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Interaction of Cylindrical Sound Waves with a Stationary Shock Wave

W. R. JOHNSON AND OTTO LAPORTE
The University of Michigan, Ann Arbor, Michigan
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The interaction of cylindrical sound waves with a stationary shock is investigated by a method analogous to that used by H. Weyl in his treatment of the propagation of radio waves over the surface of a plane earth. The incident cylindrical sound wave is represented as a superposition of plane sound waves of varying direction. Each of the plane waves in this superposition interacts with the shock giving rise to a previously determined distortion of the shock front and reflected or refracted wave field; the cylindrical wave causes a disturbance which may be written in integral form as a superposition of these plane waves. The resulting interaction integrals are evaluated asymptotically to give explicit formulas for the distortion of the shock, the sound field, and the entropy-vorticity wave.

1. INTRODUCTION

THIS paper treats the problem of the interaction of sound waves generated by a line source with a stationary shock wave in an ideal, inviscid gas.

The methods developed here may be applied to problems involving point sources and moving shocks, but for the present we restrict ourselves to the two-dimensional problem with a stationary shock.

The interaction of plane disturbances with a shock wave has been treated by various authors.[1-5] Many parts of the problem which we shall describe presently may be deduced qualitatively from the work of these authors, yet several important features, especially in the case of supersonic sources, require detailed treatment.

We shall treat the interaction problem by an adaptation of Weyl's[6] treatment of the interaction of electromagnetic waves from a dipole antenna with a plane, conducting earth. The incident sound wave is first expressed as an angular superposition of plane waves of constant frequency. Each of the plane waves in this superposition interacts with the shock, giving rise to a reflected or a refracted wave field as well as to a deformation of the shock. The entire cylindrical wave will therefore produce reflected or refracted wave fields and a deformation, all of which are superpositions of the corresponding plane disturbances.

To write down formulas for the various disturbances, we must know the relations between the angle of incidence and the angle of reflection or refraction for incident plane waves as well as the amplitudes of the resulting disturbances. That is to say, we must know the analogs of Snell's laws of reflection or refraction and the analogs of the Fresnel formulas for the plane-wave problem. Since these relations are not explicitly stated in the papers on plane-wave interactions referred to above, we must discuss them. But before we describe the plane and cylindrical wave interaction problem, let us briefly describe the differential equations and boundary conditions to be satisfied by any disturbance.

2. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Let us assume that the undisturbed state of the gas consists of a uniform flow normal to a plane shock wave which is at rest at $x = 0$. The flow enters the shock supersonically from $x < 0$, and leaves subsonically. The pressure, density, entropy, and velocity on the right (P_1, D_1, S_1, U_1) are given in terms of the flow variables on the left (P_0, D_0, S_0, U_0) by means of the relations,

$$\begin{aligned} P_1 &= (P_0/\mu)[(\mu + 1)M_0^2 - 1], \\ D_1 &= D_0\mu M_0^2/[M_0^2 + (\mu - 1)], \\ U_1 &= (U_0/\mu)[M_0^2 + (\mu - 1)], \\ S_1 &= S_0 + c_v \ln (1/\mu)[(\mu + 1)M_0^2 - 1] \\ &\quad + c_p \ln (1/\mu M_0^2)[M_0^2 + (\mu - 1)], \end{aligned} \tag{1}$$

where

$$\mu = (1/R)(c_p + c_v) \quad \text{and} \quad M_0 = U_0/C_0$$

with

$$C_0^2 = c_p P_0/c_v D_0.$$

Let us introduce a small disturbance into the

[1] H. S. Ribner, NACA TN 2864 (1953).
[2] F. K. Moore, NACA TN 2879 (1953).
[3] G. F. Carrier, Quart. Appl. Math. 6, 367 (1949).
[4] C. T. Chang, thesis, Johns Hopkins University (1955).
[5] J. M. Burgers, Koninkl. Ned. Akad. Wetenschap. Proc. 49, 274–81 (1946).
[6] H. Weyl, Ann. Physik 60, 481 (1919).

uniform flow on either side of the shock so that the flow variables become

$$\begin{aligned} \text{pressure}—p &= P + DCU\pi, \\ \text{density}—\rho &= D(1 + \delta), \\ x\text{-velocity}—u &= U(1 + \xi), \\ y\text{-velocity}—v &= U\eta, \\ \text{entropy}—s &= S + c_p\sigma, \end{aligned} \tag{2}$$

where the increments π, δ, ξ, η, and σ have been made dimensionless. The Euler equations become

$$\begin{aligned} \frac{1}{C}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\pi + \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} &= 0, \\ \frac{1}{C}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\xi + \frac{\partial \pi}{\partial x} &= 0, \\ \frac{1}{C}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\eta + \frac{\partial \pi}{\partial y} &= 0, \\ \frac{1}{C}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\sigma &= 0, \\ \delta &= M\pi - \sigma. \end{aligned} \tag{3}$$

The subscripts 0 or 1 must be attached to all constants and dependent variables occurring in Eqs. (2) and (3) to denote the side of the shock being referred to.

Disturbances on the subsonic side of the shock will be related to disturbances on the supersonic side by means of the shock conditions for a moving curved shock. If we describe the distorted shock by means of $x = f(y, t)$, its normal, tangent, and velocity vectors are given by $\mathbf{n} = (1, -f_y)$, $\mathbf{t} = (f_y, 1)$, and $\mathbf{U}_s = (f_y\dot{y} + f_t, \dot{y})$. The relative normal and tangential velocities of the flow are $U_{Rn} = (\mathbf{U} - \mathbf{U}_s)\cdot\mathbf{n} = U(1 + \xi) - f_t$ and $U_{Rt} = (\mathbf{U} - \mathbf{U}_s)\cdot\mathbf{t} = U(f_y + \eta) - \dot{y}$. Using these relative velocities in the shock conditions, we find the linear relations between the disturbances on either side of the shock and the distortion of the shock. These relations may be written:

$$\begin{bmatrix} \pi_1 \\ \xi_1 \\ \eta_1 \\ \sigma_1 \end{bmatrix} = A \begin{bmatrix} \pi_0 \\ \xi_0 \\ \eta_0 \\ \sigma_0 \\ f \end{bmatrix}, \tag{4}$$

where A is the 5×4 matrix with row vectors,

$$\begin{aligned} \mathbf{A}_1 &= \frac{1}{\mu}\frac{C_0}{C_1}\Big[(\mu - 1)M_0^2 - 1,\ 2(\mu - 1)M_0,\ 0, \\ &\qquad -(\mu - 1)M_0,\ -2(\mu - 1)\frac{1}{C_0}\frac{\partial}{\partial t}\Big], \\ \mathbf{A}_2 &= \frac{1}{\mu M_0^2}\frac{U_0}{U_1}\Big[2M_0,\ M_0^2 - (\mu - 1),\ 0,\ (\mu - 1), \\ &\qquad \frac{1}{M_0}(\mu - 1)(M_0^2 + 1)\frac{1}{C_0}\frac{\partial}{\partial t}\Big], \\ \mathbf{A}_3 &= \frac{1}{\mu M_0^2}\frac{U_0}{U_1}\Big[0,\ 0,\ 1,\ 0, \\ &\qquad (\mu - 1)(M_0^2 - 1)\frac{\partial}{\partial y}\Big], \\ \mathbf{A}_4 &= \frac{1}{\mu^2 M_0^2}\frac{C_0^2}{C_1^2}\Big[-2M_0(M_0^2 - 1)^2, \\ &\qquad 2(\mu - 1)(M_0^2 - 1)^2,\ 0, \\ &\qquad \mu^2 M_0^2 + 2(M_0^2 + (\mu - 1))(M_0^2 - 1), \\ &\qquad -\frac{2}{M_0}(\mu - 1)(M_0^2 - 1)^2\frac{1}{C_0}\frac{\partial}{\partial t}\Big]. \end{aligned} \tag{5}$$

Disturbances in the flow propagate according to Eqs. (3) and satisfy the boundary conditions (4) at $x = 0$. We shall consider in the following section plane-wave solutions of Eqs. (3) which satisfy boundary conditions (4) at $x = 0$.

