. \qquad (2.29)$$

The pressure and energy density are

$$8\pi p = \frac{C_0^2 - 4a^2C_1}{4\gamma_1^2} + \Lambda,$$
$$8\pi\epsilon = \frac{C_0^2 - 4a^2C_1}{4\gamma_1^2} + \frac{A_1(2a^2t + C_0)}{\gamma_1^{3/2}\gamma_3^{1/2}} - \Lambda. \qquad (2.30)$$

Choosing $C_0 = 0$, $A_1 = 0$, and $C_1 < 0$ gives a solution

$$8\pi p = \frac{4a^2|C_1|}{4\gamma_1^2},$$
$$8\pi\epsilon = \frac{4a^2|C_1|}{4\gamma_1^2} + \frac{A_1 2a^2t}{\gamma_1^{3/2}\gamma_3^{1/2}}. \qquad (2.31)$$

This is a vacuum Bianchi I. In Bianchi III, a more complex fluid is obtained. For $A_1 = 0$, there is a stiff fluid equation of state.[12]

III. DISCUSSION

We have seen that the metric solution can generate very different fluid pressure and densities in the two Bianchi types. A convenient indicator of other possible differences are the velocity paremeters: expansion, vorticity, acceleration, and shear. For the metric we have discussed, all of the fluids are acceleration and rotation free, but they do have expansion θ and shear $\sqrt{\sigma^2}$ given by[25]

$$\theta = \dot\gamma_3/2\gamma_3 + \dot\gamma_1/\gamma_1,$$
$$\sigma^2 = \tfrac{1}{8}(\dot\gamma_1/\gamma_1 - \dot\gamma_3/\gamma_3)^2. \qquad (3.1)$$

We find the shear can be very type dependent for our solutions. For example, in Solution 1, the effect of Bianchi type is largest at small times. For small times we find

$$\sigma^2 \cong \frac{C_0}{12}\left(\frac{1.5A_2 + 8a^2}{A_1 + A_2 + 8a^2}\right)^2. \qquad (3.2)$$

The model we gave, $A_2 = 0$, gives a shear-free vacuum in Bianchi I and a nonzero shear in Bianchi III. It is clear that choosing $A_2 = -16a^2/3$ would reverse this. Solution 3 shows similar effects for large times.

In conclusion, we have presented four combined fluid solutions to Bianchi I and III space-times. The behavior of the fluid is type dependent and can be physically reasonable. Using metric solutions parametrized over several Bianchi types is a useful tool in constructing and studying fluid cosmologies.

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[19] This equation for Bianchi I space-time can also be written in the following convenient form:

$$\frac{1}{\gamma_1\gamma_2}\frac{d}{dt}(\gamma_1\dot\gamma_2) - \frac{1}{2}\frac{d}{dt}\left(\frac{\dot\gamma_1}{\gamma_1}\ln\gamma_1\gamma_2^3\right) + \frac{d}{dt}\left(\frac{\dot\gamma_1}{\gamma_1}\right)\left[-1 + \frac{1}{2}\ln\gamma_1\gamma_2^3\right] = 0.$$

This is analogous to the Tolman equation given for the relativistic fluid sphere field equation.

[20] Here A_i, C_i, and B are constants of integration.

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