An Analysis of the Maser Oscillator Equations*

D. M. SINNETT
The University of Michigan, Institute of Science and Technology, Ann Arbor, Michigan
(Received October 23, 1961)

This paper analyses the maser oscillator equations which describe the interaction between the resonant cavity and the inverted population of the electron spin system of the paramagnetic substance. It is shown that these equations will not allow periodic solutions thus refuting the theory, based on computer solutions, that this interaction is responsible for the pulsed mode of operation of the oscillator.

Characteristics of solutions of these equations are determined analytically, and the ambiguity of computer solutions is discussed with the aid of examples. Numerical solutions are presented which show that periodic solutions may be induced by supplementing the spin system equation with an additional term.

INTRODUCTION

THE treatment of the differential equations describing the operation of the three-level paramagnetic maser oscillator by analog or digital techniques, encounters difficulties when questions of periodicity and stability arise.[1,2] A direct analytic investigation of these problems is called for. This paper presents a proof of the nonexistence of periodic solutions and limit cycles, and discusses the form of soultions with the aid of numerical results.

EQUATIONS

According to Makhov,[3] the maser oscillator equations are of the form

$$dx/dt=-c_1xy+c_2(x_0-x)+c_5f(x) \quad (1)$$

$$dy/dt=c_3xy-c_4y \quad c_i>0, \quad (2)$$

where x is the population difference in the spin system, and y is the magnetic energy in the cavity. The remaining quantities are defined in references 1 and 3. Since $dy/dt<0$ for $x<x_1$, it is necessary to supplement the system with the constraint equation

$$0<y_0\lesssim y, \quad (3)$$

where y_0 can be taken as noise in the system.

DISCUSSION OF THE EQUATIONS

In the following three sections the coefficient c_5 in Eq. (1) is assumed to be zero. The system has two singular points (x_1,y_1) and (x_2,y_2) where

$$x_1=c_4/c_3 \quad (4)$$

$$y_1=c_2(x_0-x_1)/c_1x_1 \quad (5)$$

$$x_2=x_0$$

$$y_2=0.$$

The area of interest for solutions is in the region

$$0<x<x_0$$

$$0<y.$$

Note that if a solution starts at $x=0$, $t=0$, $y=y_0$, no oscillations are possible until $x=x_1$, $t=t_1$, $y=y_0$. All further discussion assumes, as initial conditions

$$x=x_1$$

$$t=t_1$$

$$y=y_0.$$

Analog computer solutions have led to the supposition[1] that periodic solutions may exist about the point (x_1,y_1). It will be shown that no such solutions exist, but characteristics of the equations will be pointed out which show why computer solutions may lead to this conclusion.

NONEXISTENCE OF PERIODIC SOLUTIONS AND LIMIT CYCLES

On letting

$$0<x_1<x_0$$

$$0<y_0<y_1<y_3,$$

substituting $z=x-x_1$ in the system, and eliminating t by division,

$$\frac{dy}{dx}=\frac{dy}{dz}=\frac{c_3yz}{-c_1yz-c_2z+[c_2(x_0-x_1)-c_1x_1y]}=f(z,y). \quad (6a)$$

From (5) $c_1x_1y_1=c_2(x_0-x_1)$, therefore,

$$f(z,y_1)=-k \quad k>0. \quad (6b)$$

Let $y=y_0+\int_0^z f[z,y(z)]dz=F(z)$ describe a traejctory followed by a solution in the (z,y) plane. If a closed path exists, a trajectory must start at some point $(0,y_0)$, where $0<y_0<y_1$; and as t increases in a positive sense the trajectory must follow some path about $(0,y_1)$ and return to $(0,y_0)$. By expanding $F(z)$ in a Maclaurin series we find that

$$F(-z)<F(z) \quad 0<z \quad (6c)$$

in a small neighborhood of $z=0$; and by substituting y_3

* This work was conducted by Project MICHIGAN under Department of the Army Contract (DA-36-039-78801) administered by the U. S. Army Signal Corps.

[1] H. Statz and G. DeMars, *Quantum Electronics* (Columbia University Press, New York, 1960), p. 530.

