Structure of Weak Non-Hugoniot Shocks

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As the curvature of shock waves increases, the shock structure becomes two dimensional, and the usual Hugoniot jump conditions no longer hold. An equation has been derived for the structure of such a two-dimensional non-Hugoniot shock in the case of weak shocks with Mach numbers close to one. The development of this equation from the Navier–Stokes equations is based on the assumptions that the vertical velocity is of order $(M_1^* - 1)^{3/2}$ and that the flow within the shock is irrotational. From the derivation it appears that the non-Hugoniot region behaves as an acoustic wave driven by higher-order viscous effects. The properties of the above equation, which has been called the viscous–transonic or V–T equation have been investigated. The V–T equation appears to be a combination of Burgers' equation for weak normal shock structure and the transonic equation. It is shown that the structure of oblique shocks is a similarity solution of the V–T equation. Proper formulation of boundary conditions is considered and a uniqueness proof is given for a particular restricted boundary value problem.

I. INTRODUCTION

THE ratio of thickness to radius of curvature is usually so small that, locally, curved shocks may be treated as oblique shock waves along which tangential variations of velocity, pressure, etc., are negligible. The structure of oblique waves is one dimensional for they appear normal to an observer moving along the shock with the tangential component of the upstream velocity. In most aerodynamic problems structure is unimportant and shocks are treated as discontinuities separating different portions of an inviscid flow across which the Rankine–Hugoniot jump conditions hold. The jump conditions across such shocks, which are frequently called Rankine–Hugoniot shock waves, are derived by applying the inviscid conservation equations across the shock, subject to the assumption of uniform upstream and downstream flow.

The situation changes when curved shock waves are relatively thick as in transonic or low density flows, for then cases arise in which tangential variations are no longer negligible. The shock structure then becomes two dimensional, and the Rankine–Hugoniot jump conditions no longer are are applicable. In the present paper, an equation describing two-dimensional structure is developed for weak shock waves and the properties of this equation are discussed.

Sternberg[1] first studied such two-dimensional shocks, which he calls non Rankine–Hugoniot or non R–H shock waves, in his investigation of Mach reflection. In the case of weak incident waves (see Fig. 1) the perfect fluid theory in which shock waves are R–H discontinuities is in serious disagreement with experiment. Careful study showed that the inviscid theory leads to infinite shock curvature at the triple point, which is inconsistent with the existence of Rankine–Hugoniot discontinuities. Sternberg then suggested that viscous effects alter the boundary conditions at the triple intersection, and postulated that there must be a finite non Rankine–Hugoniot region though which, as shown in Fig. 1, there is a transition from the structure of the waves above and below the triple point. Across this region the usual Rankine–Hugoniot conditions do not hold so that the triple point boundary conditions must be modified. Though this non R–H region itself is relatively small, it may influence a much larger portion of the flow because of the changed triple point conditions, and because of the interaction between the non R–H region and the downstream inviscid flow.

There are indications that a region of non R–H flow must be introduced to resolve some difficulties encountered by Emmons[2,3] in his perfect fluid analysis of the transonic flow near the throat of a de Laval nozzle, and over a transonic airfoil. Emmons' application of perfect fluid theory in which shocks are R–H discontinuities leads to infinite curvature where the shock terminating a supersonic region touches the curved wall of either the airfoil or nozzle. Upstream of the shock pressure increases as one moves away from the wall because of wall curvature. On the other hand, if the upstream Mach number is close to unity and if R–H conditions hold the downstream pressure actually will decrease as one moves away from the wall. The streamline

[1] J. Sternberg, Phys. Fluids 2, 179 (1959).

[2] H. W. Emmons, NACA TN 1003 (1946).

[3] H. W. Emmons, NACA TN 1746 (1948).

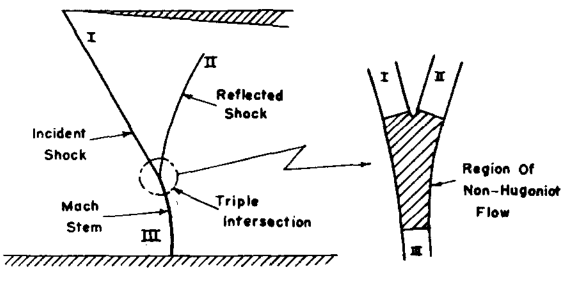


FIG. 1. Non-Hugoniot region at the triple intersection in the Mach reflection of weak shock waves.

curvature thus is singular in that it changes discontinuously at the foot of the shock wave. Emmons' analysis takes care of this difficulty by a rapid expansion behind the shock wave; however, Emmons suggests that a proper mathematical description of this behavior requires the inclusion of viscous effects in the conservation equations. Thus from Emmons' results it appears plausible that there must be a region of non R–H flow at the base of the terminating shock through which there must be a nonsingular transition from supersonic to subsonic flow.

Emmons' analysis assumes an ideal fluid without a boundary layer. In a real fluid the flow at the base of the shock wave terminating a region of supersonic flow is dominated by the shock wave boundary layer interaction rather than by the wall curvature. Even so, it is interesting to observe that just outside the boundary layer and at the downstream edge of such a supersonic region Ackeret, Feldman, and Rott[4] measured a sharp pressure rise, as might be expected in the case of a R–H shock wave, followed immediately by a rapid expansion. The pressure gradient in the expansion is of the same order as in the shock and the over-all pressure ratio across the combined compression and expansion is less than the R–H value. Thus it appears that there may be a non R–H region where the terminating shock touches the boundary layer. The failure of the terminating shock to satisfy the R–H conditions has also been noted by Sinnott,[5] who found that this fact must be taken into account to properly predict the position of the terminating shock waves on transonic airfoils.

