Hydrodynamic Analysis of Noise in a Finite-Temperature Electron Beam*

H. C. Hsieh
Electron Physics Laboratory, Department of Electrical Engineering, The University of Michigan, Ann Arbor, Michigan
(Received 4 March 1965)

On the basis of a small-signal, one-dimensional analysis, a set of basic macroscopic differential equations, governing the fluctuations in quantities such as the electron-beam temperature, the mean velocity, and the current density, has been derived by taking moments of the Liouville equation with respect to the velocity variable. This set of differential equations expresses the conservations of charge, momentum, and energy, and is valid for an arbitrary amount of velocity spreading and includes the effect of heat conduction.

A system of differential equations, governing the correlation among the fluctuations in the mean velocity, current density, and beam temperature, is also derived. The relationship among the various noise parameters along the electron beam is obtained in the form of a system of differential equations whose solution gives detailed information on the variation of the noisiness parameter along the beam. The solution of the system of differential equations thus derived is also discussed.

I. INTRODUCTION

THE analysis of a multivelocity electron beam by the density-function method has been discussed by Siegman,[1] and using this method of analysis the noise propagation in a one-dimensional space-charge-limited diode has been investigated numerically by Siegman, Watkins, and Hsieh.[2] The result of their numerical analysis shows that the noise parameters defined by Haus[3] do not remain invariant as the beam passes through a multivelocity region, which suggests that both the self-power and cross-power density spectra of shot noise fluctuations can undergo considerable modification in propagation through the potential minimum region. In particular, the quantity $(S-\Pi)$, which determines the theoretical minimum noise figure of a beam-type amplifier, decreases considerably below its value at the cathode.

Although the microscopic density-function method of analysis of Siegman *et al.*[2] is rigorous, it is also intricate, and depends upon solving a complicated partial differential equation for representative solutions. On the other hand, there exists a simpler macroscopic "hydrodynamical" model of an electron beam introduced by Hahn,[4] which may also describe at least the first-order effects of velocity spread. This model has been used by Parzen[5,6] and Goldstein[5] in a discussion of traveling-wave-tube gain, and later by Berghammer and Bloom[7] in their discussion of the nonconservation of the noise parameters in a multivelocity electron beam with sufficiently small but nonzero velocity spread. These latter authors have demonstrated the possibility of obtaining an equivalent transmission-line equation for a beam with a small velocity spread and have also discussed the case of a drifting beam in some detail.

* This work was supported by the Rome Air Development Center under Contract No. AF30(602)-3569.

[1] A. E. Siegman, J. Appl. Phys. **28**, 1132 (1957).
[2] A. E. Siegman, D. A. Watkins, and H. C. Hsieh, J. Appl. Phys. **28**, 1138 (1957).
[3] H. A. Haus, J. Appl. Phys. **26**, 560 (1955).
[4] W. C. Hahn, Proc. IRE **36**, 1115 (1948).
[5] P. Parzen and L. Goldstein, J. Appl. Phys. **22**, 398 (1951).
[6] P. Parzen, J. Appl. Phys. **23**, 394 (1952).
[7] J. Berghammer and S. Bloom, J. Appl. Phys. **31**, 454 (1960).

In this paper an attempt is made to develop a method of analysis of noise in a multivelocity electron beam based on a hydrodynamic model, which adequately takes into account the effect of heat conduction as well as temperature fluctuations along the beam.

Based on a small-signal, one-dimensional analysis, a set of basic differential equations governing the fluctuations in the mean electron beam velocity, the current density, and the electron beam temperature is derived by taking the moments of Liouville's equation (collision-free Boltzmann equation) with respect to the velocity variable. The relationships among the various noise parameters along the electron beam are derived in the form of a system of ordinary differential equations whose solution yields the desired information on the variation of noise parameters. The solution of the system of differential equations, thus derived, is discussed briefly.

II. DERIVATION OF THE BASIC DIFFERENTIAL EQUATIONS GOVERNING A MULTIVELOCITY ELECTRON BEAM

The Boltzmann equation for a one-dimensional, nonrelativistic, collision-free electron beam is written as

$$\frac{\partial F(x,u,t)}{\partial t}+u\frac{\partial F(x,u,t)}{\partial x}+\eta E(x,t)\frac{\partial F(x,u,t)}{\partial u}=0, \quad (1)$$

where $E(x,t)$ is the longitudinal electric field intensity and η is the charge-to-mass ratio, with m being the electronic mass and e the electronic charge which is taken as a negative value. The distribution function $F(x,u,t)dxdu$ denotes the charge density in the interval dx at the instant t due to electrons with velocities between u and $u+du$. Taking the zero-, first-, and second-order moments of Eq. (1) with respect to the velocity variable u, then integrating by parts, with the assumption that $F(x,\pm\infty,t)=0$, and in view of the fact that u and x are independent variables, yields the following three macroscopic equations. These express the idea of conservation of charge, conservation of momentum, and the conservation of energy, respectively;

$$\partial\rho/\partial t+\partial J/\partial x=0, \tag{2}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho\langle u^2\rangle)-\eta\rho E=0, \tag{3}$$

and

$$\frac{\partial}{\partial t}(\rho\langle u^2\rangle)+\frac{\partial}{\partial x}(\rho\langle u^3\rangle)-2\eta\rho vE=0, \tag{4}$$

where the macroscopic charge density, mean velocity, and convection current density of the electron beam are defined, respectively, as

$$\rho(x,t)=\int_{-\infty}^{\infty}Fdu,$$

$$v(x,t)=\frac{1}{\rho}\int_{-\infty}^{\infty}uFdu,$$

and

$$J(x,t)=\int_{-\infty}^{\infty}uFdu=\rho v; \tag{5}$$

and the mean values of u^n are defined by

$$\langle u^n\rangle=\frac{1}{\rho}\int_{-\infty}^{\infty}u^nFdu. \tag{6}$$

It is to be noted that

$$\langle u\rangle=v, \tag{7a}$$

$$\langle u^2\rangle=v^2+\langle(u-v)^2\rangle, \tag{7b}$$

and

$$\langle u^3\rangle=v^3+3v\langle(u-v)^2\rangle+\langle(u-v)^3\rangle. \tag{7c}$$