3. PLANE-WAVE INTERACTIONS

A plane-wave solution of the differential Eqs. (3) is described by:

$$(\pi, \xi, \eta, \sigma) = (\pi^\circ, \xi^\circ, \eta^\circ, \sigma^\circ)e^{i\kappa(\alpha x + \beta y) - i\omega t} \tag{6}$$

where α and β are the direction cosines of the wave normal. In the remainder of this section, the factor $e^{-i\omega t}$ will be omitted for the sake of brevity. Substituting this ansatz into Eqs. (3), we find the following possibilities for κ:

$$\kappa = \pm k/(1 \pm M\alpha), \qquad \kappa = k/M\alpha,$$

where

$$k = \omega/C. \tag{7}$$

Corresponding to the first choice of κ, we have

$$(\pi^\circ, \xi^\circ, \eta^\circ, \sigma^\circ) = (1, \pm\alpha, \pm\beta, 0)A, \tag{8}$$

where A is a constant. This wave is a longitudinal, isentropic sound wave, convected downstream with the flow. The wave corresponding to the second choice of κ satisfies:

$$(\pi^\circ, \xi^\circ, \eta^\circ, \sigma^\circ) = (0, -\beta, \alpha, 0)B + (0, 0, 0, 1)C \tag{9}$$

where B and C are constants. This is an isobaric, transverse wave called (after Carrier) the entropy-vorticity wave.

Let us now turn to the plane-wave interaction problem. A plane sound wave falling onto the shock will give rise to a reflected and a refracted sound wave as well as to a deformation of the shock. We wish to show that when the sound is incident from the subsonic side of the shock ($x > 0$), there can be no refracted wave, while when the sound is incident from the supersonic side ($x < 0$), there can be no reflected wave. We have either reflection or refraction but not both.

To prove these assertions, it will be necessary to use the notion of the Poynting vector $\mathbf{\Sigma}$ of a disturbance. $\mathbf{\Sigma}$ arises from system (3) and occurs in a conservation theorem analogous to the electromagnetic Poynting theorem except that all time derivatives must be replaced by convective time derivatives. In particular $\mathbf{\Sigma} = C[\pi\xi + (M/2)(\pi^2 + \xi^2 + \eta^2), \pi\eta]$.

A. Subsonic Incidence of Plane Waves

To demonstrate that there are no refracted waves in the case of subsonic incidence, let us consider the Poynting vector $\mathbf{\Sigma}$ for sound. It may be shown that the time average of this vector for plane waves is related to the time average of the energy density by means of $\mathbf{\Sigma}_{\text{Av}} = E_{\text{Av}}(C\mathbf{n} + U\mathbf{n}')$, where $\mathbf{n} = (\alpha, \beta)$ is the wave normal and $\mathbf{n}' = (1, 0)$ is a unit vector in the direction of the flow. It should be noted that if the flow is supersonic, $\mathbf{\Sigma}_{\text{Av}}$ always has a positive x component, whereas if the flow is subsonic, $\mathbf{\Sigma}_{\text{Av}}$ has a positive x component if, and only if, $\alpha > -M$. The angle $\theta_g = \cos^{-1}(-M)$ takes the place of the $\pi/2$ grazing angle of electromagnetic theory.

With these facts in mind, we see that a sound wave incident from the subsonic side has its wave normal in the range $\pi \geq \theta \geq \theta_g$, while a reflected sound wave has its wave normal in the range $\theta_g \geq \theta \geq 0$. Refracted sound waves are clearly impossible, since they would be waves in the supersonic flow, and therefore would transport energy to the right or toward the shock, not away from it. Refracted entropy-vorticity waves are also impossible; since such waves are always convected with the flow, they could not be created at the shock and transported into the supersonic flow. Thus if we have a sound wave incident from the subsonic side of the shock, it will distort the shock and generate a reflected sound wave and a reflected entropy-vorticity wave.

Since in the case of incidence from the subsonic side of the shock there is no disturbance on the supersonic side, we have $\pi_0 = \xi_0 = \eta_0 = \sigma_0 = 0$, and on the subsonic side:

$$\begin{pmatrix} \pi_1 \\ \xi_1 \\ \eta_1 \\ \sigma_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \alpha_0 & \alpha_1 & -\beta_2 & 0 \\ \beta_0 & \beta_1 & \alpha_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \epsilon \exp[ik_1(\alpha_0 x + \beta_0 y)/(1 + M_1\alpha_0)] \\ A \exp[ik_1(\alpha_1 x + \beta_1 y)/(1 + M_1\alpha_1)] \\ B \exp[ik_1(\alpha_2 x + \beta_2 y)/M_1\alpha_2] \\ C \exp[ik_1(\alpha_2 x + \beta_2 y)/M_1\alpha_2] \end{pmatrix}, \tag{10}$$

where $\mathbf{n}_0 = (\alpha_0, \beta_0)$ is the incident sound wave normal, $\mathbf{n}_1 = (\alpha_1, \beta_1)$ is the reflected sound wave normal, $\mathbf{n}_2 = (\alpha_2, \beta_2)$ is the reflected entropy-vorticity wave normal, and ϵ = incident sound wave amplitude, A = reflected sound wave amplitude, B = reflected vorticity wave amplitude, C = reflected entropy wave amplitude.

Let us assume that the distortion has the form $x = f(y, t) = a\, e^{i\kappa y - i\omega t}$. The linearized shock conditions (4) give four equations for the unknown amplitudes a, A, B, and C in terms of the incident amplitude ϵ. In order that these equations have solutions which are independent of y, we must equate the coefficients of y in the exponentials. This gives the analogs of Snell's law of reflection,

$$\kappa = k_1\beta_2/M_1\alpha_2 = k_1\beta_1/(1 + M_1\alpha_1) = k_1\beta_0/(1 + M_1\alpha_0). \tag{11}$$

From (11) we find

$$\alpha_1 = -[(1 + M_1^2)\alpha_0 + 2M_1]/L^2, \qquad \beta_1 = (1 - M_1^2)\beta_0/L^2, \tag{12}$$

$$\alpha_2 = (1 + M_1\alpha_0)/L, \qquad \beta_2 = M_1\beta_0/L, \tag{13}$$

where $L = (1 + M_1^2 + 2M_1\alpha_0)^{\frac{1}{2}}$. Substituting ansatz (10) into the linearized shock conditions (4), and using (12) and (13), one obtains

$$\begin{aligned} a\Delta &= i(\mathbf{n}_1\cdot\mathbf{n}_2 - \mathbf{n}_0\cdot\mathbf{n}_2)\epsilon, \\ A\Delta &= [\omega a_{25}\alpha_2 - \kappa a_{35}\beta_2 - \omega a_{15}(\mathbf{n}_0\cdot\mathbf{n}_2)]\epsilon, \\ B\Delta &= [\kappa a_{35}(\alpha_0 - \alpha_1) + \omega a_{25}(\beta_0 - \beta_1) \\ &\qquad + \omega a_{15}(\beta_1\alpha_0 - \alpha_1\beta_0)]\epsilon, \\ C\Delta &= \omega a_{45}(\mathbf{n}_1\cdot\mathbf{n}_2 - \mathbf{n}_0\cdot\mathbf{n}_2)\epsilon, \end{aligned} \tag{14}$$

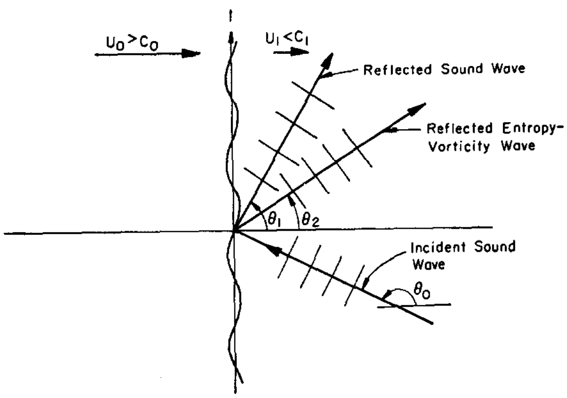

FIG. 1. The reflection of a plane sound wave incident from the subsonic side of a shock wave for $M_0 = 2$, $\mu = 6$.

where $\Delta = \omega a_{15}(\mathbf{n}_1 \cdot \mathbf{n}_2) + \kappa a_{35}\beta_2 - \omega a_{25}\alpha_2$. The a_{ij} are the numerical components of the matrix A of Eq. (5), and the α_1, β_1, α_2, β_2 are to be considered as functions of α_0, β_0 given by Eqs. (12) and (13). Since κ is proportional to ω, the amplitudes A, B, and C are independent of frequency, and a is inversely proportional to frequency. Equations (14) are the analogs of the Fresnel formulas for reflection. An illustration of the reflected waves is shown in Fig. 1, and a graph for the determination of the angles of reflection is given in Fig. 2.

B. Supersonic Incidence of Plane Waves

If a sound wave is incident on the shock from the supersonic side, a refracted sound wave and a refracted entropy-vorticity wave occur but no reflected wave. The truth of this statement may be established by consideration of the Poynting vector. In this case waves with normals in the range $0 \leq \theta_0 \leq \pi$ must be considered as incident waves since all these transport energy toward the shock. On the other hand, the refracted sound normals are restricted to the range $0 \leq \theta_1 \leq \theta_g$, since only waves in this range carry energy away from the shock.

