An Analysis of the Maser Oscillator Equations*

D. M. Sinnett

The University of Michigan, Institute of Science and Technology, Ann Arbor, Michigan
(Received October 23, 1961)

This paper analyses the maser oscillator equations which describe the interaction between the resonant cavity and the inverted population of the electron spin system of the paramagnetic substance. It is shown that these equations will not allow periodic solutions thus refuting the theory, based on computer solutions, that this interaction is responsible for the pulsed mode of operation of the oscillator.

Characteristics of solutions of these equations are determined analytically, and the ambiguity of computer solutions is discussed with the aid of examples. Numerical solutions are presented which show that periodic solutions may be induced by supplementing the spin system equation with an additional term.

INTRODUCTION

THE treatment of the differential equations describing the operation of the three-level paramagnetic maser oscillator by analog or digital techniques, encounters difficulties when questions of periodicity and stability arise. A direct analytic investigation of these problems is called for. This paper presents a proof of the nonexistence of periodic solutions and limit cycles, and discusses the form of solutions with the aid of numerical results.

EQUATIONS

According to Makhov, the maser oscillator equations are of the form

\[ \frac{dx}{dt} = -c_1 xy + c_2 (x_0 - x) + c_3 f(x) \]  \hspace{1cm} (1)

\[ \frac{dy}{dt} = c_4 xy - c_5 y \quad c_4 > 0, \]  \hspace{1cm} (2)

where \( x \) is the population difference in the spin system, and \( y \) is the magnetic energy in the cavity. The remaining quantities are defined in references 1 and 3. Since \( dy/dt < 0 \) for \( x < x_1 \), it is necessary to supplement the system with the constraint equation

\[ 0 < y_0 < y, \]  \hspace{1cm} (3)

where \( y_0 \) can be taken as noise in the system.

DISCUSSION OF THE EQUATIONS

In the following three sections the coefficient \( c_1 \) in Eq. (1) is assumed to be zero. The system has two singular points \((x_1, y_1)\) and \((x_2, y_2)\) where

\[ x_1 = \frac{c_4}{c_3} \]  \hspace{1cm} (4)

\[ y_1 = \frac{c_2 (x_0 - x_1)}{c_1 x_1} \]  \hspace{1cm} (5)

\[ x_2 = x_0 \]

\[ y_2 = 0. \]

The area of interest for solutions is in the region

\[ 0 < x < x_0 \]

\[ 0 < y. \]

Note that if a solution starts at \( x = 0, t = 0, y = y_0 \), no oscillations are possible until \( x = x_1, t = t_1, y = y_0 \). All further discussion assumes, as initial conditions

\[ x = x_1 \]

\[ t = t_1 \]

\[ y = y_0. \]

Analog computer solutions have led to the supposition that periodic solutions may exist about the point \((x_1, y_1)\). It will be shown that no such solutions exist, but characteristics of the equations will be pointed out which show why computer solutions may lead to this conclusion.

NONEXISTENCE OF PERIODIC SOLUTIONS AND LIMIT CYCLES

On letting

\[ 0 < x_1 < x_0 \]

\[ 0 < y_0 < y_1 < y_2, \]

substituting \( z = x - x_1 \) in the system, and eliminating \( t \) by division,

\[ \frac{dy}{dz} = \frac{c_3 y_0}{dx} \]

\[ = \frac{c_3 y_0}{-c_1 y_0 - c_2 + \left[ c_2 (x_0 - x_1) - c_1 x_1 y \right]} = f(z, y). \]  \hspace{1cm} (6a)

From (5) \( c_1 x_1 y_1 = c_2 (x_0 - x_1) \), therefore,

\[ f(z, y_1) = -k \quad k > 0. \]  \hspace{1cm} (6b)

Let \( y = y_0 + \int f(z, y)dz = F(z) \) describe a trajectory followed by a solution in the \((z, y)\) plane. If a closed path exists, a trajectory must start at some point \((0, y_0)\), where \( 0 < y_0 < y_1 \), and as \( t \) increases in a positive sense the trajectory must follow some path about \((0, y_1)\) and return to \((0, y_0)\). By expanding \( F(z) \) in a Maclaurin series we find that

\[ F(-z) < F(z) \quad 0 < z \]  \hspace{1cm} (6c)

in a small neighborhood of \( z = 0 \); and by substituting \( y_2 \),

\[ 0 < y_2 < y_1, \]

\[ \frac{dy}{dz} < 0, \]

\[ k > 0. \]