In view of the fact that the electron beam temperature $T(x,t)$ is related to the mean-square deviation of the velocity by

$$kT(x,t)/m=\langle(u-v)^2\rangle, \tag{8}$$

where k is the Boltzmann constant, Eqs. (2)–(4) can be written as follows:

$$\partial\rho/\partial t+\partial J/\partial x=0, \tag{9}$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(Jv)-\eta\rho E=-\frac{\partial}{\partial x}\left(\rho\frac{kT}{m}\right), \tag{10}$$

and

$$\frac{\partial}{\partial t}(\rho v^2)+\frac{\partial}{\partial x}(Jv^2)-2\eta JE=-\frac{\partial}{\partial t}\left(\rho\frac{kT}{m}\right)-\frac{\partial}{\partial x}\left[3J\left(\frac{kT}{m}\right)+\rho\langle(u-v)^3\rangle\right]. \tag{11}$$

It is noticed that the right-hand sides of Eqs. (10) and (11) indicate the effect of the presence of beam velocity spreading and they vanish as the velocity spread approaches zero, leading to the familiar form of the equation of motion and the kinetic power theorem of the single-velocity theory. Furthermore, the last term on the right-hand side of Eq. (11), when it is divided by a factor (-2η), represents the divergence of the energy flow density. The first member of this term represents the internal energy carried by the average velocity which is often referred to as convection and the second member corresponds to the energy carried by (heat) conduction.

For convenience, let us define the thermal current density Q (i.e., the rate of transfer of kinetic energy associated with the random motion per unit area per second) as follows:

$$Q(x,t)=\frac{1}{-2\eta}\rho\langle(u-v)^3\rangle=\frac{1}{-2\eta}\int_{-\infty}^{\infty}(u-v)^3Fdu. \tag{12}$$

Then, by multiplying Eq. (10) by $(2v)$ and subtracting from Eq. (11), with the aid of Eq. (9), Eq. (11) can be written in the following manner:

$$2v\frac{\partial}{\partial x}\left(\rho\frac{kT}{m}\right)=\frac{\partial}{\partial t}\left(\rho\frac{kT}{m}\right)+3\frac{\partial}{\partial x}\left(J\frac{kT}{m}\right)-2\eta\frac{\partial Q}{\partial x}. \tag{13}$$

Assume that all quantities of interest have the following form:

$$G(x,t)=G_0(x)+G_1(x)e^{j\omega t}, \tag{14}$$

with ω being the angular radian frequency. Equations (5), (9), (10), and (13) yield the following set of dc equations:

$$J_0=\rho_0v_0, \tag{15}$$

$$dJ_0/dx=0, \tag{16}$$

$$\frac{d}{dx}(J_0v_0)-\eta\rho_0E_0=-\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right), \tag{17}$$

and

$$2v_0\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)=3\frac{d}{dx}\left(J_0\frac{kT_0}{m}\right)-2\eta\frac{dQ_0}{dx}, \tag{18}$$

and the following set of ac equations (under the small-signal assumption):

$$J_1=\rho_0v_1+v_0\rho_1, \tag{19}$$

$$j\omega\rho_1+dJ_1/dx=0, \tag{20}$$

$$j\omega J_1+\frac{d}{dx}(J_0v_1+J_1v_0)-\eta(\rho_0E_1+\rho_1E_0)=-\frac{d}{dx}\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right), \tag{21}$$

and

$$2v_1\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)+2v_0\frac{d}{dx}\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right)$$
$$=j\omega\left(\rho_0\frac{kT_1}{m}+\rho_1\frac{kT_0}{m}\right)+3\frac{d}{dx}\left(J_0\frac{kT_1}{m}+J_1\frac{kT_0}{m}\right)-2\eta\frac{dQ_1}{dx}. \quad (22)$$

The set of dc equations can be solved with the aid of the electrostatic scalar potential function, which satisfies Poisson's equation, and the dc density function. The ac quantities, v_1, ρ_1, J_1, T_1, Q_1, and E_1 are of interest to us in the study of noise in the electron beam.

For a one-dimensional beam (or in an open-circuited diode) the total alternating current density may be considered to be zero, so that the alternating convection current density J_1 and the ac electric field E_1 are related by

$$E_1=-J_1/j\omega\epsilon_0, \quad (23)$$

where ϵ_0 is the dielectric constant *in vacuo*.

Let us now assume that the alternating thermal current density Q_1 is invariant along the beam, i.e.,

$$dQ_1/dx=0 \quad (24)$$

so that it is only necessary to use three ac quantities to characterize the ac behavior of the beam, in view of relations (19), (23), and (24). In the present paper it has been decided to work with the quantities v_1, J_1, and T_1 and for convenience consider the ratio of the ac to dc quantities, namely, (J_1/J_0), (v_1/v_0), and (T_1/T_0).