[2] G. Makhov, "On the theory of the three-level paramagnetic maser oscillator," Conference on Electron Tube Research, Mexico City, 1959 (unpublished).

[3] G. Makhov, J. Appl. Phys. (to be published).

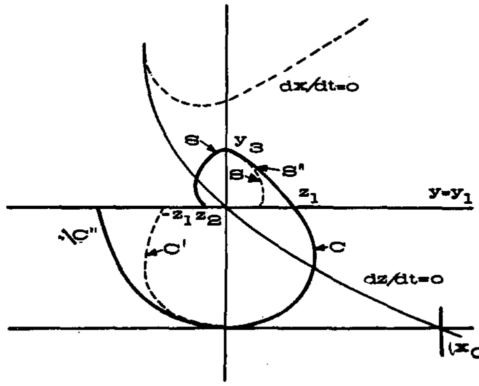

FIG. 1. Diagram to clarify arguments eliminating periodic solutions.

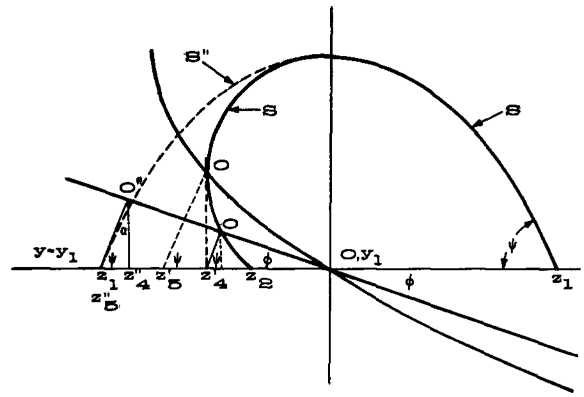

FIG. 2. Diagram showing construction of the lower bound for damping.

for y_0 we find that this is true for any section of a trajectory which crosses the y axis.

The level curve $dz/dt=0$, shown in Fig. 1, is a branch of a hyperbola with asymptotes $z=-x_1$ and $y=-c_2/c_1$. At points on a trajectory which lie above the hyperbola $dz/dt<0$, and at points below the hyperbola $dz/dt>0$. Also, the derivative changes sign in the regions $z>0$, $y<y_1$, and $z<0$, $y>y_1$. Choosing points (z_i,y_i), $(-z_i,y_i)$: $z_i>0$, $y_0<y_i<y_1$, we find that

$$|f(-z_i,y_i)|<|f(z_i,y_i)|, \tag{6d}$$

and therefore the magnitude of the slope of a trajectory at any such point (z_i,y_i) must be greater than the magnitude of the slope of the trajectory at the point's reflection with respect to the y axis.

In Fig. 1, C is an arbitrary trajectory which is a solution for the equations, C' is the reflection of C with respect to the y axis, and C'' is the extension of C in the region $y<y_1$, $z<0$. We know that the slope of C' changes sign, while the slope of C'' does not. For a closed path we know [Eq. (6c)] that C'' must be below C' near the y axis. We know that C'' cannot cross C' from the inside to the outside as t increases, since the magnitude of the slope of C'' would then be greater than that of C', in contradiction of (6d). Therefore, since C'' must be outside C' to close the path, and since it cannot cross C' from the inside to the outside, we conclude that if a closed path exists, C'' must always be outside C'. Now if (z_1,y_1) is on C, and (z_2,y_1) is on C'', it is obvious that

$$|z_2|>|z_1| \quad z_2<0<z_1, \tag{6e}$$

if a closed path exists.

We now let S'' (Fig. 1) be the extension of C in the region $z>0$, $y>y_1$, intersecting the y axis at $(0,y_3)$. S is the trajectory determined by the point $(0,y_3)$ in the region $y>y_1$, $z<0$, and S' is the reflection of S with respect to the y axis. Using analogous arguments we find that if S is part of the same trajectory as C, and if (z_2,y_1) is a point on S, and (z_1,y_1) is a point on C,

$$|z_2|<|z_1| \quad \begin{cases} z_1>0 \\ z_2<0 \\ y=y_1. \end{cases}$$

From the second argument we conclude that the distance from the trajectory to the y axis on the line $y=y_1$ becomes successively smaller with each consecutive half-cycle. The first argument states that, for finite z, the distance must initially become larger to admit a closed path. Since the trajectory must cross the line $y=y_1$ with a constant angle of inclination $\neq\pi/2$ (with respect to the z axis), a closed path consisting of the y axis is eliminated. The level curve $dy/dt=0$ is the y axis; all trajectories must cross this line horizontally, and therefore a closed path cannot consist of a line at an angle other than $\pi/2$.