It is also necessary to introduce a region of non R–H flow to explain the flow at the leading edge of the boundary layer generated by a shock wave moving past a stationary flat plate. For weak, and hence thick, shocks, Sichel[6] has shown that the boundary layer approximation can be extended to the foot of the shock. The leading edge region thus can be divided into a boundary layer or shear region near the wall and a free stream or shock region. A schematic diagram of this leading edge flow from the point of view of a coordinate system fixed to the shock wave is shown in Fig. 2. The shear layer and the shock region flows can be properly matched only if there is a region of non R–H flow at the outer edge of the boundary layer. In an approximate analysis of this shock wave–boundary layer interaction problem in which the shock region was replaced by an oblique shock wave, the shock is followed by a rapid compression with a velocity gradient of the same order as within the shock itself. If this compression is considered part of the shock the usual R–H conditions are not satisfied.

The mathematical formulation of non R–H flows such as discussed above is explored in the present paper. In Sec. II, a series expansion of the Navier–Stokes equation is used to develop a potential equation for non R–H flow. Properties of this equation are discussed in Sec. III, and the formulation of boundary conditions is considered in Sec. IV.

II. DERIVATION OF EQUATION FOR NON RANKINE–HUGONIOT FLOW

It is assumed that the fluid behaves as a continuum described by the Navier–Stokes equations. In the transonic regime where shock thicknesses are many mean free paths, this assumption appears reasonable. In subscript notation the continuity, momentum, and energy equations for steady flow are

$$\frac{\partial}{\partial \bar{x}_i}(\bar{\rho}\bar{u}_i) = 0, \qquad (1)$$

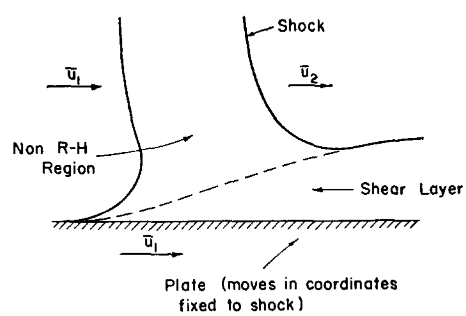


FIG. 2. Theoretical model for the leading edge of a shock-induced boundary layer, with coordinates fixed to free stream shock.

[4] J. Ackeret, F. Feldman, and N. Rott, NACA TM 1113 (1946).
[5] E. S. Sinnott, J. Aerospace Sci. 27, 767 (1960).
[6] M. Sichel, Phys. Fluids 5, 1168 (1962).

$$\bar{\rho}\bar{u}_j \frac{\partial \bar{u}_i}{\partial \bar{x}_j} = \bar{\rho}\bar{X}_i - \frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{\partial}{\partial \bar{x}_j}\left[\bar{\mu}\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right) + (\bar{\mu}'' - 2\bar{\mu})\left(\frac{\partial \bar{u}_k}{\partial \bar{x}_k}\right)\delta_{ij}\right], \quad (2)$$

$$\bar{\rho}\bar{T}\bar{u}_i \frac{\partial \bar{S}}{\partial \bar{x}_i} = \frac{\partial}{\partial \bar{x}_i}\left(\bar{k}\,\frac{\partial \bar{T}}{\partial \bar{x}_i}\right) + \tfrac{1}{2}\bar{\mu}\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right) \cdot\left(\frac{\partial \bar{u}_i}{\partial \bar{x}_j} + \frac{\partial \bar{u}_j}{\partial \bar{x}_i}\right) + (\bar{\mu}'' - 2\bar{\mu})\left(\frac{\partial \bar{u}_j}{\partial \bar{x}_j}\right)^2. \quad (3)$$

The bars here denote dimensional quantities. $\bar{\mu}$ is the conventional viscosity, and $\bar{\mu}''$, the longitudinal viscosity, is related to the bulk viscosity $\bar{\mu}'$ by[7]

$$\bar{\mu}'' = \tfrac{4}{3}\bar{\mu} + \bar{\mu}'. \quad (4)$$

The use of a bulk viscosity implies that relaxation effects are small.

The following thermodynamic equations,[8] valid for any fluid, must also be included:

$$d\bar{p} = \bar{a}^2\, d\bar{\rho} + (\bar{\alpha}\bar{T}\bar{\rho}\bar{a}^2/\bar{C}_p)\, d\bar{S}, \quad (5)$$

$$\bar{C}_p\, d\bar{T} - \bar{T}\, d\bar{S} = (\bar{\alpha}\bar{T}/\bar{\rho})\, d\bar{p}, \quad (6)$$

$$\bar{\alpha}\bar{T} = (\gamma - 1)\bar{C}_p/\bar{a}^2\bar{\alpha}, \quad (7)$$

where $\bar{\alpha}$ is the coefficient of thermal expansion. For a perfect gas, $\bar{\alpha} = (1/\bar{T})$.

A two-dimensional non R–H region is considered with uniform upstream flow in the x direction with a Mach number of M_1, and with $\bar{u}$ and $\bar{v}$ the velocity components in the $\bar{x}$ and $\bar{y}$ directions. To attack the system of equations above an expansion in a small parameter combined with appropriate stretching of the coordinates will be used. This procedure, which is equivalent to an order of magnitude analysis, has been discussed by Guderley.[9] The flow variables are made dimensionless using critical conditions as a reference except for dimensionless pressure which is defined as

$$p = \bar{p}/\rho_* a_*^2.$$

The reference values, which are indicated by an asterisk, are those which could be reached if the upstream flow expanded isentropically to a Mach number of unity.

Since the flow under consideration is transonic, a logical expansion parameter to use is

$$\epsilon = M_1^* - 1. \quad (8)$$

[7] W. D. Hayes, *Fundamentals of Gas Dynamics* (Princeton University Press, Princeton, New Jersey, 1958), Sec. D.

[8] F. W. Sears, *Thermodynamics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953), pp. 147–151.

[9] K. G. Guderley, *Theorie schallnaher Strömungen* (Springer-Verlag, Berlin, 1957).