After some algebraic manipulation, the following set of differential equations is obtained (see Appendix A for the details):

$$\frac{d\tilde{X}_l(x)}{dx}=\sum_{m=1}^{3}\tilde{a}_{lm}(x)\tilde{X}_l(x) \quad l=1, 2, 3, \quad (25)$$

in which the symbol $\sim$ denotes a complex quantity and thus $\tilde{X}_{lm}(x)$ and $\tilde{a}_{lm}(x)$ are complex quantities although the independent variable x is real. In the system of Eq. (25) the dependent variables $\tilde{X}_l(x)$ are defined as

$$\tilde{X}_1(x)=\frac{\tilde{J}_1(x)}{J_0(x)},\ \tilde{X}_2(x)=\frac{\tilde{v}_1(x)}{v_0(x)},\ \text{and}\ \tilde{X}_3(x)=\frac{\tilde{T}_1(x)}{T_0(x)} \quad (26a)$$

and the coefficients $\tilde{a}_{lm}(x)$ are given by

$$\tilde{a}_{lm}(x)=b_{lm}(x)+jC_{lm}(x),\ l=1, 2, 3\ \ m=1, 2, 3, \quad (26b)$$

with

$$\tilde{a}_{11}=-j\beta_e,\quad \tilde{a}_{12}=j\beta_e,\quad \tilde{a}_{13}=0,$$
$$\tilde{a}_{21}=-\frac{\delta h}{\Delta}+j\frac{\beta_e}{\Delta}\left(h+\frac{\omega_p^2}{\omega^2}\right),$$
$$\tilde{a}_{22}=-\frac{2}{\Delta}\frac{d}{dx}\ln v_0-j\frac{\beta_e}{\Delta}(1+h),$$
$$\tilde{a}_{23}=\frac{3h}{\Delta}\frac{d}{dx}\ln v_0+j\frac{\beta_e}{\Delta}h,$$
$$\tilde{a}_{31}=\frac{\delta}{\Delta}(1-h)-j\frac{2\beta_e}{\Delta}\left(h+\frac{\omega_p^2}{\omega^2}\right),$$
$$\tilde{a}_{32}=\frac{4}{\Delta}\frac{d}{dx}\ln v_0+j\frac{2\beta_e}{\Delta}(1+h)$$

and

$$\tilde{a}_{33}=\frac{1}{\Delta}\left[\delta(1-3h)-6h\frac{d}{dx}\ln v_0\right]-j\frac{\beta_e}{\Delta}(1-h),$$

with

$$\Delta(x)=1-3h(x). \quad (26c)$$

The wavenumbers $\beta_e(x)$, plasma angular frequency $\omega_p(x)$, velocity spreading parameter $h(x)$, and heat conduction parameter $\delta(x)$ are defined as follows:

$$\beta_e(x)=\omega/v_0(x),\quad \omega_p^2(x)=\eta\rho_0(x)/\epsilon_0,$$
$$h(x)=kT_0(x)/mv_0^2(x),$$

and

$$\delta(x)=\frac{-2\eta}{J_0}\left(\frac{m}{kT_0}\right)\frac{dQ_0}{dx}. \quad (26d)$$

Now let the function $\tilde{\Phi}_{ln}(x)$ be defined as follows:

$$\tilde{\Phi}_{ln}(x)=\tilde{X}_l(x)\tilde{X}_n^*(x)\quad l=1, 2, 3\quad n=1, 2, 3, \quad (27)$$

where the symbol * denotes the complex conjugate. It is to be noted that, in a language of the generalized harmonic analysis,[8] $\tilde{\Phi}_{ln}(x)$ represents the spectra of the correlation; for example, if $l=n$ it represents the spectrum of the autocorrelation of a random function, e.g., the current-, velocity-, or beam-temperature fluctuation in our case, and if $l\neq n$ it represents the spectrum of the cross-correlation of the random functions. These spectra and their respective correlation functions are related by a Fourier transform pair.

Since $\tilde{\Phi}_{ln}(x)$ is a complex quantity it can always be expressed in the following form:

$$\tilde{\Phi}_{ln}(x)=\Pi_{ln}(x)+j\Lambda_{ln}(x), \quad (28)$$

where Π_{ln} and Λ_{ln} are real quantities. Then the functions $\tilde{\Phi}_{ln}(x)$, $\Pi_{ln}(x)$, and $\Lambda_{ln}(x)$ can be shown to have

[8] Y. W. Lee, *Statistical Theory of Communication* (John Wiley & Sons, Inc., New York, 1960), Chap. 2.

the following properties:

$$\tilde{\Phi}_{ln}=(\tilde{\Phi}_{nl})^*,\quad \Pi_{ln}=\Pi_{nl},\quad \Lambda_{ln}=-\Lambda_{nl}\quad \text{for } l=1,2,3,\quad n=1,2,3,$$

$$\Phi_{ll}=\Pi_{ll} \text{ and } \Lambda_{ll}=0 \text{ for } l=1,2,3. \tag{29}$$

Upon differentiating Eq. (27) with respect to the real variable x and using Eqs. (25) and (29) one obtains

$$\frac{d\tilde{\Phi}_{ln}(x)}{dx}=\sum_{m=1}^{3}[\tilde{a}_{lm}(x)\tilde{\Phi}_{mn}(x)+\tilde{a}_{nm}^*(x)\tilde{\Phi}_{ml}^*(x)]\quad l=1,2,3 \text{ and } n=1,2,3. \tag{30}$$

The following system of first-order ordinary real differential equations is then obtained with the aid of Eq. (28):

$$\frac{d\Pi_{ln}}{dx}=\sum_{m=1}^{3}[(b_{lm}\Pi_{mn}+b_{nm}\Pi_{ml})-(C_{lm}\Lambda_{mn}+C_{nm}\Lambda_{ml})]\quad l=1,2,3,\quad n=1,2,3 \tag{30a}$$

and

$$\frac{d\Lambda_{ln}}{dx}=\sum_{m=1}^{3}[(C_{lm}\Pi_{mn}-C_{nm}\Pi_{ml})+(b_{lm}\Lambda_{mn}-b_{nm}\Lambda_{ml})]\quad l\neq n. \tag{30b}$$

It is observed that there are nine correlation functions which need to be considered, namely auto- and cross-correlation of the current fluctuations, velocity fluctuations and temperature fluctuations. Although there are 18 parameters Π_{ln} and Λ_{ln}, for $l=1, 2, 3$ and $n=1, 2, 3$ involved, since Eq. (29) represents nine conditions of constraint, it is necessary only to use nine parameters to specify the correlations. Consequently, the conditions are to be imposed on Eq. (30) in such a way that Eq. (30a) gives six equations and Eq. (30b) gives three equations.