The incident sound wave on the supersonic side of the shock has the form:

$$(\pi_0, \xi_0, \eta_0, \sigma_0) = (1, \alpha_0, \beta_0, 0)\epsilon \cdot \exp\,[ik_0(\alpha_0 x + \beta_0 y)/(1 + M_0\alpha_0)], \qquad (15)$$

where ϵ is the incident wave amplitude and $\mathbf{n}_0 = (\alpha_0, \beta_0)$ is the incident wave normal. On the subsonic side we have the refracted wave field:

$$\begin{pmatrix} \pi_1 \\ \xi_1 \\ \eta_1 \\ \sigma_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ \beta_1 & \alpha_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A \exp\,[ik_1(\alpha_1 x + \beta_1 y)/(1 + M_1\alpha_1)] \\ B \exp\,[ik_1(\alpha_2 x + \beta_2 y)/M_1\alpha_2] \\ C \exp\,[ik_1(\alpha_2 x + \beta_2 y)/M_1\alpha_2] \end{pmatrix} \qquad (16)$$

where $\mathbf{n}_1 = (\alpha_1, \beta_1)$ is the refracted sound wave

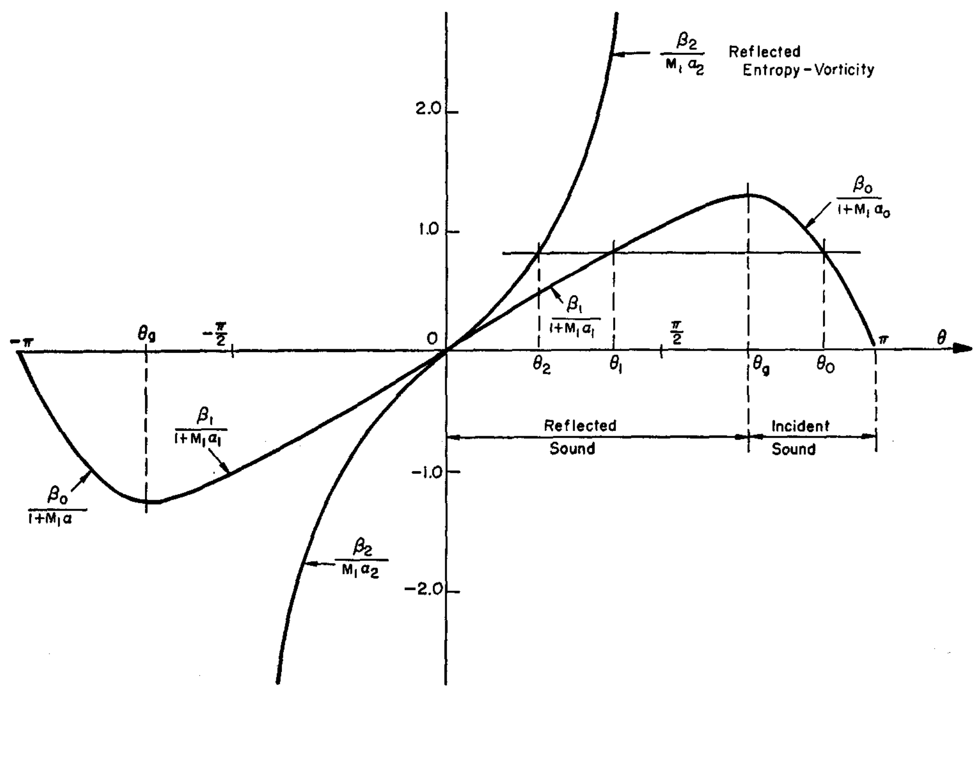

FIG. 2. The angles of the reflected sound and entropy-vorticity waves are determined from the graph for $M_0 = 2$, $\mu = 6$.

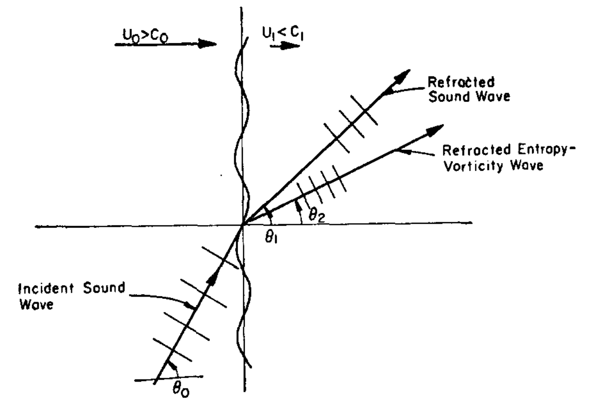

FIG. 3. The refraction of a plane sound wave incident from the supersonic side of a shock wave for $M_0 = 2$, $\mu = 6$.

normal, $\mathbf{n}_2 = (\alpha_2, \beta_2)$ is the refracted entropy-vorticity wave normal, and A = refracted sound wave amplitude, B = refracted vorticity wave amplitude, and C = refracted entropy wave amplitude.

The waves (15) and (16) are related across the shock by the linearized shock conditions (4). We again assume that $x = f(y, t) = a\, e^{i\kappa y - i\omega t}$ and demand that the shock conditions give solutions for a, A, B, and C which are independent of y. We therefore must equate the coefficients of y in the exponentials of (15) and (16),

$$\kappa = k_1\beta_2/M_1\alpha_2 = k_1\beta_1/(1 + M_1\alpha_1) = k_0\beta_0/(1 + M_0\alpha_0). \tag{17}$$

Equations (17) are the analogs of Snell's law of refraction. Setting $\lambda = k_0/k_1$, we find

$$\alpha_1 = [-M_1\lambda^2\beta_0^2 \pm (1 + M_0\alpha_0)N]/L^2, \quad \beta_1 = \lambda\beta_0[(1 + M_0\alpha_0) \pm M_1N]/L^2 \tag{18}$$

and

$$\alpha_2 = \pm(1 + M_0\alpha_0)/L, \qquad \beta_2 = M_1\lambda\beta_0/L \tag{19}$$

with

$$L = [\lambda^2M_1^2\beta_0^2 + (1 + M_0\alpha_0)^2]^{\frac{1}{2}}$$

and

$$N = [(1 + M_0\alpha_0)^2 - (1 - M_1^2)\lambda^2\beta_0^2]^{\frac{1}{2}}.$$

In Eqs. (18) we must choose the positive sign for $(1 + M_0\alpha_0) > 0$ and the negative sign for $(1 + M_0\alpha_0) < 0$ to obtain refracted waves which carry energy downstream when α_1 and β_1 are real and which are damped when α_1 and β_1 are complex. This latter behavior occurs when $\lambda\beta_0(1 - M_1^2)^{\frac{1}{2}} > (1 + M_0\alpha_0) > -\lambda\beta_0(1 - M_1^2)^{\frac{1}{2}}$. This angular region is bounded by the two critical angles corresponding to $\lambda\beta_0(1 - M_1^2)^{\frac{1}{2}} = \pm(1 + M_0\alpha_0)$. In this critical region the Poynting vector is directed along the shock and the refracted sound field is damped in the direction $\mathbf{n}' = (1, 0)$ normal to the shock. In Eqs. (19) we choose the signs in precisely the same way [+ or − according as $(1 + M_0\alpha_0) \gtrless 0$] to achieve refracted entropy-vorticity waves which are always directed downstream.

By a process exactly analogous to the steps which advanced us from (10) to (14), we now obtain the other set of Fresnel formulas:

$$\begin{aligned}
a\Delta &= [(a_{21} + \alpha_0 a_{22})\alpha_2 + \beta_0 a_{33}\beta_2 \\
&\quad - (a_{11} + \alpha_0 a_{12})\mathbf{n}_1\cdot\mathbf{n}_2]\epsilon \\
A\Delta &= \{a_{15}\omega[(a_{21} + \alpha_0 a_{22})\alpha_2 + \beta_0 a_{33}\beta_2] \\
&\quad + (a_{11} + \alpha_0 a_{12})(a_{35}\kappa\beta_2 - a_{25}\omega\alpha_2)\}\epsilon \\
B\Delta &= \{a_{15}\omega[\alpha_1\beta_0 a_{33} - \beta_1(a_{21} + \alpha_0 a_{22})] \\
&\quad - [a_{25}\omega\beta_0 a_{33} + a_{35}\kappa(a_{11} + \alpha_0 a_{12})] \\
&\quad + (a_{11} + \alpha_0 a_{12})(a_{25}\omega\beta_1 + a_{35}\kappa\alpha_1)\}\epsilon \\
C\Delta &= [a_{45}\omega(\beta_2 a_{33}\beta_0 + \alpha_2(a_{21} + \alpha_0 a_{22}) \\
&\quad - (a_{11} + \alpha_0 a_{12})\mathbf{n}_1\cdot\mathbf{n}_2) \\
&\quad + (a_{41} + \alpha_0 a_{42})(a_{15}\omega\mathbf{n}_1\cdot\mathbf{n}_2 \\
&\quad - a_{25}\omega\alpha_2 + a_{35}\kappa\beta_2)]\epsilon
\end{aligned} \tag{20}$$

where $\Delta = \omega a_{15}(\mathbf{n}_1\cdot\mathbf{n}_2) + \kappa a_{35}\beta_2 - \omega a_{25}\alpha_2$. The a_{ij} are given constants from Eqs. (5) and the α_1, β_1, α_2, β_2 are functions of α_0, β_0 by Eqs. (18) and (19). It should be noted that A, B, and C are independent of frequency and a is inversely proportional to frequency. An illustration of the refracted waves is shown in Fig. 3, and a graph for the determination of the angles of refraction is given in Fig. 4.