The choice of this parameter, which also is used in transonic flow theory,[9] was in part dictated by the fact that dimensionless Hugoniot conditions across a normal shock then take the simple form

$$u_1 = \bar{u}_1/a_* = 1 + \epsilon = M_1^*,$$

$$u_2 = \bar{u}_2/a_* = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots = M_2^*, \quad (9)$$

where 1 and 2 refer to upstream and downstream conditions. Equation (9) and the fact that across the transonic non R–H region flow variables remain close to critical conditions suggest that u, T, μ, μ'', C_p, α, and k be expanded in series of the form

$$L = 1 + \epsilon L^{(1)} + \epsilon^2 L^{(2)} + \cdots + \epsilon^n L^{(n)} + \cdots, \quad (10)$$

where L symbolizes the parameter in question.

Pressure p, and vertical velocity v are the only exceptions to this expansion scheme. Because $\rho_* a_*^2$ rather than p_* is used as a reference the expansion for p takes the form

$$p = (1/\gamma) + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \cdots. \quad (11)$$

γ, the ratio of specific heats, is assumed constant. Since in transonic flow[9] and in transonic oblique shock waves the veritcal velocity v is of order $\epsilon^{\frac{3}{2}}$, the expansion for v is taken as

$$v = \epsilon^{\frac{1}{2}}(\epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots). \quad (12)$$

From the discussion of various non R–H flows in Sec. I above, it is evident that the characteristic length in the $\bar{x}$ direction will be of the order of a shock thickness. For weak shock waves the thickness will be of the order[10]

$$\bar{\lambda} \sim O(\nu_*''/a_*\epsilon), \quad (13)$$

where ν_*'' is the kinematic viscosity based on μ_*'', and where a_* is the critical speed of sound. Using (13) a dimensionless or stretched $\bar{x}$ coordinate defined by

$$x = \bar{x}/\bar{\lambda} = \bar{x}a_*\epsilon/\nu_*'' \quad (14)$$

is introduced.

There remains the problem of choosing a characteristic length or stretch factor for the $\bar{y}$ coordinate. With uniform upstream conditions the flow within and downstream of weak Hugoniot shock waves is irrotational since entropy changes are of higher order. Thus it appears reasonable to assume that at least to first order the flow in the transonic non R–H region, though viscous, is also irrotational.

[10] M. J. Lighthill, "Viscosity in Waves of Finite Amplitude," in *Surveys in Mechanics*, edited by G. K. Batchelor and R. M. Davies (Cambridge University Press, New York, 1956).

Then there must exist a potential ϕ which can be expanded as

$$\phi = x + \epsilon\phi^{(1)}(x, \hat{y}), \tag{15}$$

where initially the shock thickness, $\bar{\lambda}$, has also been used to stretch the $\bar{y}$ coordinate so that

$$\hat{y} = \bar{y}/\bar{\lambda} = \bar{y}a_*\epsilon/\nu''_*. \tag{16}$$

In terms of this potential the velocities u and v now become

$$u = \phi_x = 1 + \epsilon\phi_x^{(1)},$$
$$v = \phi_{\hat{y}} = \epsilon\phi_{\hat{y}}^{(1)}, \tag{17}$$

where the subscript denotes partial differentiation. To satisfy the requirement $v \sim O(\epsilon^{3/2})$ it is necessary that $\phi_{\hat{y}}^{(1)} \sim O(\epsilon^{1/2})$. The expansion for ϕ is thus inconsistent for both $\phi_x^{(1)}$ and $\phi_{\hat{y}}^{(1)}$ should be $O(1)$. The difficulty is resolved by introducing a new coordinate y defined by

$$y = \epsilon^{1/2}\hat{y}$$

so that

$$\epsilon\phi_{\hat{y}}^{(1)} = \epsilon^{3/2}\phi_y^{(1)}.$$

From this argument, which was originally used by Cole and Messiter[11] to derive the inviscid transonic equation, it is concluded that the proper stretched $\bar{y}$ coordinate to use is

$$y = \bar{y}a_*\epsilon^{3/2}/\nu''_*. \tag{18}$$

The assumption of irrotational flow, which is a key step in the present development, cannot be rigorously justified *a priori*. As Rae[12] points out, this assumption implies that only the compressive viscous stress due to gradients in volume dilation is important in the first-order flow; the viscous stress due to the shearing of the fluid being negligible. In the structure of shock waves, it is the compressive stress which plays the dominant role, and this provides the physical basis for assuming that the same is true in non R–H shock waves, i.e., that the flow is irrotational. Obviously, the approximations made here never will apply to flow in a boundary layer where shear stresses are dominant. Finally, it should be mentioned that in the case of the linearized Navier–Stokes equations, which are obtained by an expansion in a small parameter without coordinate stretching, it can be rigorously demonstrated that the flow can be split into an irrotational and a rotational component,[13] corresponding to longitudinal and transverse waves, and of course the non R–H region is closely related to these longitudinal waves.

Introducing the above series expansions and stretched coordinates into the Navier–Stokes equations (1)–(3) and the thermodynamic relations (5) and (6), eliminating the entropy, and equating coefficients of the lowest power of ϵ, the following equations are obtained for the first-order coefficients:

continuity: $$\frac{\partial\rho^{(1)}}{\partial x} + \frac{\partial u^{(1)}}{\partial x} = 0; \tag{19a}$$

x momentum: $$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial p^{(1)}}{\partial x} = 0; \tag{19b}$$

y momentum: $$\frac{\partial v^{(1)}}{\partial x} + \frac{\partial p^{(1)}}{\partial y} = 0; \tag{19c}$$

energy: $$\alpha_* T_* \frac{\partial T^{(1)}}{\partial x} - (\gamma - 1)\frac{\partial p^{(1)}}{\partial x} = 0; \tag{19d}$$

thermodynamic:

$$\gamma\frac{\partial p^{(1)}}{\partial x} = \frac{\partial\rho^{(1)}}{\partial x} + \alpha_* T_* \frac{\partial T^{(1)}}{\partial x}. \tag{19e}$$