It is interesting to note that in the case of a single-velocity beam there are only four parameters needed to specify the correlation; however, nine are needed here.

The conventionally defined noise parameters, Ψ, Φ, Π, Λ, and S, introduced by Haus[3] are related to the Π_{ln} and Λ_{ln} as follows (on the basis of per unit bandwidth and per unit beam cross-sectional area):

$$\begin{aligned}\Psi&=(4\pi)^{-1}J_0^2\Pi_{11},\\ \Phi&=(4\pi)^{-1}(v_0^4/\eta^2)\Pi_{22},\\ \Pi&=(4\pi)^{-1}(v_0^2J_0/\eta)\Pi_{21},\\ \Lambda&=(4\pi)^{-1}(v_0^2J_0/\eta)\Lambda_{21},\\ S&=(4\pi)^{-1}(v_0^2J_0/\eta)S_{21},\end{aligned} \tag{31a}$$

where

$$S_{21}=[\Pi_{22}\Pi_{11}-\Lambda_{21}^2]^{\frac{1}{2}} \tag{31b}$$

and the noisiness parameter $N(x)$ can be expressed as

$$N(x)=\frac{2\pi}{kT_0}(S-\Pi)=\frac{V_0I_0}{kT_0}n(x)=\left(\frac{J_0}{2e}\right)\frac{1}{h(x)}n(x). \tag{32a}$$

The dc kinetic voltage V_0, the direct beam current I_0, and the dimensionless parameter $n(x)$ are defined by

$$V_0=-v_0^2/2\eta,\quad I_0=-J_0$$

and

$$n(x)=S_{21}(x)-\Pi_{21}(x). \tag{32b}$$

The theoretical minimum noise figure for a beam-type amplifier may be written as

$$F_{\min}=1+(V_0I_0/kT_0)n(x), \tag{33}$$

where $T_0(x)$ is the dc electron beam temperature.

In order to know how $N(x)$ varies along the beam, it is necessary to find out the variations of Π_{11}, Π_{22}, Π_{21}, and Λ_{21} with distance by solving the system of differential equations given by Eqs. (30a) and (30b), with the coefficients $\tilde{a}_{lm}$ being given by Eq. (26c).

It is also of interest to note that, upon differentiating Eq. (31a), and with the aid of Eq. (30a) and (30b), and using the fact that $\Lambda_{ll}=0$ for $l=1, 2, 3$, $b_{1m}=0$ for $m=1, 2, 3$, and $C_{13}=0$, the following relationship is obtained governing the spatial rate of change of the conventionally defined noise parameters:

$$\begin{aligned}d\Psi/dx&=-(J_0^2/2\pi)C_{12}\Lambda_{21},\\ \frac{d\Phi}{dx}&=\left(\frac{4}{v_0}\frac{dv_0}{dx}\right)\Phi+\frac{v_0^4}{2\pi\eta^2}[b_{21}\Pi_{21}+b_{22}\Pi_{22}+C_{21}\Lambda_{21}+b_{23}\Pi_{32}-C_{23}\Lambda_{32}],\\ \frac{d\Pi}{dx}&=\left(\frac{2}{v_0}\frac{dv_0}{dx}\right)\Pi+\frac{v_0^2J_0}{4\pi\eta}[b_{21}\Pi_{11}+b_{22}\Pi_{21}+(C_{11}-C_{22})\Lambda_{21}+b_{23}\Pi_{31}-C_{23}\Lambda_{31}],\\ \frac{d\Lambda}{dx}&=\left(\frac{2}{v_0}\frac{dv_0}{dx}\right)\Lambda+\frac{v_0^2J_0}{4\pi\eta}[C_{21}\Pi_{11}-C_{12}\Pi_{22}+(C_{22}-C_{11})\Pi_{21}+b_{22}\Lambda_{21}+C_{23}\Pi_{31}+b_{23}\Lambda_{31}].\end{aligned} \tag{34}$$

It is observed that the rate of change of Ψ, Φ, Π, and Λ does depend upon the functions $\tilde{\Phi}_{31}$ and $\tilde{\Phi}_{32}$, which represent the spectrum of the cross-correlation of the beam temperature fluctuations and the current fluctuations, and that of the temperature fluctuations and velocity fluctuations, respectively.

III. DISCUSSION OF THE SOLUTION OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS DERIVED

The systems of ordinary linear differential equations (25) and (30) can be solved if the coefficients $\tilde{a}_{lm}(x)$ are known, and the fluctuation in the quantities such as the current, the velocity, and the beam temperature, and their correlation along the electron beam can be determined when the input-plane boundary conditions are specified. Since $v_0(x)$, $\rho_0(x)$, and $T_0(x)$ are obtainable from $F_0(x,u)$, the coefficients $\tilde{a}_{lm}(x)$ can be determined once the dc density function $F_0(x,u)$ is known.