Let \( y = y_0 + \int f(z, y)dz = F(z) \) describe a trajectory followed by a solution in the \((z, y)\) plane. If a closed path exists, a trajectory must start at some point \((0, y_0)\), where \( 0 < y_0 < y_1 \), and as \( t \) increases in a positive sense the trajectory must follow some path about \((0, y_1)\) and return to \((0, y_0)\). By expanding \( F(z) \) in a Maclaurin series we find that

\[ F(-z) < F(z) \quad 0 < z \]  \hspace{1cm} (6c)

in a small neighborhood of \( z = 0 \); and by substituting \( y_2 \),
for $y_0$ we find that this is true for any section of a trajectory which crosses the $y$ axis.

The level curve $dz/dt=0$, shown in Fig. 1, is a branch of a hyperbola with asymptotes $z=-x_1$ and $y=-x_0/c_1$. At points on a trajectory which lie above the hyperbola $dz/dt<0$, and at points below the hyperbola $dz/dt>0$. Also, the derivative changes sign in the regions $z>0$, $y<y_0$, and $z<0$, $y>y_1$. Choosing points $(z_0,y_0)$, $(-z_0,y_0)$: $z_1>0$, $y_0<y_1<y_1$, we find that

$$|f(-z_0,y_0)| < |f(z_0,y_0)|,$$

and therefore the magnitude of the slope of a trajectory at any such point $(z_0,y_0)$ must be greater than the magnitude of the slope of the trajectory at the point's reflection with respect to the $y$ axis.

In Fig. 1, $C$ is an arbitrary trajectory which is a solution for the equations, $C'$ is the reflection of $C$ with respect to the $y$ axis, and $C''$ is the extension of $C$ in the region $y<y_0$, $z<0$. We know that the slope of $C'$ changes sign, while the slope of $C''$ does not. For a closed path we know [Eq. (6c)] that $C''$ must be below $C'$ near the $y$ axis. We know that $C''$ cannot cross $C'$ from the inside to the outside as $t$ increases, since the magnitude of the slope of $C''$ would then be greater than that of $C'$, in contradiction of (6d). Therefore, since $C''$ must be outside $C'$ to close the path, and since it cannot cross $C'$ from the inside to the outside, we conclude that if a closed path exists, $C''$ must always be outside $C'$. Now if $(z_1,y_1)$ is on $C$, and $(z_2,y_2)$ on $C''$, it is obvious that

$$|z_2| > |z_1|, z_2 < 0 < z_1,$$

if a closed path exists.

We now let $S''$ (Fig. 1) be the extension of $C$ in the region $z>0$, $y>y_1$, intersecting the $y$ axis at $(0,y_1)$. A is the trajectory determined by the point $(0,y_0)$ in the region $y>y_1$, $z<0$, and $S'$ is the reflection of $S$ with respect to the $y$ axis. Using analogous arguments we find that if $S$ is part of the same trajectory as $C$, and if $(z_2,y_2)$ is a point on $S$, and $(z_1,y_1)$ is a point on $C$,

$$|z_2| < |z_1| \begin{cases} z_1 > 0 \\ z_2 < 0 \\ y = y_1. \end{cases}$$

From the second argument we conclude that the distance from the trajectory to the $y$ axis on the line $y=y_1$ becomes successively smaller with each consecutive half-cycle. The first argument states that, for finite $z$, the distance must initially become larger to admit a closed path. Since the trajectory must cross the line $y=y_1$ with a constant angle of inclination $\neq \pi/2$ (with respect to the $z$ axis), a closed path consisting of the $y$ axis is eliminated. The level curve $dy/dt=0$ is the $y$ axis; all trajectories must cross this line horizontally, and therefore a closed path cannot consist of a line at an angle other than $\pi/2$.

Finally, as $|z|\to 0$, $y=y_1$, then $|y_1-y|\to 0$, $y=0$, and therefore periodic solutions are not possible.