Viscous and conductive terms are absent from Eq. (19) indicating that dissipative effects are of higher order. The system (19) is redundant for with (19a) and (19b) the thermodynamic equation (19e) can be reduced to

$$\alpha_* T_* \frac{\partial T^{(1)}}{\partial x} = (\gamma - 1)\frac{\partial p^{(1)}}{\partial x},$$

which is identical with the energy equation. Consequently (19) does not yield a solution for the first-order coefficients; however, since the upstream flow is uniform integration of (19a, b, and d) does yield the relations

$$\rho^{(1)} + u^{(1)} = 0,$$
$$u^{(1)} + p^{(1)} = 0, \tag{20}$$
$$\alpha_* T_* T^{(1)} - (\gamma - 1)p^{(1)} = 0.$$

It may be shown that the relations (20) are identical to those which hold in a leftward propagating acoustic wave. To first order the free stream flow is thus like a sound wave, and the redundancy of Eq. (19) is in accord with this for the form of an undamped acoustic wave is arbitrary, and is determined by its source.

[11] J. D. Cole and A. F. Messiter, Guggenheim Aeronautical Laboratory, California Institute of Technology Rept. TN 56-1 (1956).

[12] W. J. Rae, Cornell University, Rept. TN 60-409 (1960).

[13] P. A. Lagerstrom, J. D. Cole, and L. Trilling, "Problems in the Theory of Viscous Compressible Fluids," Guggenheim Aeronautical Laboratory, California Institute of Technology (1949).

From Eqs. (19c) and (20) it follows that

$$\frac{\partial v^{(1)}}{\partial x} = \frac{\partial u^{(1)}}{\partial y} \tag{21}$$

so that the first-order flow is irrotational. Equation (21) is consistent with the assumption used above to determine the $\bar{y}$ coordinate stretch factor.

To obtain the additional relation needed to determine the first-order coefficients, it is necessary to proceed to the second-order equations. Hence, by equating the coefficients of the next higher power of ϵ and using Eqs. (20), second-order continuity, x momentum, y momentum, energy, and thermodynamic equations are obtained as follows:

$$\frac{\partial \rho^{(2)}}{\partial x} + \frac{\partial u^{(2)}}{\partial x} - 2u^{(1)}\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \tag{22a}$$

$$\frac{\partial u^{(2)}}{\partial x} = -\frac{\partial p^{(2)}}{\partial x} + \frac{\partial^2 u^{(1)}}{\partial x^2}, \tag{22b}$$

$$\frac{\partial v^{(2)}}{\partial x} + \frac{\partial p^{(2)}}{\partial y} = \frac{\partial^2 v^{(1)}}{\partial x^2}, \tag{22c}$$

$$\frac{(\gamma-1)}{\alpha_* T_*}\frac{\partial T^{(2)}}{\partial x} - \frac{\partial p^{(2)}}{\partial x} - C_p^{(1)}\frac{\partial u^{(1)}}{\partial x} = \left[\alpha^{(1)} + \left(1 - \frac{\gamma-1}{\alpha_* T_*}\right)u^{(1)}\right]\frac{\partial u^{(1)}}{\partial x} - \frac{1}{\text{Pr}''}\frac{\partial^2 u^{(1)}}{\partial x^2}, \tag{22d}$$

$$\gamma\left(\frac{\partial p^{(2)}}{\partial x} - u^{(1)}\frac{\partial u^{(1)}}{\partial x}\right) - (\gamma-1) \cdot\left[2\alpha^{(1)} + 2a^{(1)} - \frac{(\gamma-1)}{\alpha_* T_*}u^{(1)} - C_p^{(1)}\right]\frac{\partial u^{(1)}}{\partial x} = \frac{\partial \rho^{(2)}}{\partial x} - (u^{(1)} + 2a^{(1)})\frac{\partial u^{(1)}}{\partial x} + \alpha_* T_*\frac{\partial T^{(2)}}{\partial x} - (\gamma-1)(\alpha^{(1)} + 2a^{(1)})\frac{\partial u^{(1)}}{\partial x}. \tag{22e}$$

In Eq. (22d), Pr″ is the longitudinal Prandtl number $C_{v*}\mu''_*/k_*$, and is assumed to remain constant.

Equations (22) are redundant in the quantities $u^{(2)}$, $\rho^{(2)}$, $p^{(2)}$, and $T^{(2)}$; therefore, when these equations are combined, second-order quantities drop out and it is found that the following nonlinear equation relating $u^{(1)}$ and $v^{(1)}$ must be satisfied:

$$2(u^{(1)} - a^{(1)})\frac{\partial u^{(1)}}{\partial x} - \left(1 + \frac{\gamma-1}{\text{Pr}''}\right)\frac{\partial^2 u^{(1)}}{\partial x^2} - \frac{\partial v^{(1)}}{\partial y} = 0. \tag{23}$$

For the general gas Hayes[7] has shown that to the present order of approximation

$$(\bar{u} - \bar{a})/a_* = u^{(1)} - a^{(1)} = \Gamma(\bar{u} - a_*)/a_* = \Gamma u^{(1)}$$

where Γ is the thermodynamic quantity,

$$\Gamma = (1/a)[\partial(\rho a)/\partial \rho]_S,$$

which may be evaluated at reference conditions. For a perfect gas $\Gamma = \frac{1}{2}(\gamma + 1)$. Equation (23) thus reduces to

$$\left(1 + \frac{\gamma-1}{\text{Pr}''}\right)\frac{\partial^2 u^{(1)}}{\partial x^2} - 2\Gamma u^{(1)}\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} = 0, \tag{24}$$

which in combination with the irrotationality equation (21) and suitable boundary conditions is sufficient to determine $u^{(1)}(x, y)$ and $v^{(1)}(x, y)$. Equation (24) has been derived without assuming a perfect gas or constant specific heat, and is therefore quite general.