For an electron beam in a drift region, where $dv_0/dx=0$, the coefficients $\tilde{a}_{lm}$ become independent of the position variable x, and δ also becomes zero. Consequently, the solution to the systems [Eqs. (25) and (30)] is obtained by a standard Laplace-transform technique, which is rather straightforward and simple. On the other hand, for a beam in an accelerating region, once the coefficients $\tilde{a}_{lm}(x)$ as functions of the position x are known, a numerical method, such as the Runge–Kutta method,[9] can be employed for solving the systems Eqs. (25) and (30).

Case I. Drifting Beam

For a drifting beam, the system of Eq. (25) has a traveling-wave solution, which can be easily shown as follows: After taking the Laplace transformation of the system (25) with respect to the spatial variable x, the following set of algebraic equations is obtained:

$$\sum_{m=1}^{3} D_{lm}(p)y_m(p)=X_l(0) \quad l=1,2,3, \tag{35}$$

where

$$y_l(p)=\int_0^\infty X_l(x)e^{-px}dx \tag{36a}$$

and

$$D_{lm}(p)=(\delta_{lm}p-a_{lm}), \tag{36b}$$

in which δ_{lm} is the Kroneker delta, equal to one for $l=m$, and to zero for $l\neq m$. The term $X_l(0)$ appearing in Eq. (35) denotes the values of $X_l(x)$ at $x=0$, the input plane to the drift region.

From Cramer's rule the solution of the set of Eq. (35) can be expressed as follows:

$$y_m(p)=\sum_{l=1}^{3}\frac{N_{lm}(p)}{D(p)}X_l(0) \quad m=1,2,3, \tag{37}$$

where $D(p)$ is the determinant of the set of the transformed Eq. (35) with an order of 3, and $N_{lm}(p)$ is the cofactor of the element D_{lm} in the determinant $D(p)$, which is formed from $D(p)$ by striking out the row and column containing the element D_{lm} and prefixing the sign factor $(-1)^{l+m}$.

After taking the inverse transformation of the system (37), it is found that

$$X_m(x)=\sum_{l=1}^{3}X_l(0)\sum_{k=1}^{3}\frac{N_{lm}(p_k)}{D'(p_k)}e^{p_kx} \qquad 0\leq x \qquad m=1,2,3 \tag{38a}$$

where

$$D'(p_k)=dD/dp\,|_{p=p_k} \tag{38b}$$

provided that the rational fraction $[N_{lm}(p)/D(p)]$ has only a first-order pole, where p_k is the root of the characteristic equation $D(p)=0$.

[9] J. B. Scarborough, *Numerical Mathematical Analysis* (The John Hopkins Press, Baltimore, Maryland, 1962), 5th ed., p. 301.

In view of the fact that in a drift region, $dv_0/dx=0$, and from Eqs. (17), (18), and (26d), δ must be zero, so that the coefficients $\tilde{a}_{lm}$ become purely imaginary quantities, and it can be easily shown that the characteristic equation has the following form:

$$D(p)=p^3+ap^2+bp+c=0, \tag{39}$$

in which

$$a=j\beta_eA,\quad b=(j\beta_e)^2B,\quad \text{and}\quad c=(j\beta_e)^3C, \tag{40a}$$

with

$$\begin{aligned}A&=1+2/\Delta,\\ B&=\Delta^{-1}(3-\omega_p^2/\omega^2),\\ C&=\Delta^{-1}(1-\omega_p^2/\omega^2).\end{aligned} \tag{40b}$$

Now by letting

$$p=j\beta_e\gamma, \tag{41}$$

Eq. (39) becomes

$$\gamma^3+A\gamma^2+B\gamma+C=0, \tag{42}$$

which can be arranged in the following form, when A, B, and C are given by Eq. (40b),

$$(\gamma+1)[\Delta\gamma^2+2\gamma+(1-\omega_p^2/\omega^2)]=0. \tag{43}$$

Note that Eq. (43) has three distinct roots and consequently Eq. (39) has the following roots:

$$\begin{aligned}p_1&=-j\beta_e\\ p_2&=-j(\beta_e/\Delta)[1-\{1-\Delta(1-\omega_p^2/\omega^2)\}^{\frac{1}{2}}]\\ p_3&=-j(\beta_e/\Delta)[1+\{1-\Delta(1-\omega_p^2/\omega^2)\}^{\frac{1}{2}}].\end{aligned} \tag{44a}$$

Furthermore note that as $h\to 0$, $\Delta\to 1$ and

$$\begin{aligned}p_2&\to -j\beta_e(1-\omega_p/\omega)=-j(\beta_e-\beta_p),\\ p_3&\to -j\beta_e(1+\omega_p/\omega)=-j(\beta_e+\beta_p),\end{aligned} \tag{44b}$$

which are the familiar expressions for the single-velocity theory, where $\beta_p=\omega_p/v_0$ is the plasma wavenumber. In view of the fact that the time harmonic variation has been assumed in the present discussion with the aid of Eq. (44a), Eq. (38a) represents the superposition of three propagating waves, all in the positive x direction, but with different phase velocities. There is one kinematic wave with phase velocity equal to the dc beam velocity, and the other two corresponding to the fast- and the slow-space-charge waves. Thus it can be concluded that a drifting beam, with an arbitrary amount of velocity spread, can support one kinematic and two space-charge waves.