For a discussion of numerous cases—often strange—of reflected and refracted plane waves, the reader is referred to the monograph of F. K. Moore.[2] It is interesting that in obtaining the analogs of Snell's laws and Fresnel's formulas, the use of $\mathbf{\Sigma}$ was indispensable.

4. CYLINDRICAL SOUND WAVES IN A MOVING GAS

Now we wish to use the results obtained concerning plane waves to determine the behavior of cylindrical waves. For this purpose it is necessary to decompose the incident cylindrical sound wave into a "Weyl" type of superposition of plane waves. A velocity potential $\Phi(x, y, t)$, from which the flow

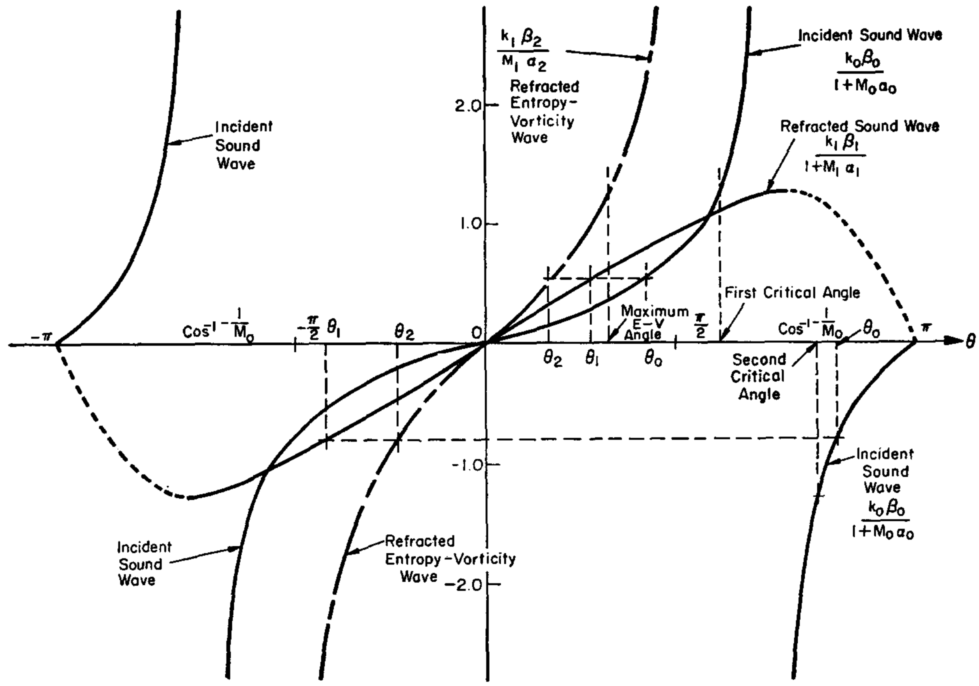

FIG. 4. The angles of the refracted sound and entropy-vorticity waves are determined from the graph for $M_0 = 2$, $\mu = 6$.

variables are derived through $\xi = (\partial\Phi/\partial x)$, $\eta = (\partial\Phi/\partial y)$, $\pi = -1/C[(\partial/\partial t) + U(\partial/\partial x)]\Phi$, satisfies by virtue of (3) a convective wave equation,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{1}{C^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\Phi = 0.$$

The corresponding inhomogeneous equation for the waves generated by an oscillating line source of strength $4\pi\epsilon$ located at the origin is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{1}{C^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\Phi = 4\pi\epsilon\ \delta(x)\ \delta(y)e^{-i\omega t}. \quad (21)$$

The substitution $\Phi(x, y, t) = \varphi(x, y)\exp[i(Mkx/(M^2-1) - \omega t)]$ reduces (21) to a time-independent equation,

$$(1 - M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + [k^2/(1 - M^2)]\varphi = 4\pi\epsilon\ \delta(x)\ \delta(y). \quad (22)$$

Setting

$$x' = x/(|1 - M^2|)^{\frac{1}{2}}, \qquad y' = y,$$
$$k' = k/(|1 - M^2|)^{\frac{1}{2}}, \quad (23)$$
$$\epsilon' = \epsilon/(|1 - M^2|)^{\frac{1}{2}},$$

Eq. (22) becomes, for $M \neq 1$,

$$\pm\left(\frac{\partial^2\varphi}{\partial x'^2} + k'^2\varphi\right) + \frac{\partial^2\varphi}{\partial y'^2} = 4\pi\epsilon'\ \delta(x')\ \delta(y') \quad (24)$$

where a + or − sign is chosen for $M < 1$ or $M > 1$, respectively. A solution to (24) in the form of a Fourier integral is

$$\varphi(x', y') = \pm(\epsilon'/\pi) \cdot \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_x \frac{e^{i(k_x x' + k_y y')}}{k'^2 \mp k_y^2 - k_x^2}. \quad (25)$$

The path of integration will be discussed presently.

Let us consider the k_x integral first. In particular, set

$$\psi(x') = \pm\int_{-\infty}^{+\infty} \frac{e^{ik_x x'}}{K^2 - k_x^2} dk_x, \quad K = (k'^2 \mp k_y^2)^{\frac{1}{2}}. \quad (26)$$

This integral is to be evaluated in the k_x plane along the real axis. The method of surrounding the poles at $\pm K$ must be specified in both cases ($M < 1$ and $M > 1$). For $M < 1$ we choose the path P_1

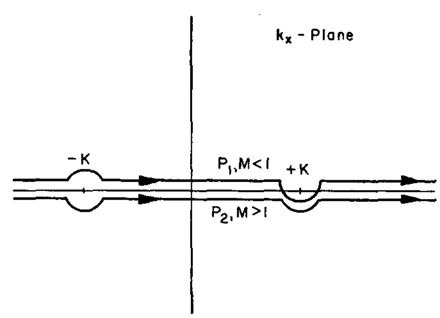

FIG. 5. The path of integration for $\psi(x')$ in the complex k_x plane is P_1 for $M < 1$ and P_2 for $M > 1$.

shown in Fig. 5 so that the Sommerfeld radiation condition is satisfied at $M = 0$. For $M > 1$ we choose the path P_2 to obtain a solution of the equation which vanishes upstream of the source, i.e., for $x < 0$.

Having specified the paths of integration, the integral (26) may be evaluated

for $M < 1$

$$\psi(x') = (\pi/ik)e^{ik|x'|}$$

for $M > 1$ (27)

$$\psi(x') = \begin{cases} (i\pi/k)(e^{ikx'} - e^{-ikx'}), & x' > 0 \\ 0, & x' < 0. \end{cases}$$

Using (27) the integral (25) may be rewritten:

for $M < 1$

$$\varphi(x', y') = \frac{\epsilon'}{i} \int_{-\infty}^{+\infty} \frac{\exp\,[i((k'^2 - k_y^2)^{\frac{1}{2}}\,|x'| + k_y y')]}{(k'^2 - k_y^2)^{\frac{1}{2}}}\, dk_y$$

for $M > 1$ (28)

$$\varphi(x', y') = \begin{cases} -\dfrac{\epsilon'}{i} \displaystyle\int_{-\infty}^{+\infty} \left(\frac{\exp\,[i((k'^2 + k_y^2)^{\frac{1}{2}}x' + k_y y')]}{(k'^2 + k_y^2)^{\frac{1}{2}}} - \frac{\exp\,[-i((k'^2 + k_y^2)^{\frac{1}{2}}x' - k_y y']}{(k'^2 + k_y^2)^{\frac{1}{2}}}\right) dk_y, & x' > 0 \\ 0, & x' < 0. \end{cases}$$

For $M < 1$ we transform (28) to the complex θ plane by means of $k_y = k' \sin\theta$ to find

$$\varphi(x', y') = \frac{\epsilon'}{i} \int_P e^{ik'(\cos\theta|x'| + \sin\theta y')}\, d\theta, \qquad (29)$$

where P is the path of integration shown in Fig. 6(a). For $M > 1$ we must use $k_y = ik' \sin\theta$ and $(k'^2 + k_y^2)^{\frac{1}{2}} = \pm k' \cos\theta$ for the first and second parts of the integrand, with the result:

$$\varphi(x', y') = \begin{cases} -\epsilon' \displaystyle\int_{P_1+P_2} e^{ik'(\cos\theta x' + i\sin\theta y')}\, d\theta, & x' > 0 \\ 0, & x' < 0 \end{cases} \qquad (30)$$

where P_1 and P_2 are the paths illustrated in Fig. 6(b).