A key feature of the expansion scheme used to develop equation (24) is that the coordinate stretching factors defined by (14) and (18) vary with the expansion parameter ϵ. Because of this the nonlinear convection term is retained in the final equation (24) in contrast to the more usual linearizations in which all nonlinear terms are of higher order.

Upon introducing a potential defined by

$$u^{(1)} = \phi_x; \qquad v^{(1)} = \phi_y, \tag{25}$$

Eqs. (21) and (24) can be reduced to the single potential equation

$$[1 + (\gamma - 1)/\text{Pr}'']\phi_{xxx} - 2\Gamma\phi_x\phi_{xx} + \phi_{yy} = 0. \tag{26}$$

If the initial assumptions that $v \sim O(\epsilon^{\frac{3}{2}})$ and that the first-order flow is irrotational in the non R–H shock region are valid, then Eq. (26) is a mathematical description of the flow in such a non R–H region. Equation (26) was first derived by Cole[14] in order to describe the flow about an airfoil in the transonic range. Cole's derivation was based on the concept of a fluid which has only compression viscosity so that it can still slip over the airfoil surface as in inviscid flow.

The need to go to the second-order equations (22) in order to derive Eq. (24) is an interesting feature of the above derivation, and seems to indicate that the non R–H flow behaves as an acoustic wave driven by higher-order dissipative and convective effects. Sternberg's macroscopic analysis[1] of the non R–H region in the Mach reflection problem (Fig. 1) supports this interpretation. Sternberg concluded that viscous stresses and conduction at the surfaces of the non R–H region are negligible. The jump conditions across the non R–H region depend only on the structure of the bounding Mach

[14] J. D. Cole, "Problems in Transonic Flow," Ph.D. thesis, California Institute of Technology (1949).

stem and incident and reflected waves. Thus in Sternberg's analysis the non R–H flow, though inviscid, depends upon the balance of the higher-order dissipative and convective effects that determines the structure of the intersecting waves. Application of an expansion similar to that above to the structure of a normal shock wave yielded results that were in good agreement with exact analytical solutions.[15]

The potential equation (26) may be reduced to the universal form

$$\tilde{\phi}_{XXX} - \tilde{\phi}_X \tilde{\phi}_{XX} + \tilde{\phi}_{YY} = 0 \tag{27}$$

which is independent of the fluid properties, with the transformation

$$X = Ax; \qquad Y = Ay\Gamma^{\frac{1}{2}}; \qquad \tilde{\phi} = 2A\phi,$$

where

$$A = \Gamma/[1 + (\gamma - 1)(\mathrm{Pr}'')^{-1}].$$

III. PROPERTIES OF THE EQUATION FOR NON R-H SHOCK WAVES

Equation (24) or (26) represents a balance between the combined effects of viscous stress, heat conduction, convection, and mass conservation. The term $\partial^2 u^{(1)}/\partial x^2$ is due to heat conduction and viscous stress. The quantity $(\gamma - 1)/\mathrm{Pr}''$ in the coefficient of $\partial^2 u^{(1)}/\partial x^2$ represents heat conduction while the term unity is from the viscous stress. The proportionate effects of heat conduction and viscosity are the same as in an attenuating acoustic wave.[7] $u^{(1)}(\partial u^{(1)}/\partial x)$ is the convective term which, in one-dimensional flow, is responsible for the steepening of compression waves. $(\partial v^{(1)}/\partial y)$ is a mass conservation term.

In those parts of the flow where $v^{(1)}$ vanishes, Eq. (24) reduces to

$$\left(1 + \frac{\gamma - 1}{\mathrm{Pr}''}\right)\frac{\partial^2 u^{(1)}}{\partial x^2} - 2\Gamma u^{(1)}\frac{\partial u^{(1)}}{\partial x} = 0, \tag{28}$$

which is the steady form of Burgers' equation.[7] The solution

$$u^{(1)} = -\tanh Ax \tag{29}$$

of Eq. (28) satisfies the first-order Hugoniot conditions

$$u^{(1)}(-\infty) = +1, \qquad u^{(1)}(+\infty) = -1$$

across a weak normal shock wave and is identical with Taylor's weak shock solution. Burgers' equation expresses the balance between the wave attenuating effect of viscosity and conduction and the wave steepening effect of convection, and it is this balance which is responsible for the existence of a steady state shock structure.[10] From the above it appears conceivable that Eq. (24) could describe the transition from a normal shock to a region of non R–H flow.

Without the dissipative term $(\partial^2 u^{(1)}/\partial x^2)$, Eq. (24) reduces to

$$\frac{\partial v^{(1)}}{\partial y} - 2\Gamma u^{(1)}\frac{\partial u^{(1)}}{\partial x} = 0, \tag{30}$$

which is a form of the inviscid transonic equation.[9] Equation (24) thus represents a combination of the processes which take place in a weak, one-dimensional shock wave and in transonic flow suggesting that an appropriate name for (24) or (26) might be "viscous–transonic" or V–T equation.