Case II. Space-Charge-Limited Diode

It is obvious, from Eqs. 5 and 14, that the density function $F(x,u,t)$ is of the form

$$F(x,u,t)=F_0(x,u)+F_1(x,u)\cdot e^{j\omega t}.$$

It is well known that the dc density function $F_0(x,u)$ which satisfies the dc part of Eq. (1) and at the same time meets the proper boundary condition at the cathode has the following form:

$$F_0(x,u)=2\alpha J_0 S(u-w)e^{-\alpha(u^2-w^2)}, \quad (45a)$$

where

$$\alpha=m/2kT_c$$

and

$$J_0=J_0(x_m)=J_s \exp[-e\varphi_0(x_m)/kT_c], \quad (45b)$$

in which J_s is the saturation or total value of emission current density, and T_c is the cathode temperature (in degrees Kelvin) and $\varphi_0(x_m)$ is the dc potential at the potential minimum, $x=x_m$.

The function $S(u-w)$ is the usual unit step function:

$$S(u-w)=0 \quad \text{for} \quad (u-w)<0$$
$$=1 \quad \text{for} \quad (u-w)\geq 0. \quad (45c)$$

The function $w(x)$ is defined and related to the dc electric potential function $\varphi_0(x)$ as follows:

$$w(x)=\mp w_0(x) \quad (45d)$$

with

$$w_0(x)=[-2\eta\{\varphi_0(x)-\varphi_0(x_m)\}]^{\frac{1}{2}}. \quad (45e)$$

In Eq. (45d) the upper sign is to be used for the α region, which is between the cathode and potential minimum, and the lower sign is for the β region, which is between the potential minimum and the anode.

Having assumed the form of the dc density function, the quantities ρ_0, v_0, (kT_0/m), and Q_0 can be obtained and expressed as follows:

$$\rho_0(x)=\int_{-\infty}^{\infty} F_0 du=(\pi\alpha)^{\frac{1}{2}}J_0 e^{\alpha w^2}[1-\text{erf}(\alpha^{\frac{1}{2}}w)], \quad (46a)$$

$$v_0(x)=\frac{1}{\rho_0}\int_{-\infty}^{\infty} uF_0 du=\frac{1}{(\pi\alpha)^{\frac{1}{2}}}\left[\frac{e^{-\alpha w^2}}{1-\text{erf}(\alpha^{\frac{1}{2}}w)}\right], \quad (46b)$$

$$\frac{kT_0(x)}{m}=\frac{1}{\rho_0}\int_{-\infty}^{\infty}(u-v_0)^2F_0 du$$
$$=(kT_c/m)-v_0^2+v_0 w, \quad (46c)$$

and

$$Q_0(x)=\frac{1}{-2\eta}\int_{-\infty}^{\infty}(u-v_0)^3F_0 du$$
$$=(J_0/-2\eta)[(1/\alpha)+w^2-v_0^2-3kT_0/m], \quad (46d)$$

where $w(x)$ is given by Eq. (45d) and the error function $\text{erf}(Y)$ is defined as

$$\text{erf}(Y)=\frac{2}{\sqrt{\pi}}\int_0^Y e^{-y^2}dy. \quad (46e)$$

It is to be noted that in the above equations, ρ_0, v_0, (kT_0/m), and Q_0 are expressed essentially in terms of the dc potential function $\varphi_0(x)$ through Eqs. (45d) and (45e) and these quantities are continuous at the potential minimum $x=x_m$, where $w=0$. On the other hand, $\varphi_0(x)$ must satisfy Poisson's equation:

$$d^2\varphi_0/dx^2=-(\pi\alpha)^{\frac{1}{2}}(J_0/\epsilon_0)e^{\alpha w^2}[1-\text{erf}(\alpha^{\frac{1}{2}}w)] \quad (47)$$

which has been solved numerically by Langmuir.[10]

In view of the fact that the electrostatic field intensity $E_0(x)$ is derivable from the dc potential function $\varphi_0(x)$ by

$$E_0(x)=-d\varphi_0/dx \quad (48)$$

which is consistent with the requirement that $\varphi_0(x)$ must satisfy Poisson's equation, and upon substitution of Eq. (48) into Eq. (17), it is found that

$$\eta\frac{d\varphi_0}{dx}+v_0\frac{dv_0}{dx}=\frac{-1}{\rho_0}\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right), \quad (49)$$

which is equivalent to the following equation:

$$dv_0/dw=2\alpha(v_0-w)v_0, \quad (50)$$

in which v_0 is considered to be a function of w, since ρ_0, (kT_0/m), and φ_0 are expressible in terms of w. Similarly upon substitution of Eq. (46d) into Eq. (18), Eq. (50) is again obtained.

It is important to observe that the function $v_0(w)$ given by Eq. (46b) does satisfy the differential equation (50). Thus it indicates two interesting facts:

1. The form assumed for the dc density function $F_0(x,u)$ given by Eq. (45a) is consistent with the assumption that the dc potential function must satisfy Poisson's equation.
2. The quantities ρ_0, v_0, (kT_0/m), and Q_0 do satisfy the differential equations (16), (17), and (18).

These facts, in turn, ensure that once the potential function φ_0, which satisfies Poisson's equation, is specified, the quantities ρ_0, v_0, (kT_0/m), and Q_0 are properly determined and are given by Eqs. (46a–d), respectively, in such a way that the laws of conservation of charge, momentum, and energy are satisfied.

Therefore, once the dc potential distribution in the region under consideration is specified, the functions $h(x)$ and $\delta(x)$ are determined from Eq. (26d), and so are the coefficients $\tilde{a}_{lm}(x)$ in the systems (25) and (30). Having determined the coefficients $\tilde{a}_{lm}(x)$, the systems (25) and (30) can be solved by the Runge–Kutta method with the properly imposed input-boundary conditions.