Finally, as $|z|\to0$, $y=y_1$, then $|y_1-y|\to0$, $z=0$, and therefore periodic solutions are not possible.

To eliminate limit cycles and to understand the change in the damping which results from a change in the position of the singular point (x_1,y_1), it is necessary to find a lower bound for the quantity $(|z_1|-|z_2|)$. In Fig. 2, let S be a trajectory in the region $y>y_1$, $z>0$, such that (z_1,y_1) is a point on S. S'' is the reflection of S with respect to the y axis, and S' is the extension of S in the region $y>y_1$, $z<0$. O is the point at which S' intersects the hyperbola $dz/dt=0$. The tangent to S' is parallel to the y axis at this point; and extended to the line $y=y_1$, it intersects at (z_4',y_1).

Let $\theta=\tan^{-1}(dy/dz)$ on S. It can be shown that θ is bounded on S, $\pi/2<\tan^{-1}(-k)\leq\theta\leq\pi$. Let $\psi=\pi-\tan^{-1}(-k)$. Clearly $|z_4'|>|z_2|$. From point O construct a line which interesects the line $y=y_1$ with angle of inclination ψ, and call the point of intersection (z_5',y_1). Now $|z_5'|<|z_1|$, since θ is bounded on S'' and S'' must also intersect the line $y=y_1$ with angle of inclination ψ. We now have a lower bound on the damping. Letting $|z_5'|-|z_4'|=\Delta z$,

$$|z_1|-|z_2|>\Delta z.$$

We now find a simpler and more useful bound. Construct the straight line defined by the points $(0,y_1)$ and $([x_0-x_1],0)$ and intersecting the line $y=y_1$ with angle ϕ. This line, as shown in Fig. 2, is always below the hyperbola in the region $y>y_1$, $z<0$, and always above the hyperbola in the region $y<y_1$, $z>0$. Preserving

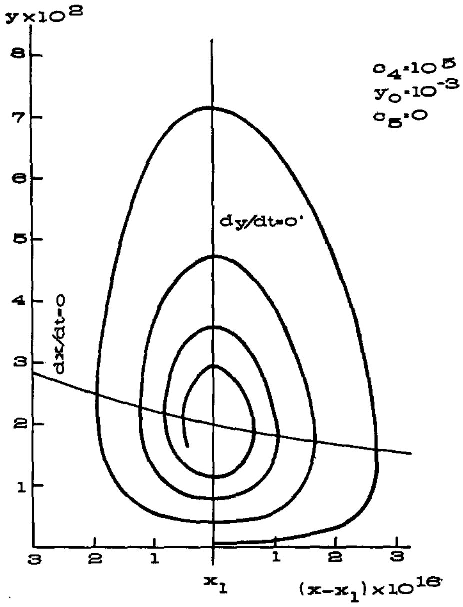

FIG. 3. Phase plane plot of a solution with high damping.

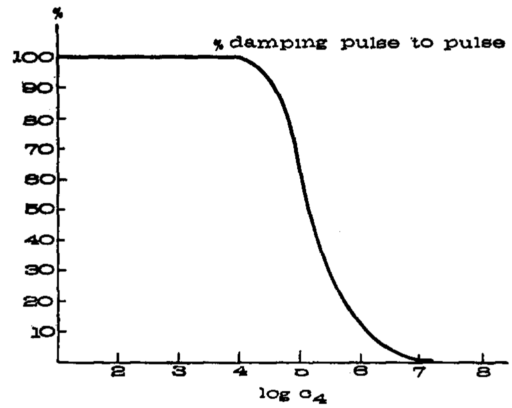

FIG. 5. Percent damping versus log c_4. The change in amplitude of successive pulses in the (y, t) plane.

angles, we now let the point O follow the trajectory until it reaches O', at which point the trajectory intersects the straight line. Let O' follow the straight line in the direction of decreasing z, still preserving angles, until the point (z_5'', y_1) coincides with the point $(-z_1, y_1)$.