The integral (29) is Sommerfeld's representation of $H_0^{(1)}$, and therefore for $M < 1$ the line source solution may be written:

$M < 1$

$$\Phi(x, y, t) = \frac{\epsilon\pi}{i(1 - M^2)^{\frac{1}{2}}} H_0^{(1)}\left(\frac{k[x^2 + (1 - M^2)y^2]^{\frac{1}{2}}}{1 - M^2}\right) \cdot \exp\left(-i\frac{Mkx}{1 - M^2} - i\omega t\right). \qquad (31)$$

On the other hand, for $M > 1$ we must consider two cases. For $|y'| > x'$, we may close the paths of integration since in the region $0 \leq \mathrm{Re}\,\theta \leq \pi$ the integrand $\to 0$ as $\mathrm{Im}\,\theta \to \pm\infty$. Thus for $|y'| > x'$ the integral in (30) $= 0$. However, for $|y'| < x'$ the paths may not be closed as the integrand is unbounded in the strip $0 \leq \mathrm{Re}\,\theta \leq \pi$. We may nevertheless evaluate the integral (30). We find in this case that the integral is the Sommerfeld representation for the Bessel function. Therefore:

$M > 1$

$$\Phi(x, y, t) = \begin{cases} -\dfrac{2\pi\epsilon}{(M^2 - 1)^{\frac{1}{2}}} J_0\left(\dfrac{k[x^2 - (M^2 - 1)y^2]^{\frac{1}{2}}}{M^2 - 1}\right) \cdot \exp\left(i\dfrac{Mkx}{M^2 - 1} - i\omega t\right), & |y| < \dfrac{x}{(M^2 - 1)^{\frac{1}{2}}} \\ 0, & \text{otherwise.} \end{cases} \qquad (32)$$

Note that the disturbance (32) is confined to the interior of the Mach wedge and has stationary zeros, a property usually associated with standing waves. It is also interesting to observe the 90° phase shift of the sound circles across their point of tangency with the Mach wedge. This is analogous to the 90° phase shift of light waves passing through a focal line predicted by Debye.[7] The waves de-

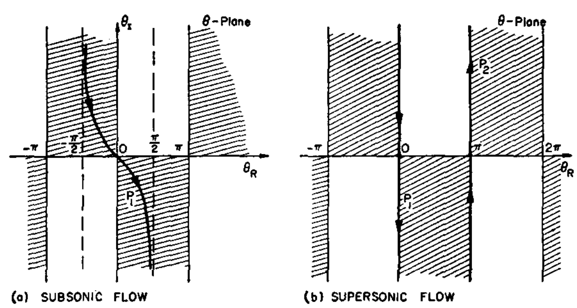

FIG. 6. The path of integration for $\Phi(x, y, t)$ in the complex θ plane is P_1 for $M < 1$ and $P_1 + P_2$ for $M > 1$.

[7] A. Sommerfeld, *Lectures on Theoretical Physics*, Vol. 4, *Optics*, translated by O. Laporte and P. Moldauer (Academic Press, Inc., New York, 1954), p. 319.

scribed by Eqs. (31) and (32) are illustrated in Fig. 7.

The corresponding formulas for the case of a point source may be found by a simple extension of the above arguments. They are

for $M < 1$,

$$\Phi(x, y, z, t) = -\epsilon \frac{\exp\left(i\,\dfrac{k([x^2 + (1 - M^2)(y^2 + z^2)]^{\frac{1}{2}} - Mx)}{1 - M^2} - i\omega t\right)}{[x^2 + (1 - M^2)(y^2 + z^2)]^{\frac{1}{2}}}$$

for $M > 1$, (33)

$$\Phi(x, y, z, t) = \begin{cases} 2\epsilon \dfrac{\cos\left(\dfrac{k}{M^2 - 1}[x^2 - (M^2 - 1)(y^2 + z^2)]^{\frac{1}{2}}\right)\exp\left(i\,\dfrac{kMx}{M^2 - 1} - i\omega t\right)}{[x^2 - (M^2 - 1)(y^2 + z^2)]^{\frac{1}{2}}} \\ x > (M^2 - 1)^{\frac{1}{2}}(y^2 + z^2)^{\frac{1}{2}} \\ 0, \quad \text{otherwise.} \end{cases}$$

The decomposition into Weyl type of integrals is accomplished most easily directly from Eqs. (29) and (30). The integral (29) may be transformed to the θ_0 plane by means of the aberration formulas

$$\cos\theta_0 = -\frac{\cos\theta + M}{1 + M\cos\theta}$$
$$\sin\theta_0 = \frac{(1 - M^2)^{\frac{1}{2}}\sin\theta}{1 + M\cos\theta} \tag{34}$$

to give (for $x < 0$, the region of interest to us):

$$\Phi(x, y, t) = -\frac{\epsilon}{i}\int_{P_0} \frac{\exp\left[ik\,\dfrac{(\alpha_0 x + \beta_0 y)}{1 + M\alpha_0} - i\omega t\right]}{1 + M\alpha_0}\,d\theta_0, \tag{35}$$

where P_0 is the image of P under the transformation $\theta \to \theta_0$ and $\alpha_0 = \cos\theta_0$, $\beta_0 = \sin\theta_0$.

The integral (30) is transformed to the θ_0 plane by means of

$$\cos\theta_0 = -\frac{\cos\theta + M}{1 + M\cos\theta},$$
$$\sin\theta_0 = -\frac{i(M^2 - 1)^{\frac{1}{2}}\sin\theta}{1 + M\cos\theta} \tag{36}$$

to give [for $x > (M^2 - 1)^{\frac{1}{2}}\,|y|$]:

$$\Phi(x, y, t) = \frac{\epsilon}{i}\int_{P_{01}+P_{02}} \frac{\exp\left[ik\,\dfrac{(\alpha_0 x + \beta_0 y)}{1 + M\alpha_0} - i\omega t\right]}{1 + M\alpha_0}\,d\theta_0, \tag{37}$$

where P_{01} and P_{02} are the images of P_1 and P_2, and where $\alpha_0 = \cos\theta_0$, $\beta_0 = \sin\theta_0$. From expressions (35) and (37), one may calculate the pressure π and the velocities ξ and η of the disturbance. When doing this it is seen that in the subsonic case we may identify (35) with a weighted superposition of plane waves, the weighting function being $\epsilon^*(\theta_0) = -(\epsilon k)/[(1 + M\alpha_0)^2]$; and similarly (37) may be identified with a plane-wave superposition with weight function $\epsilon^*(\theta_0) = (\epsilon k)/[(1 + M\alpha_0)^2]$.

Each of the plane waves occurring in these integrals gives rise to a reflected or refracted plane-wave field of known amplitude and direction, and therefore the integrals (35) and (37) give rise to a superposition of such wave fields. We shall in the following section set up and evaluate the integrals for the resulting disturbances.