IV. CONCLUDING REMARKS

The heat conduction along the electron beam has been properly taken into account in the present paper by introducing the heat conduction parameter $\delta(x)$, which is defined in Eq. (26d). The parameter $\delta(x)$ is related to the velocity spreading parameter $h(x)$ and the dc mean velocity $v_0(x)$ by the following relation,

---

[10] I. Langmuir, Phys. Rev. **21**, 419 (1923).

from Eq. (A7):

$$\delta(x) = -[d/dx \ln h(x) + 4d/dx \ln v_0(x)], \qquad (51)$$

which is due to the law of conservation of energy, and $\delta(x)$ does depend upon the spatial rate of variation of $h(x)$ and $v_0(x)$. On the other hand, the thermal current density $Q_0(x)$, which has a dimension of joules per sec per unit area, may be put in the following form:

$$Q_0(x) = -\lambda_0(x)[dT_0(x)/dx], \qquad (52)$$

where λ_0 is the thermal conductivity of the electron beam, which in general depends upon the collision force, ρ_0 and T_0, and is governed by Eqs. (18), (26d), and (52). It is to be observed that for an adiabatic flow λ_0 can be set equal to zero so that δ will be zero also. However, for an isothermal flow, λ_0 becomes very large and δ need not be zero. It is interesting to note that for an adiabatic flow, since $\delta(x)$ can be set equal to zero, Eq. (51) implies that the quantity (hv_0^4) or $(kT_0/m)v_0^2$ is invariant along the beam, which suggests that the quantity (T_0/ρ_0^2) is also invariant.[7] On the other hand, in a drift region, since $Q_0(x)$ is independent of x, $\delta(x)$ will be zero from Eq. (26d). Thus it suggests that the thermal effect (heat conduction effect) in a drifting beam can be neglected.

It should be pointed out that no specific assumption has been made with regard to the input-plane boundary conditions in deriving the systems of equations (25) and (30). However, for a special case in which $\delta=0$, for instance, in an adiabatic flow, and when it is further assumed that the following relation holds at the input plane, for example, at the cathode surface,

$$T_1/T_0 = 2\rho_1/\rho_0 \quad \text{at} \quad x=0, \qquad (53)$$

the system of equations is reduced to that obtained by Berghammer and Bloom,[7] which is demonstrated in Appendix B.

While the density-function method involves solving a rather complicated partial differential equation, which must also deal with the Dirac delta function, the present method of analysis of signal and noise propagation along the electron beam involves solving a system of linear ordinary first-order differential equations, whose solution is obtainable by relatively simple and straightforward methods.

ACKNOWLEDGMENT

The author is very grateful to Professor J. E. Rowe for his contributions and reading the manuscript.

APPENDIX A. DERIVATION OF THE SYSTEM OF EQUATION (25)

First note the following identity:

$$\frac{dC}{dx} - \frac{dD}{dx} = D\frac{d}{dx}\left(\frac{C}{D}\right) + \left(\frac{C}{D} - 1\right)\frac{dD}{dx}. \qquad (A1)$$

From Eqs. (15) and (19)

$$J_1/J_0 = v_1/v_0 + \rho_1/\rho_0. \qquad (A2)$$

Subtraction of Eq. (16) from Eq. (20) with the aid of Eqs. (A1) and (A2) gives

$$j\beta_e\left(\frac{J_1}{J_0}\right) - j\beta_e\left(\frac{v_1}{v_0}\right) + \frac{d}{dx}\left(\frac{J_1}{J_0}\right) = 0, \qquad (A3)$$

where

$$\beta_e = \omega/v_0.$$

Subtraction of Eq. (17) from Eq. (21), with the aid of Eqs. (A1) and (23) gives, after using Eqs. (16), (17), and (A2):

$$j\beta_e\left(1 - \frac{\omega_p^2}{\omega^2}\right)\left(\frac{J_1}{J_0}\right) + \frac{d}{dx}\left(\frac{v_1}{v_0} + \frac{J_1}{J_0}\right) + h\frac{d}{dx}\left(\frac{T_1}{T_0} + \frac{J_1}{J_0} - \frac{v_1}{v_0}\right)$$

$$+ \frac{2}{v_0}\frac{dv_0}{dx}\left(\frac{v_1}{v_0}\right) + \left[\frac{1}{J_0 v_0}\frac{d}{dx}\left(\rho_0\frac{kT_0}{m}\right)\right]\left(\frac{T_1}{T_0}\right) = 0, \qquad (A4)$$

where

$$\omega_p^2 = \eta\rho_0/\epsilon_0, \quad h = kT_0/mv_0^2.$$

Similarly, first subtracting Eq. (18) from Eq. (22), and with the aid of Eq. (A1), then dividing it through by the factor $J_0(kT_0/m)$ yields, after using Eqs. (18) and (A2),

$$\frac{d}{dx}\left(\frac{J_1}{J_0} + \frac{2v_1}{v_0} + \frac{T_1}{T_0}\right) + j\beta_e\left(\frac{T_1}{T_0} + \frac{J_1}{J_0} - \frac{v_1}{v_0}\right)$$

$$+ 2\eta\left(\frac{m}{J_0 kT_0}\right)\frac{dQ_0}{dx}\left(\frac{T_1}{T_0} + \frac{J_1}{J_0}\right) - 2\eta\left(\frac{m}{J_0 kT_0}\right)\frac{dQ_1}{dx} = 0. \qquad (A5)$$

Defining the heat conduction parameter $\delta(x)$ as

$$\delta(x) = \frac{-2\eta}{J_0}\left(\frac{m}{kT_0}\right)\frac{dQ_0}{dx}, \qquad (A6)$$

Eq. (18) can be written as follows, after it is divided through by a factor $J_0(kT_0/m)$:

$$\frac{d}{dx}\ln\left(\rho_0\frac{kT_0}{m}\right) = -\delta(x) - 3\frac{d}{dx}\ln v_0. \qquad (A7)$$