The entropy variation through the non R–H region may be determined from Eq. (3), the entropy form of the energy equation. If a dimensionless entropy is defined by

$$S = \bar{S}/C_{p*},$$

and if the series expansions and stretched coordinates are substituted in Eq. (3), there results the equation

$$\mathrm{Pr}''\left(\epsilon\frac{\partial S}{\partial x} + \epsilon^3 v^{(1)}\frac{\partial S}{\partial y}\right) = \epsilon^3\frac{\partial^2 T^{(1)}}{\partial x^2}. \tag{31}$$

Only the largest term has been retained on the right side of (31). In order that $\epsilon^3(\partial^2 T^{(1)}/\partial x^2)$ be balanced by a term on the left side of (31), it is necessary that $S \sim O(\epsilon^2)$ so that the expansion for S takes the form

$$S = \epsilon^2 S^{(2)} + \epsilon^3 S^{(3)} + \cdots. \tag{32}$$

Now substituting (32), the series expansions, and stretched coordinates in Eq. (3) and equating coefficients of the lowest power of ϵ the equation

$$\frac{\partial S^{(2)}}{\partial x} = \frac{1}{\mathrm{Pr}''}\frac{\partial^2 T^{(1)}}{\partial x^2} \tag{33}$$

relating $S^{(2)}$ and $T^{(1)}$ is obtained. Equation (33) shows that $S^{(2)}$ is an entropy transport term for if flow upstream of the non R–H region is uniform, Eq. (33) may be integrated to yield

$$S^{(2)} = \frac{1}{\mathrm{Pr}''}\frac{\partial T^{(1)}}{\partial x} + \text{const.} \tag{34}$$

If $(\partial T^{(1)}/\partial x) = 0$, as $x \to \pm\infty$, the net change in $S^{(2)}$ across the non R–H region is zero.

If it is assumed that $\bar{k} \sim \bar{T}$ so that $k^{(1)} = T^{(1)}$, and the coefficients of the next higher power of ϵ are equated, the equation

[15] M. Sichel, J. Aerospace Sci. 27, 635 (1960).

$$\frac{\partial S^{(3)}}{\partial x} = \frac{1}{\mathrm{Pr}''}\left(\frac{\partial^2 T^{(2)}}{\partial x^2} + \frac{\partial^2 T^{(1)}}{\partial y^2}\right) + (\gamma - 1)\left[1 + \frac{\gamma - 1}{\mathrm{Pr}''}\right]\left(\frac{\partial u^{(1)}}{\partial x}\right)^2 \qquad (35)$$

follows. The second-order terms on the right side represent entropy transport in the x and y directions; however, the last term of (35) is a dissipative term which is always positive so that the net change of $S^{(3)}$ across the non R–H region must also be positive. Assuming that the order of magnitude analysis above is applicable, it would appear that as in weak shock waves entropy changes across the non R–H region are third order in ϵ, while entropy variations within the non R–H region will be of second order due to entropy transport.

Within a transonic oblique shock, the flow is irrotational and $v \sim O(\epsilon^{\frac{3}{2}})$; therefore, it is to be expected that the structure of an oblique shock wave is a solution of the V–T equation. Since an oblique shock may be derived from a normal shock by superimposing a tangential velocity, it is readily shown[6] that the expressions

$$u^{(1)}(x, y) = \frac{\alpha^2}{\gamma + 1} - \left(1 - \frac{\alpha^2}{\gamma + 1}\right) \cdot \tanh A\left(1 - \frac{\alpha^2}{\gamma + 1}\right)(x + \alpha y),$$
$$v^{(1)}(x, y) = -\alpha\left(1 - \frac{\alpha^2}{\gamma + 1}\right) \cdot \left[1 + \tanh A\left(1 - \frac{\alpha^2}{\gamma + 1}\right)(x + \alpha y)\right], \qquad (36)$$

represent the first-order structure of weak oblique shock waves. Since the angle σ between the shock and the vertical is of $O(\epsilon^{\frac{1}{2}})$ in transonic flow, an obliquity parameter α, defined by $\sigma = \alpha\epsilon^{\frac{1}{2}}$ has been introduced in the equations above. By direct substitution it can be shown that $u^{(1)}$ and $v^{(1)}$ given above satisfy the V–T equation (24) and the irrotationality condition (21). Equation (36) is essentially a similarity solution of the V–T equation with similarity variable $(x + \alpha y)$.

A characteristic feature of non R–H shocks, as discussed in Sec. I, is that the usual Hugoniot conditions do not hold. This can be demonstrated qualitatively by a partial integration of the V–T equation (24). Since $u^{(1)}(-\infty, y) = +1$ integration of (24) from $-\infty$ to x yields the result:

$$(u^{(1)})^2 = \frac{[1 + (\gamma - 1)(\mathrm{Pr}'')^{-1}]}{\Gamma}\frac{\partial u^{(1)}}{\partial x} + 1 + \frac{1}{\Gamma}\int_{-\infty}^{x} \frac{\partial v^{(1)}}{\partial y}\, dx. \qquad (37)$$

Assuming that the flow downstream of the non R–H region above, where $\partial u^{(1)}/\partial x = 0$, is subsonic, it follows from (37) that

$$u^{(1)}(+\infty, y) = -\left[1 + \frac{1}{\Gamma}\int_{-\infty}^{\infty} \frac{\partial v^{(1)}}{\partial y}\, dx\right]^{\frac{1}{2}}. \qquad (38)$$

In a normal shock $v^{(1)} = 0$ and then (38) yields the usual result that $u^{(1)}(+\infty) = -1$. In an oblique shock $\partial v^{(1)}/\partial y < 0$ always, hence from (38) it follows that $u^{(1)}(+\infty, y) > -1$, as is of course actually the case. In the case of the normal shock terminating a supersonic region it is to be expected that $v^{(1)} > 0$ within the shock wave due to presence of the boundary layer. Since this disturbance dies out with increasing y $\partial v^{(1)}/\partial y < 0$ and (38) then indicates that $u^{(1)}(\infty, y) > -1$. This result is in accord with Emmons'[2,3] results and with experiment.[4,5] At the leading edge of the shock induced boundary layer $v^{(1)} < 0$ and $\partial v^{(1)}/\partial y > 0$. This result suggests that $u^{(1)}(+\infty, y) < -1$ and agrees with the approximate solution of the leading edge problem[6] which also indicates that $u^{(1)}(+\infty, y)$ overshoots the downstream Hugoniot value.

IV. BOUNDARY CONDITIONS

A uniqueness proof for the V–T equation which is applicable to all possible boundary conditions, is not available. Consequently, the two specific examples of the non R–H region at the leading edge of the shock induced boundary layer, and of the boundary value problem for a finite rectangular domain in the x–y plane are discussed below, and should indicate the nature of the viscous–transonic boundary value problem.