5. INTERACTION PROBLEM FOR CYLINDRICAL WAVES

A. Source on the Subsonic Side of the Shock

If a line source is located at $x_0 > 0$, the incident sound wave will be described by Eq. (35) with x replaced by $x - x_0$. The resulting distortion and the reflected wave fields may be written down immediately,

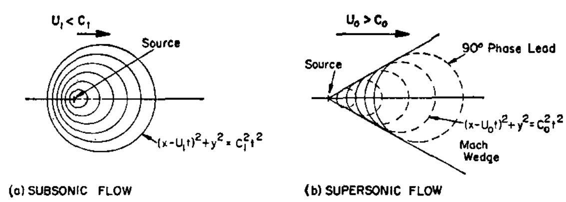

FIG. 7. The surfaces of constant phase for cylindrical waves in a moving gas.

$$
\begin{aligned}
f(y,t) &= -k_1 \int_{P_0} \frac{a}{(1+M_1\alpha_0)^2} \exp\left[ik_1 \frac{(-\alpha_0 x_0 + \beta_0 y)}{1+M_1\alpha_0} - i\omega t \right] d\theta_0 \\
\Phi_1(x,y,t) &= i \int_{P_0} \frac{1+M_1\alpha_1}{(1+M_1\alpha_0)^2} A \exp\left[-i \frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1 \frac{(\alpha_1 x + \beta_1 y)}{1+M_1\alpha_1} - i\omega t \right] d\theta_0 \\
\chi_1(x,y,t) &= -i \int_{P_0} \frac{M_1\alpha_2}{(1+M_1\alpha_0)^2} B \exp\left[-i \frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t \right] d\theta_0 \\
\sigma_1(x,y,t) &= -k_1 \int_{P_0} \frac{1}{(1+M_1\alpha_0)^2} C \exp\left[-i \frac{k_1\alpha_0 x_0}{1+M_1\alpha_0} + ik_1 \frac{(\alpha_2 x + \beta_2 y)}{M_1\alpha_2} - i\omega t \right] d\theta_0 .
\end{aligned} \tag{38}
$$

The expressions defining a, A, B, and C are given in Eqs. (14) and those defining α_1, β_1 and α_2, β_2 are given in Eqs. (12) and (13). The path of integration is that defined for the integral (35). The function $\chi_1(x, y, t)$ introduced in (38) for the sake of brevity to describe the vorticity field is defined by

$$
-\frac{\partial \chi_1}{\partial y} = \xi_{\text{vort.}}, \qquad \frac{\partial \chi_1}{\partial x} = \eta_{\text{vort.}}. \tag{39}
$$

To evaluate these integrals, it is convenient to transform the path of integration to the Sommerfeld path P of Fig. 6(a) by means of the relations inverse to (34). Equations (38) thereby reduce to:

$$
\begin{aligned}
f(y,t) &= k_1 \frac{\exp\left(i \frac{k_1 M_1 x_0}{1-M_1^2} - i\omega t\right)}{(1-M_1^2)^{\frac{1}{2}}} \int_P (1+M_1\alpha) a(\alpha) \exp\left[ik_1 \frac{\alpha x_0 + \beta(1-M_1^2)^{\frac{1}{2}} y}{1-M_1^2} \right] d\theta, \\
\Phi_1(x,y,t) &= \frac{\exp\left[i \frac{k_1 M_1 (x-x_0)}{1-M_1^2} - i\omega t \right]}{i(1-M_1^2)^{\frac{1}{2}}} \int_P \frac{1+M_1\alpha}{1-M_1\alpha} A(\alpha) \exp\left[ik_1 \frac{\alpha(x+x_0) + \beta(1-M_1^2)^{\frac{1}{2}} y}{1-M_1^2} \right] d\theta, \\
\chi_1(x,y,t) &= \frac{iM_1 \exp\left[i \frac{k_1 M_1 x_0}{1-M_1^2} + i \frac{k_1}{M_1}(x - U_1 t) \right]}{1-M_1^2} \int_P B(\alpha,\beta) \exp\left[ik_1 \frac{\alpha x_0 + \beta(1-M_1^2)^{\frac{1}{2}} y}{1-M_1^2} \right] d\theta, \\
\sigma_1(x,y,t) &= k_1 \frac{\exp\left[i \frac{k_1 M_1 x_0}{1-M_1^2} + i \frac{k_1}{M_1}(x - U_1 t) \right]}{(1-M_1^2)^{\frac{1}{2}}} \int_P (1+M_1\alpha) C(\alpha) \exp\left[ik_1 \frac{\alpha x_0 + \beta(1-M_1^2)^{\frac{1}{2}} y}{1-M_1^2} \right] d\theta,
\end{aligned} \tag{40}
$$

$a(\alpha)$, $A(\alpha)$, $B(\alpha, \beta)$, and $C(\alpha)$ being the appropriately transformed Fresnel formulas.

We shall use the saddle-point method to evaluate the integrals in Eqs. (40). To use the standard saddle-point formula it is first necessary to verify that the integrands are slowly varying functions of the independent variable near the saddle point. That this is so may be seen by examining the poles of the various integrands and verifying that they are isolated from the saddle point. It may be seen by direct calculation of a, A, B, and C that all the integrands have poles at the points where

$$M_0^2\alpha^2 + 2M_0^2 M_1\alpha + 1 = 0$$

or at

$$\alpha = -M_1 \pm (M_1^2 - 1/M_0^2)^{\frac{1}{2}}.$$

It is easy to see that these poles lie outside the strip $-\pi/2 \leq \operatorname{Re}\theta \leq \pi/2$ and are indeed isolated from the saddle points of the integrands. The integrands for $f(y, t)$ and $\sigma_1(x, y, t)$ have no other poles, while the integrand for $\Phi_1(x, y, t)$ has poles at $1 - M_1\alpha = 0$ and the integrand for $\chi_1(x, y, t)$ has poles at $1 + M_1\alpha = 0$. The first of these possibilities, $1 - M_1\alpha = 0$, gives poles on the imaginary axis, which can be disregarded since $M_1 < 1$. The second possibility, $1 + M_1\alpha = 0$, gives poles outside the strip $-\pi/2 \leq \operatorname{Re}\theta \leq \pi/2$ and also may be eliminated from further consideration.[8]

We may immediately write down the first term in the asymptotic series for the integrals in Eqs. (40) as

[8] If the poles had not been isolated from the saddle points of the integrands, we would have been confronted with the difficulties of the sort which Ott and Van der Waerden encountered in electromagnetic diffraction. No surface waves were found in this investigation. See H. Ott, Ann. Physik **43**, 393 (1943); B. L. Van der Waerden, Appl. Sci. Research **B2**, 33 (1951).

$$
\begin{aligned}
f(y,t) &\sim \left(\frac{2\pi}{k_1\rho}\right)^{\frac{1}{2}} \frac{k_1(\rho + M_1x_0)}{(1-M_1^2)\rho}\, a\left(\frac{x_0}{\rho}\right) \exp\left[\left(i\frac{k_1}{1-M_1^2}\right)(\rho + M_1x_0) - i\omega t - i\frac{\pi}{4}\right],\\
\Phi_1(x,y,t) &\sim \frac{1}{i}\left(\frac{2\pi}{k_1\rho_1}\right)^{\frac{1}{2}} \frac{\rho_1 + M_1(x+x_0)}{\rho_1 - M_1(x+x_0)}\, A\left(\frac{x+x_0}{\rho}\right) \exp\left[\left(i\frac{k_1}{1-M_1^2}\right)(\rho_1 + M_1(x-x_0)) - i\omega t - i\frac{\pi}{4}\right],\\
\chi_1(x,y,t) &\sim i\left(\frac{2\pi}{k_1\rho}\right)^{\frac{1}{2}} \frac{M_1}{(1-M_1^2)^{\frac{1}{2}}}\, B\left[\frac{x_0}{\rho}, \frac{(1-M_1^2)^{\frac{1}{2}}y}{\rho}\right] \exp\left[\left(i\frac{k_1}{1-M_1^2}\right)(\rho + M_1x_0) + i\frac{k_1}{M_1}(x - U_1t) - i\frac{\pi}{4}\right],\\
\sigma_1(x,y,t) &\sim \left(\frac{2\pi}{k_1\rho}\right)^{\frac{1}{2}} \frac{k_1(\rho + M_1x_0)}{(1-M_1^2)\rho}\, C\left(\frac{x_0}{\rho}\right) \exp\left[\left(i\frac{k_1}{1-M_1^2}\right)(\rho + M_1x_0) + i\frac{k_1}{M_1}(x - U_1t) - i\frac{\pi}{4}\right],
\end{aligned}
\tag{41}
$$

where $\rho = [x_0^2 + (1-M_1^2)y^2]^{\frac{1}{2}}$ and $\rho_1 = [(x+x_0)^2 + (1-M_1^2)y^2]^{\frac{1}{2}}$ and the functions a, A, B, and C are the Fresnel coefficients evaluated at the saddle points of the integrands. It is important to realize that a, A, B, and C are functions of the transformed variables α and β in Eqs. (41) and not functions of the original variables α_0 and β_0 as in Eqs. (14).

The surfaces of constant phase for the reflected sound potential are given by

$$(x + x_0 - U_1t_1)^2 + y^2 = C_1^2t_1^2 \tag{42}$$

with

$$t_1 = t - \frac{2M_1x_0}{C_1(1-M_1^2)}.$$

These are, of course, circles originating at the virtual source $-x_0$ and carried downstream with the flow. The surfaces of constant phase for the reflected entropy and vorticity waves, on the other hand, are:

$$\left(x + \frac{M_1^2x_0}{1-M_1^2} - U_1t\right)^2 - \frac{M_1^2}{1-M_1^2}y^2 = \frac{M_1^2x_0^2}{1-M_1^2}, \tag{43}$$

which represent a family of hyperbolas transported downstream with the flow. The waves discussed here are illustrated in Fig. 8.