Now $|z_5''| = |z_1|$ and $|z_4''| > |z_2|$ by reasoning used previously, and $\Delta z'$ is still a lower bound, and the magnitude can be determined easily:

$$\Delta z' = |z_1| - |z_4''| = K z_1$$

$$K = \sin\phi \cos\psi / \sin(\pi - \psi - \phi) = 1/[1 + (\tan\psi/\tan\phi)] < 1.$$

If z_3, z_4, . . . are successive intersections of the trajectory with the line $y = y_1$, repetition of this argument shows that $z_n = z_1(1-k)^{n-1}$, and the trajectory damps in an exponential manner.

COMMENTS

It is apparent that although y_0 will determine the amplitude and frequency of the oscillation, it will have no effect on the type of oscillation (i. e., damped or undamped).

An examination of the expression $f(z,y)$ indicates that the damping is a result of the term $[c_2(x_0 - x_1) - c_1 x_1 y]$. Now as $x_1 \rightarrow x_0$, we see from Eq. (5), that $y_1 \rightarrow 0$, and the trajectories become more symmetric. When $x_1 = x_0$ the trajectories represent degenerate limit cycles at the point $(x_0, 0)$. The expression for $\Delta z'$ indicates that this does happen. Note that as $x_1 \rightarrow x_0$ the singular point (x_1, y_1) moves to a region in which the level curve $dx/dt = 0$ is more nearly horizontal. As $x_1 \rightarrow x_0$ the angle $\phi \rightarrow \pi$, $\psi \rightarrow \pi/2$, and therefore $K \rightarrow 0$. Assuming that the damping decreases as the lower bound approaches zero, it would be expected that the trajectories would closely approximate periodic solutions for x_1 close to x_0.

This is verified by numerical solutions. Figures 3 and 4 are graphs of solutions which exhibit this property.

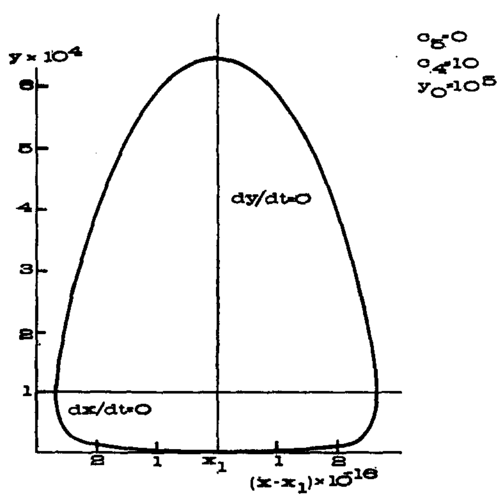

FIG. 4. Phase plane plot of a solution with low damping.

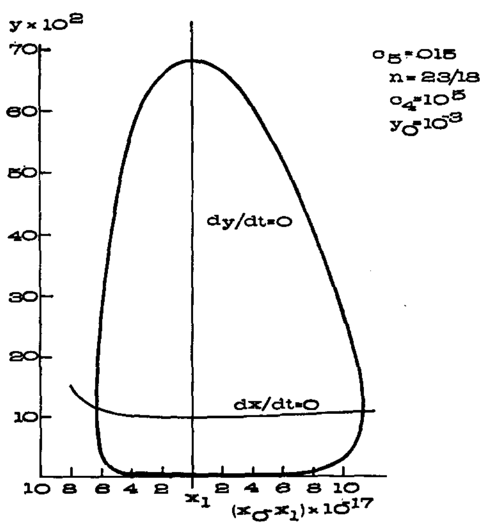

FIG. 6. Phase plane plot of a solution with damping corrected.