At first it is revealing to discuss the boundary conditions needed for the, perhaps trivial, cases of weak, normal and oblique shock waves. In both cases the upstream boundary condition

$$u^{(1)}(-\infty, y) = +1 \qquad (39)$$

is singular in that it is applied at $x = -\infty$. Since the normal shock is governed by the second-order Burgers' equation (28), an additional boundary condition is needed. The condition $u_x^{(1)}(-\infty) = 0$ is redundant since it is already implied by (39). Bounded solutions of Burgers' equation are such that with the specification of condition (39) the functional form of $u^{(1)}(x)$ is completely determined.

The second boundary condition therefore merely locates the Taylor shock with respect to the coordinate system and might take the form

$$u^{(1)}(0) = 0. \tag{40}$$

A similar situation exists in the case of oblique shocks, except that it is also necessary to specify the downstream vertical velocity $v^{(1)}(\infty, y)$, or $\phi_y^{(1)}(\infty, y)$. When the downstream inviscid flow is subsonic, the oblique shock, and downstream flow will interact. There is no freedom regarding the variation of $v^{(1)}(x, y)$ within the shock; $v^{(1)}(x, y)$ is set by the nature of the oblique shock similarity solution of the V–T equation.

In the case of the shock induced leading edge problem,[6] discussed in the Introduction and shown in Fig. 2, the upstream flow is uniform so that as for the normal and oblique shocks

$$\phi_x(-\infty, y) = +1, \tag{41}$$

where now the potential equation (26) will be considered. As $y \rightarrow \infty$ any upstream disturbance caused by the leading edge interaction must die out so that

$$\phi_y(-\infty, \infty) = 0. \tag{42}$$

From Eqs. (24), (41), and (42), it then follows that

$$\phi_y(-\infty, y) = 0. \tag{43}$$

The boundary condition at the base of the non R–H region depends on the flow at the outer edge of the shear layer as is evident in Figure 2. The shear layer generates a vertical velocity[6] which at its outer edge is of order $\epsilon^{\frac{3}{2}}$, i.e., $v^{(1)}(\bar{x}, \bar{\delta}) \sim O(\epsilon^{\frac{3}{2}})$, where $\bar{\delta}$, the shear layer thickness is $O[\nu_* \nu_*''/a_* \epsilon^{\frac{1}{2}}]$. In the free stream, assuming $\partial v^{(1)}/\partial y \sim O(1)$,

$$\frac{\partial \bar{v}}{\partial \bar{y}} \sim O\left(\frac{a_*^2 \epsilon^3}{\nu_*''}\right)$$

therefore, the change in $v^{(1)}$ over a distance of the same order as the shear layer thickness $\bar{\delta}$, will be

$$\Delta v^{(1)} \sim O[\epsilon(\nu_*/\nu_*'')^{\frac{1}{2}}]$$

which is of higher order. The mean surface approximation is therefore applicable and the shear layer boundary condition can be applied at $y = 0$. It has been shown[6] that $\phi_y(x, 0)$, the velocity at the outer edge of the shear layer, is given by

$$\begin{aligned} v^{(1)}(x, 0) &= \phi_y(x, 0) \\ &= (\nu_*/\nu_*''\pi)^{\frac{1}{2}}[1 + (\gamma - 1)\ \mathrm{Pr}^{-\frac{1}{2}}] \\ &\quad \cdot \int_{-\infty}^{x} (x - \xi)^{-\frac{1}{2}} \phi_{xx}(\xi, 0)\, d\xi, \end{aligned} \tag{44}$$

and this is the boundary condition at the base of the non R–H region. From (44) it is evident that the free stream and shear layer flows interact for $v^{(1)}(x, 0)$ depends upon $u_x^{(1)}$ or ϕ_{xx} in the free stream and the boundary condition (44) expresses this interaction mathematically.

In the case of the transonic and Laplace equations, only one boundary condition can be specified at $y = 0$ if the solution is to remain bounded as $y \rightarrow \infty$. From the close relation between the V–T and transonic equations, it appears reasonable that one boundary condition at $y = 0$ will also be sufficient in the present case, i.e., the V–T equation cannot simultaneously satisfy a boundary condition for $v^{(1)}(x, 0)$ and the no-slip condition $u^{(1)}(x, 0) = 0$, and of course a no-slip condition implies shear stresses which are not permitted by the V–T equation. The no-slip condition at the plate in the present problem is satisfied by interposing a shear layer between the non R–H region and the plate.

There remains the boundary condition downstream of the interaction region. The order of magnitude considerations upon which the derivation of the V–T equation is based, are truly valid only in the interaction region at the base of the shock wave. Downstream of the non R–H region the flow will be subsonic and inviscid, and the non R–H and inviscid flows must be matched at some point x_m. Thus, the downstream boundary condition of the V–T equation will be

$$\phi_y(x_m, y) = f(y), \tag{45}$$

where $f(y)$ is the variation of $v^{(1)}$ as given by the downstream inviscid solution.

Changes in the coordinate x_m should not have major effects upon the solution of the interaction problem. The matching point x_m must be chosen sufficiently downstream that $u_x^{(1)}$ and $u_{xx}^{(1)}$ are small compared to values within the non R–H region so that the location of x_m will not affect the solution. As in the case of conventional shocks, x_m should be such that flow is uniform with respect to a distance of the order of a shock thickness. It is not possible to let $x_m \rightarrow \infty$ for even though the shear layer solution, $v^{(1)}(x, 0)$, has been shown to be valid[6] as $x \rightarrow \infty$, it cannot be *a priori* shown that the same is true for solutions of the V–T equation. The non R–H and inviscid flows will interact for $f(y)$ in (45) will depend upon $\phi_x(x_m, y)$ from the solution of the V–T equation.