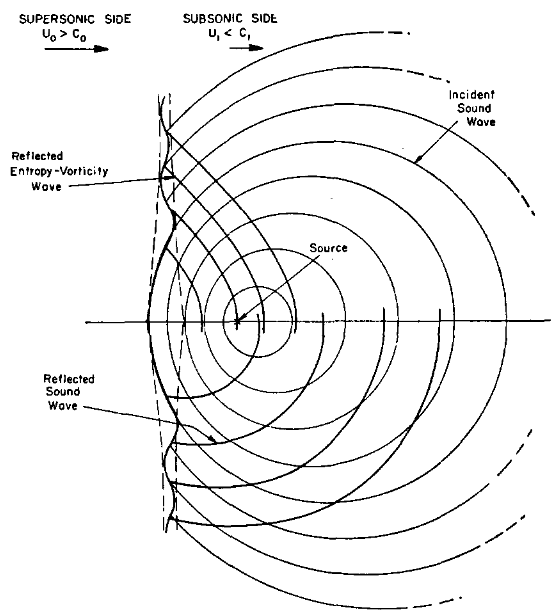

FIG. 8. The reflection of cylindrical sound waves incident from the subsonic side of a shock wave for $M_0 = 2$, $\mu = 6$.

B. Source on the Supersonic Side of the Shock

If our line source is located at $-x_0$, the incident sound wave will be described by Eq. (37) with x replaced by $x + x_0$. The corresponding distortion and refracted wave field may be written down immediately as:

$$
\begin{aligned}
f(y,t) &= k_0 \int_{P_{01}+P_{02}} \frac{a}{(1+M_0\alpha_0)^2} \exp\left(ik_0\frac{\alpha_0x_0 + \beta_0y}{1+M_0\alpha_0} - i\omega t\right) d\theta_0,\\
\Phi_1(x,y,t) &= \frac{\lambda}{i}\int_{P_{01}+P_{02}} \frac{1+M_1\alpha_1}{(1+M_0\alpha_0)^2} A \exp\left(i\frac{k_0\alpha_0x_0}{1+M_0\alpha_0} + ik_1\frac{\alpha_1x + \beta_1y}{1+M_1\alpha_1} - i\omega t\right) d\theta_0,\\
\chi_1(x,y,t) &= i\lambda\int_{P_{01}+P_{02}} \frac{M_1\alpha_2}{(1+M_0\alpha_0)^2} B \exp\left(i\frac{k_0\alpha_0x_0}{1+M_1\alpha_0} + ik_1\frac{\alpha_2x + \beta_2y}{M_1\alpha_2} - i\omega t\right) d\theta_0,\\
\sigma_1(x,y,t) &= k_0\int_{P_{01}+P_{02}} \frac{1}{(1+M_0\alpha_0)^2} C \exp\left(i\frac{k_0\alpha_0x_0}{1+M_0\alpha_0} + ik_1\frac{\alpha_2x + \beta_2y}{M_1\alpha_2} - i\omega t\right) d\theta_0.
\end{aligned}
\tag{44}
$$

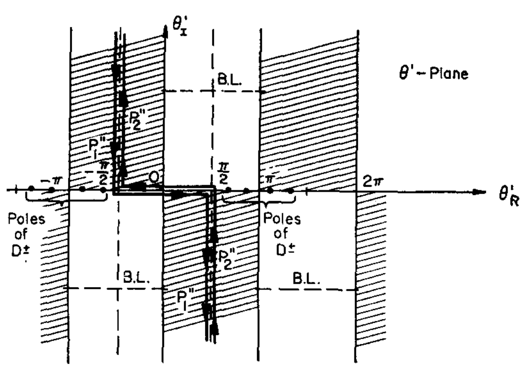

FIG. 9. The paths of integration for $\Phi(x, y, t)$ in the complex θ' plane for supersonic incidence.

The Fresnel coefficients a, A, B, and C are given in Eqs. (20), and the expressions for α_1, β_1 and α_2, β_2 are given in Eqs. (18) and (19). The paths of integration P_{01} and P_{02} are, of course, those used in Eq. (37).

It is convenient to transform the paths of integration to the Sommerfeld paths P_1 and P_2 of Fig. 6(b) by means of the inverses of Eqs. (36) to evaluate the integrals for $f(y, t)$, $\chi_1(x, y, t)$, and $\sigma_1(x, y, t)$, but this procedure is not particularly helpful for the sound potential $\Phi_1(x, y, t)$. However, the sound integral may be reduced to a convenient form by transformation to the θ' plane by means of:

$$\begin{aligned}\alpha_0 &= [-M_0\beta'^2 \pm \lambda(1-M_1^2)^{\frac{1}{2}}N']/L'^2,\\ \beta_0 &= \beta'[\lambda(1-M_1^2)^{\frac{1}{2}} \mp M_0N']/L'^2,\end{aligned} \tag{45}$$

with

$$\alpha' = \cos\theta', \qquad \beta' = \sin\theta',$$

where

$$L' = [\lambda^2(1-M_1^2) + M_0^2\beta'^2]^{\frac{1}{2}}$$

and

$$N' = [\lambda^2(1-M_1^2) + (M_0^2-1)\beta'^2]^{\frac{1}{2}},$$

where the $+$ and $-$ signs are chosen for the paths P_{01} and P_{02}, respectively. The images of the paths P_{01} and P_{02} which we denote by P_1' and P_2' are illustrated in Fig. 9.

The transformed integrals may be written

$$\begin{aligned}
f(y,t) &= ik_0\frac{\exp\left(i\dfrac{k_0M_0x_0}{M_0^2-1}-i\omega t\right)}{(M_0^2-1)^{\frac{1}{2}}}\int_{P_1+P_2}(1+M_0\alpha)a(\alpha,\beta)\exp\left[ik_0\frac{\alpha x_0+i\beta(M_0^2-1)^{\frac{1}{2}}y}{M_0^2-1}\right]d\theta\\
\Phi_1(x,y,t) &= i\lambda(1-M_1^2)^{\frac{1}{2}}\frac{\exp\left(i\dfrac{k_0M_0x_0}{M_0^2-1}-i\dfrac{k_1M_1x}{1-M_1^2}-i\omega t\right)}{M_0^2-1}\\
&\quad\cdot\left\{\int_{P_1'}D^+(\alpha',\beta')\exp\left[i\frac{k_0x_0N'(\alpha',\beta')}{(M_0^2-1)\lambda(1-M_1^2)^{\frac{1}{2}}}\right]\right.\\
&\quad\left.-\int_{P_2'}D^-(\alpha',\beta')\exp\left[-i\frac{k_0x_0N'(\alpha',\beta')}{(M_0^2-1)\lambda(1-M_1^2)^{\frac{1}{2}}}\right]\right\}\exp\left[ik_1\frac{\alpha'x+\beta'(1-M_1^2)^{\frac{1}{2}}y}{1-M_1^2}\right]d\theta'\\
\chi_1(x,y,t) &= i\lambda\frac{\exp\left[i\dfrac{k_0M_0x_0}{M_0^2-1}+i\dfrac{k_1}{M_1}(x-U_1t)\right]}{M_0^2-1}\int_{P_1+P_2}\frac{M_1(1+M_0\alpha)}{[(M_0^2-1)-M_1^2\lambda^2\beta^2]^{\frac{1}{2}}}B(\alpha,\beta)\\
&\quad\cdot\exp\left[ik_0\frac{\alpha x_0+i\beta(M_0^2-1)^{\frac{1}{2}}y}{M_0^2-1}\right]d\theta\\
\sigma_1(x,y,t) &= ik_0\frac{\exp\left[i\dfrac{k_0M_0x_0}{M_0^2-1}+i\dfrac{k_1}{M_1}(x-U_1t)\right]}{(M_0^2-1)^{\frac{1}{2}}}\int_{P_1+P_2}(1+M_0\alpha)C(\alpha,\beta)\\
&\quad\cdot\exp\left[ik_0\frac{\alpha x_0+i\beta(M_0^2-1)^{\frac{1}{2}}y}{M_0^2-1}\right]d\theta,
\end{aligned} \tag{46}$$

where $a(\alpha, \beta)$, $B(\alpha, \beta)$, and $C(\alpha, \beta)$ are the Fresnel coefficients of Eqs. (20) transformed to the θ plane by means of the inverses of Eqs. (36), while

$$D^{\pm}(\alpha',\beta') = \frac{\alpha'[\lambda(1-M_1^2)^{\frac{1}{2}} \pm M_0N'(\alpha',\beta')]}{(1-M_1\alpha')N'(\alpha',\beta')}A(\alpha',\beta'). \tag{47}$$

Here $A(\alpha', \beta')$ is the transform of the Fresnel coefficient for sound under (45). In Eq. (47), the $+$ sign is to be used when integrating along the path P_1' and the $-$ sign is used along the path P_2'.