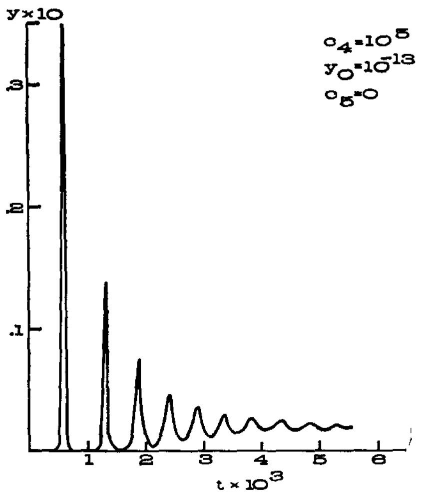

FIG. 7. Shows a damped solution in the (y, t) plane.

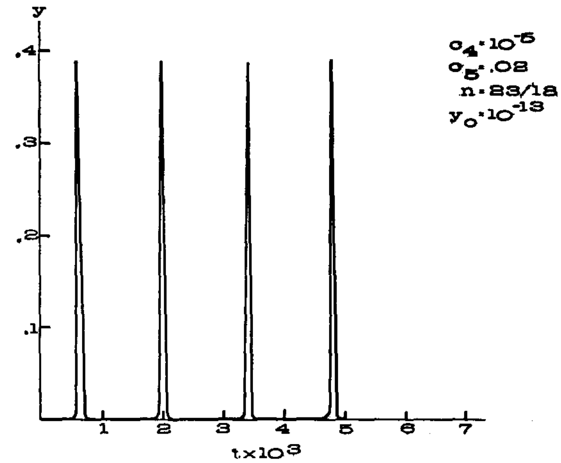

FIG. 8. A solution in the (y, t) plane with damping corrected.

Since c_4 is the constant most easily varied in the oscillator (being inversely proportional to the cavity Q), numerical results have been obtained varying only c_4, c_5, and $f(x)$. The values of the remianing constants are: $c_1=10^5$, $c_2=10$, $c_3=10^{-12}$, $x_0=2\times10^{19}$.

Figure 5 shows the percentage of damping, pulse-to-pulse, which results from varying c_4 with $c_5=0$. Note that the damping for realistic values of c_4 is of the order of 1%. Due to lack of resolution, analog or digital equipment may not detect this amount of damping, and consequently computer solutions may very easily lead to erroneous conclusions.

EFFECT OF THE TERM $c_5f(x)$

It has previously been shown,[3] that introducing the term $c_5f(x)$, where $f(x)=x^n$, will make a limit cycle possible. To illustrate this effect numerically, a value of c_4 was selected which would produce considerable damping (50%) and c_5 and n were adjusted to eliminate the damping. Figure 6 is a graph of the solution in the phase plane with damping corrected. Figures 7 and 8 show, in the (y,t) plane, the damped solution, and the solution with $c_5\, f(x)$ just sufficient to eliminate the damping. For these graphs a realistic value of y_0 was used, so that comparison might be made with the experimental results.[4]

The introduction of $c_5f(x)$ has other interesting effects on the solutions. The dashed line in Fig. 1 shows the effect of the term on the curve $dx/dt=0$. It has been observed numerically that a solution can be initially damped and spiral in to a limit cycle. It is also possible to produce negative damping, and without Eq. (3) the solution spirals out, and the amplitude and frequency become infinite. With Eq. (3) in the system, c_5 and n can be varied to produce identical pulses of arbitrary amplitude or frequency.

METHOD OF NUMERICAL SOLUTION

All computation was performed on the IBM 709. Solutions were obtained using a variable step method which utilizes the predictor-corrector equations of Adams and Moulton.[5] The numerical accuracy was checked by parallel solution of many cases using the Runge–Kutta method, which is much slower, but is stable in cases where predictor-corrector methods may not be.[6]

ACKNOWLEDGMENTS

The author wishes to thank G. Makhov, under whose guidance the analysis was performed, J. Ullman, J. Riordan, and C. Schensted for invaluable suggestions, and J. Mack for reproducing the figures.

[4] G. Makhov, R. Terhune, J. Lambe, and L. Cross, J. Appl. Phys. 31, 936 (1960).

[5] F. J. Murray, "The numerical solution of systems of differential equations on a sequence calculator," Summer Conference, University of Michigan College of Engineering, 1959 (unpublished).

[6] M. Urabe, MRC Technical Summary Report No. 183, University of Wisconsin, October, 1960.