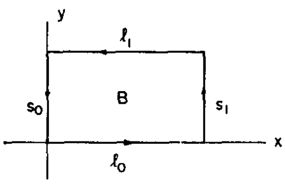
Finally, in analogy to the normal and oblique shocks, the free stream shock wave and non R–H regions must be located with respect to the coordinate system with the condition

$$\phi_x(0, \infty) = 0. \tag{46}$$

With (46) the specification of boundary conditions is complete. The mathematical difficulty of the boundary value problem posed by the leading edge makes it unlikely that an exact solution can be found, rather iterative or approximate methods would have to be employed to attack the complicated boundary value problem above.

As a second example, the boundary value problem for a finite rectangular domain B such as shown in Fig. 3 has been chosen, since an exact uniqueness

FIG. 3. The finite rectangular domain B with boundaries S_0, S_1, l_0, and l_1.



theorem can be developed for this case. Following a procedure described by Courant and Hilbert,[16] it is shown in the Appendix that if ϕ_x is specified on the boundary S_0 and ϕ is given on the boundaries S_0, l_0, S_1, and l_1, of the domain B then the solution of

$$\phi_{xxx} - \phi_x\phi_{xx} + \phi_{yy} = 0 \tag{47}$$

in B is unique provided that $\phi_x < 0$ in B. The universal form of the V–T equation has been used above.

The boundary conditions above also imply a knowledge of $u^{(1)}$ or ϕ_x on S_0, l_0, and l_1 and a knowledge of $v^{(1)}$ or ϕ_y on S_0 and S_1. The condition $\phi_x < 0$, which restricts the validity of the proof to subsonic flows, results from the nonlinearity and does not mean that there are no unique solutions for $\phi_x > 0$. The above boundary value problem does not have any particular physical significance and of course is different from the shock induced leading edge problem. It nevertheless is interesting to note that as in the leading edge case, only one boundary condition can be specified on the boundaries parallel to the free stream which in the present case are l_0 and l_1.

The viscous–transonic equation is parabolic for it has the triple characteristic $y =$ const. Hence, it is interesting that ϕ must be specified over the entire boundary of B in contrast to the parabolic equation for unsteady heat conduction:

$$z_{xx} - z_y = 0,$$

for which z can only be specified on an open boundary. The higher order of the V–T equation as compared to the heat equation is responsible for this result. The parabolic nature of the V–T equation is reflected in the fact that ϕ_x in addition to ϕ must be given on S_0.

V. CONCLUSIONS

By assuming irrotationality and that $v \sim O(\epsilon^{3/2})$ an equation describing the structure of non R–H shock waves, and called the viscous–transonic or V–T equation has been derived. From this derivation, it appears that the non R–H region behaves as a sound wave driven by higher-order viscous effects. A partial integration of the V–T equation gives a qualitative indication of the manner in which conditions across non R–H regions deviate from the usual Hugoniot jump conditions.

The type of boundary conditions required by this equation have been discussed by considering two specific examples. Even though the equation is parabolic, the fact that it is of third order requires that boundary conditions be specified on a closed contour.

The V–T equation appears to be a combination of Burgers' equation and the transonic equation. Unfortunately, the hodograph and the Hopf–Cole[17] transformations which greatly simplify the treatment of the Burgers' and transonic equations do not work here. Thus, the solution of the V–T equation, particularly for a boundary value problem such as the shock induced leading edge, poses an extremely difficult mathematical problem. It has, however, been demonstrated that the structure of an oblique shock wave represents an exact similarity solution of the V–T equation, and other similarity solutions have been investigated.[18]

It would be of interest to find other solutions of the V–T equation. A question of particular interest in whether the V–T equation can provide an explanation for the development of the shock wave which terminates a region of supersonic flow.

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APPENDIX

The uniqueness theorem stated in Sec. IV is proved below. Let f and g be two functions which satisfy Eq. (27), and for which f_x and g_x are given on S_0 and f and g are given on the entire boundary of the domain B (see Fig. 3). Then if $(g - f) = \Delta$, it follows from (27) that

$$\Delta_{xxx} + \Delta_{yy} - (g_x g_{xx} - f_x f_{xx}) = 0. \qquad (A.1)$$

Multiplying (A.1) by Δ and integrating over the domain B it is possible to obtain the result

$$\iint_B \left\{ \left[\Delta\Delta_{xx} - \tfrac{1}{2}\Delta_x^2 - \frac{\Delta\Delta_x}{2}(f_x + g_x) \right]_x + (\Delta\Delta_y)_y \right\} dx\, dy + \iint_B \left[-\Delta_y^2 + \frac{\Delta_x^2}{2}(g_x + f_x) \right] dx\, dy = 0. \qquad (A.2)$$

Applying the divergence theorem to (A.2) and taking line integrals on the boundary of B in the direction indicated in Fig. 3, there results

$$\int_{S_0+S_1} [\Delta\Delta_{xx} - \tfrac{1}{2}\Delta_x^2 - \tfrac{1}{2}\Delta\Delta_x(f_x + g_x)]\, dy - \int_{l_0+l_1} \Delta\Delta_y\, dx + \iint_B \left[-\Delta_y^2 + \frac{\Delta_x^2}{2}(g_x + f_x) \right] dx\, dy = 0. \qquad (A.3)$$

From the boundary conditions satisfied by f and g it follows that $\Delta_x = 0$ on S_0 and $\Delta = 0$ on S_0, S_1, l_0 and l_1. Consequently, (A.3) reduces to

$$-\frac{1}{2}\int_{S_1} \Delta_x^2\, dy + \iint_B \left[-\Delta_y^2 + \frac{\Delta_x^2}{2}(g_x + f_x) \right] dx\, dy = 0. \qquad (A.4)$$

If $f_x < 0$, $g_x < 0$, Eq. (A.4) can be satisfied only if $\Delta_x = \Delta_y = 0$ in B from which it follows that $\Delta = 0$ in B. The theorem is now proved.