After making the following definitions

$$X_1(x) = J_1(x)/J_0(x), \quad X_2(x) = v_1(x)/v_0(x), \quad \text{and} \quad X_3(x) = T_1(x)/T_0(x), \qquad (A8)$$

Eqs. (A3), (A4), and (A5) can be arranged into the following system of equations with the assumption (24):

$$\frac{dX_1}{dx}=\sum_{m=1}^{3} A_{1m}X_m,$$

$$(1+h)\frac{dX_1}{dx}+(1-h)\frac{dX_2}{dx}+h\frac{dX_3}{dx}=\sum_{m=1}^{3} A_{2m}X_m,$$

$$\frac{dX_1}{dx}+2\frac{dX_2}{dx}+\frac{dX_3}{dx}=\sum_{m=1}^{3} A_{3m}X_m, \quad \text{(A9)}$$

where

$$A_{11}=-j\beta_e,\quad A_{21}=-j\beta_e\left(1-\frac{\omega_p^2}{\omega^2}\right),\quad A_{31}=\delta-j\beta_e,$$

$$A_{12}=j\beta_e,\quad A_{22}=-2\frac{d}{dx}\ln v_0,\quad A_{32}=j\beta_e,$$

$$A_{13}=0,\quad A_{23}=\delta h+3h\frac{d}{dx}\ln v_0,\quad A_{33}=\delta-j\beta_e. \quad \text{(A10)}$$

Upon solving algebraically for (dX_1/dx), (dX_2/dx), and (dX_3/dx), in terms of X_1, X_2, and X_3, from the system (A9), with the aid of Cramer's rule, one obtains

$$\frac{dX_1}{dx}=\sum_{m=1}^{3} a_{1m}X_m,$$

$$\frac{dX_2}{dx}=\sum_{m=1}^{3} a_{2m}X_m, \quad \text{(A11)}$$

$$\frac{dX_3}{dx}=\sum_{m=1}^{3} a_{3m}X_m,$$

where

$$a_{1m}=A_{1m},$$
$$a_{2m}=\Delta^{-1}[A_{2m}-hA_{3m}-A_{1m}],$$
$$a_{3m}=\Delta^{-1}[(1-h)A_{3m}-2A_{2m}+(1+3h)A_{1m}],$$
$$\Delta=1-3h \quad \text{for} \quad m=1,2,3, \quad \text{(A12)}$$

and upon substituting Eq. (A10) into Eq. (A12), Eq. (26c) is obtained.

APPENDIX B. DISCUSSION OF THE SYSTEM OF EQ. (34) FOR A SPECIAL CASE

For the case $\delta=0$, Eq. (A9) becomes

$$2dX_2/dx+dX_3/dx=-j\beta_e X_3 \quad \text{(B1)}$$

and from Eqs. (A2) and (A3), one has

$$\frac{dX_2}{dx}=-j\beta_e\left(\frac{\rho_1}{\rho_0}\right)-\frac{d}{dx}\left(\frac{\rho_1}{\rho_0}\right). \quad \text{(B2)}$$

After combining Eqs. (B1) and (B2), the following is obtained:

$$\frac{d}{dx}\left(\frac{T_1}{T_0}-2\frac{\rho_1}{\rho_0}\right)+j\beta_e\left(\frac{T_1}{T_0}-2\frac{\rho_1}{\rho_0}\right)=0, \quad \text{(B3)}$$

which has a solution of the form

$$\frac{T_1}{T_0}-\frac{2\rho_1}{\rho_0}=K\exp\left(-j\int_0^x \beta_e(y)dy\right), \quad \text{(B4)}$$

where K is a constant of integration, which is to be determined by the input-plane boundary conditions.

Suppose that $K=0$, as has been used by Berghammer and Bloom[7]; then

$$T_1/T_0=2\rho_1/\rho_0 \quad \text{(B5)}$$

or equivalently

$$X_3=2(X_1-X_2). \quad \text{(B6)}$$

Now from Eqs. (27) and (28), the following relations are evolved:

$$\Pi_{31}=2[\Pi_{11}-\Pi_{21}],\quad \Lambda_{31}=-2\Lambda_{21} \quad \text{(B7)}$$

and

$$\Pi_{32}=2[\Pi_{21}-\Pi_{22}],\quad \Lambda_{32}=-2\Lambda_{21}. \quad \text{(B8)}$$

Upon substituting the relations (B7) and (B8) into the system of equations (34), with the aid of Eq. (31a) and using the fact that $b_{21}=0$ for $\delta=0$,

$$d\Psi/dx=-2B_0\Lambda,$$
$$d\Phi/dx=-2M_0\Phi+2R_0\Pi+2X_0\Lambda,$$
$$d\Pi/dx=R_0\Psi-M_0\Pi+N_0\Lambda, \quad \text{(B9)}$$
$$d\Lambda/dx=X_0\Psi-B_0\Phi-N_0\Pi-M_0\Lambda,$$

where

$$B_0=(J_0\eta/v_0^2)C_{12},$$
$$M_0=-(2/v_0)(dv_0/dx)-b_{22}+2b_{23},$$
$$R_0=(2v_0^2/\eta J_0)b_{23}, \quad \text{(B10)}$$
$$X_0=(v_0^2/\eta J_0)(C_{21}+2C_{23}),$$
$$N_0=(C_{11}-C_{22}+2C_{23}).$$

When the coefficients b_{lm} and C_{lm} given by Eq. (26c) are substituted into Eq. (B10), our Eqs. (B9) and (B10) become Eqs. (22) and (19) of Berghammer and Bloom,[7] respectively.