We shall discuss the integrals for $f(y, t)$, $\chi_1(x, y, t)$, and $\sigma_1(x, y, t)$ first since these integrals are defined in the same plane. It may be verified that the integrands of $f(y, t)$ and $\sigma_1(x, y, t)$ have no poles and that the integrand of $\chi_1(x, y, t)$ has poles at $(M_0^2 - 1) = M_1^2\lambda^2\beta^2$. In addition to these poles, all the integrands have branch points at $M_0^2 - 1 = (M_1^2 - 1)\lambda^2\beta^2$. Because of the presence of the branch points as well as of poles in the plane of integration, considerable care must be exercised in the discussion of the behavior of the integrals.

Inside the streamlines emanating from the intersection of the Mach wedge with the shock, we may use the saddle-point method again to evaluate the integrals. There are two saddle points now, one on P_1 and another on P_2.

For $|y| < (M_0^2 - 1)^{\frac{1}{2}}x_0$ the integrals reduce asymptotically to

$$\begin{aligned} f(y, t) \sim i\left(\frac{2\pi}{k_0\rho}\right)^{\frac{1}{2}}\Bigg[&\frac{k_0(\rho + M_0x_0)}{\rho(M_0^2 - 1)}\, a\left[\frac{x_0}{\rho}, i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(i\,\frac{k_0\rho}{M_0^2 - 1} - i\,\frac{\pi}{4}\right) \\ &+ \frac{k_0(\rho - M_0x_0)}{\rho(M_0^2 - 1)}\, a\left[-\frac{x_0}{\rho}, -i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(-i\,\frac{k_0\rho}{M_0^2 - 1} + i\,\frac{\pi}{4}\right)\Bigg] \exp\left(i\,\frac{k_0M_0x_0}{M_0^2 - 1} - i\omega t\right) \\ \chi_1(x, y, t) \sim i\left(\frac{2\pi}{k_0\rho}\right)^{\frac{1}{2}}\Bigg[&\frac{\lambda M_1(\rho + M_0x_0)}{(M_0^2 - 1)(\rho^2 + M_1^2\lambda^2y^2)^{\frac{1}{2}}}\, B\left[\frac{x_0}{\rho}, i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(i\,\frac{k_0\rho}{M_0^2 - 1} - i\,\frac{\pi}{4}\right) \\ &+ \frac{\lambda M_1(\rho - M_0x_0)}{(M_0^2 - 1)(\rho^2 + M_1^2\lambda^2y^2)^{\frac{1}{2}}}\, B\left[-\frac{x_0}{\rho}, -i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(-i\,\frac{k_0\rho}{M_0^2 - 1} + i\,\frac{\pi}{4}\right)\Bigg] \\ &\cdot \exp\left[i\,\frac{k_0M_0x_0}{M_0^2 - 1} + i\,\frac{k_1}{M_1}(x - U_1t)\right] \qquad (48) \\ \sigma_1(x, y, t) \sim i\left(\frac{2\pi}{k_0\rho}\right)^{\frac{1}{2}}\Bigg\{&\frac{k_0(\rho + M_0x_0)}{\rho(M_0^2 - 1)}\, C\left[\frac{x_0}{\rho}, i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(i\,\frac{k_0\rho}{M_0^2 - 1} - i\,\frac{\pi}{4}\right) \\ &+ \frac{k_0(\rho - M_0x_0)}{\rho(M_0^2 - 1)}\, C\left[-\frac{x_0}{\rho}, -i\,\frac{(M_0^2 - 1)^{\frac{1}{2}}y}{\rho}\right] \exp\left(-i\,\frac{k_0\rho}{M_0^2 - 1} + i\,\frac{\pi}{4}\right)\Bigg\} \\ &\cdot \exp\left[i\,\frac{k_0M_0x_0}{M_0^2 - 1} + i\,\frac{k_1}{M_1}(x - U_1t)\right], \end{aligned}$$

where $\rho = [x_0^2 - (M_0^2 - 1)y^2]^{\frac{1}{2}}$. It is important to remember that a, B, and C are to be regarded as functions of the transformed variables α and β according to Eqs. (20).

For $|y| > (M_0^2 - 1)^{\frac{1}{2}}x_0$, outside the edge streamlines, we may close the two paths of integration as in the discussion of the incident sound wave. Since there are no poles in the integrands for the functions $f(y, t)$ and $\sigma_1(x, y, t)$, these functions immediately reduce to zero in the region under consideration. It may also be easily verified that in this region the function $\chi_1(x, y, t)$ is exponentially damped in the $\pm y$ directions.

The integral in Eqs. (46) describing the sound potential is in a form amenable to a saddle-point expansion. There are again two saddle points and various branch points and poles (which can be shown to contribute nothing to the integral) to be accounted for. Having done this, we may write the resulting asymptotic expression for the integral as:

$$\begin{aligned} \Phi_1(x, y, t) = i\lambda\,\frac{1 - M_1^2}{M_0^2 - 1}\left(\frac{2\pi}{k_1\rho}\right)^{\frac{1}{2}}\Bigg(&D^+\left[\frac{x_0}{\rho}, \frac{(1 - M_1^2)^{\frac{1}{2}}y}{\rho}\right] \exp\left\{i\,\frac{k_0x_0}{M_0^2 - 1}\left[1 + \frac{(M_0^2 - 1)y^2}{\lambda^2\rho^2}\right]^{\frac{1}{2}}\right\} \\ &+ D^-\left[\frac{x_0}{\rho}, \frac{(1 - M_1^2)^{\frac{1}{2}}y}{\rho}\right] \exp\left\{-i\,\frac{k_0x_0}{M_0^2 - 1}\left[1 + \frac{(M_0^2 - 1)y^2}{\lambda^2\rho^2}\right]^{\frac{1}{2}}\right\}\Bigg) \qquad (49) \\ &\cdot \exp\left[ik_1\,\frac{(\rho - M_1x)}{1 - M_1^2} + i\,\frac{k_0M_0x_0}{M_0^2 - 1} - i\omega t - i\,\frac{\pi}{4}\right] \end{aligned}$$

with $\rho = [x^2 + (1 - M_1^2)y^2]^{\frac{1}{2}}$ and $D^{\pm}$ as given by (47).

Equations (48) and (49) describe the behavior of the refracted wave field. It is of interest to examine the surfaces of constant phase for these refracted waves. For the entropy and vorticity waves of Eqs. (48), the surfaces of constant phase are seen to be

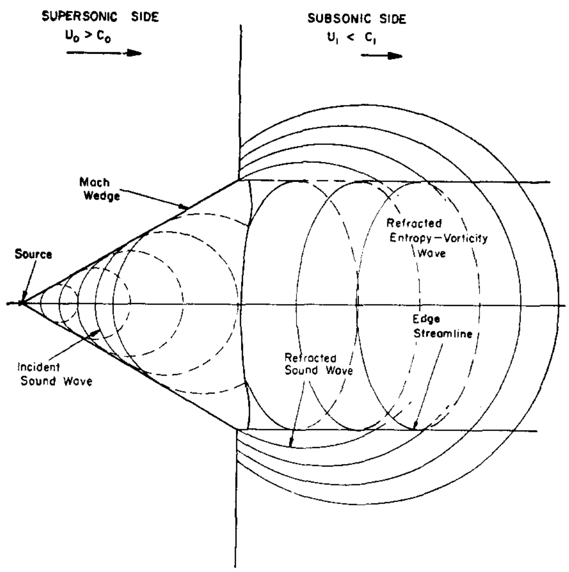

FIG. 10. The refraction of cylindrical sound waves incident from the supersonic side of a shock wave for $M_0 = 2$, $\mu = 6$.

ellipses tangent to the edge streamlines blown downstream with the flow. There is a phase difference of 90° between the front and back surfaces of these ellipses as in the case of the incident sound circles. The refracted sound wave has surfaces of constant phase which are approximately described as circles blown downstream with the flow. The deviation from circularity is of the order x_0/ρ as $\rho \rightarrow \infty$. The refraction of cylindrical waves incident on the shock from the supersonic side as illustrated in Fig. 10.

As was mentioned in the introduction, these methods may be easily extended to discuss the interaction of spherical waves with a shock if we use the appropriate Weyl expansion corresponding to incident waves of the form described by Eqs. (33). The extension to moving shocks presents greater difficulty, for the saddle points of the various integrands become quite complicated functions of position and time.

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