To eliminate limit cycles and to understand the change in the damping which results from a change in the position of the singular point $(x_1,y_1)$, it is necessary to find a lower bound for the quantity $(|z_1|-|z_2|)$. In Fig. 2, let $S$ be a trajectory in the region $y>y_1$, $z>0$, such that $(z_1,y_1)$ is a point on $S$. $S''$ is the reflection of $S$ with respect to the $y$ axis, and $S'$ is the extension of $S$ in the region $y>y_1$, $z<0$. $O$ is the point at which $S'$ intersects the hyperbola $dz/dt=0$. The tangent to $S'$ is parallel to the $y$ axis at this point; and extended to the line $y=y_1$, it intersects at $(z_2',y_1)$.

Let $\theta = \tan^{-1}(dy/dz)$ on $S$. It can be shown that $\theta$ is bounded on $S$, $\pi/2 < \tan^{-1}(-k) \leq \theta \leq \pi$. Let $\psi = \pi - \tan^{-1}(-k)$. Clearly $|z_2'| > |z_1|$. From point $O$ construct a line which intersects the line $y=y_1$ with angle of inclination $\psi$, and call the point of intersection $(z_2'',y_1)$. Now $|z_2'| > |z_1|$, since $\theta$ is bounded on $S''$ and $S''$ must also intersect the line $y=y_1$ with angle of inclination $\psi$. We now have a lower bound on the damping. Letting

$$|z_2'| - |z_2'| = \Delta x,$$

$$|z_1| - |z_2| \geq \Delta x.$$

We now find a simpler and more useful bound. Construct the straight line defined by the points $(0,y_1)$ and $(x_0-x_1,0)$ and intersecting the line $y=y_1$ with angle $\phi$. This line, as shown in Fig. 2, is always below the hyperbola in the region $y>y_1$, $z<0$, and always above the hyperbola in the region $y<y_1$, $z>0$. Preserving
angles, we now let the point $O$ follow the trajectory until it reaches $O'$, at which point the trajectory intersects the straight line. Let $O'$ follow the straight line in the direction of decreasing $z$, still preserving angles, until the point $(z_0',y_0')$ coincides with the point $(-z_0,y_0)$.

Now $|z_0'| = |z_1|$ and $|z_2| > |z_2|$ by reasoning used previously, and $\Delta z'$ is still a lower bound, and the magnitude can be determined easily:

$$\Delta z' = |z_1| - |z_2'| = K_{z_1}$$

$$K = \sin \phi \cos \psi / \sin(\pi - \psi - \phi) = 1 / [1 + (\tan \psi / \tan \phi)] < 1.$$ 

If $z_1, z_2, \ldots$ are successive intersections of the trajectory with the line $y = y_1$, repetition of this argument shows that $z_n = z_1 (1 - k)^{n-1}$, and the trajectory damps in an exponential manner.

**Fig. 3.** Phase plane plot of a solution with high damping.

**Fig. 4.** Phase plane plot of a solution with low damping.

**Fig. 5.** Percent damping versus log $c_i$. The change in amplitude of successive pulses in the $(y, t)$ plane.

**Fig. 6.** Phase plane plot of a solution with damping corrected.

**COMMENTS**

It is apparent that although $y_0$ will determine the amplitude and frequency of the oscillation, it will have no effect on the type of oscillation (i.e., damped or undamped).

An examination of the expression $f(x, y)$ indicates that the damping is a result of the term $[c_2(x_0 - x) - c_1 x_1 y]$. Now as $x_1 \rightarrow x_0$, we see from Eq. (5), that $y_1 \rightarrow 0$, and the trajectories become more symmetric. When $x_1 = x_0$ the trajectories represent degenerate limit cycles at the point $(x_0, 0)$. The expression for $\Delta z'$ indicates that this does happen. Note that as $x_1 \rightarrow x_0$ the singular point $(x_1, y_1)$ moves to a region in which the level curve $dx/dt = 0$ is more nearly horizontal. As $x_1 \rightarrow x_0$ the angle $\phi - \pi, \psi - \pi/2$, and therefore $K \rightarrow 0$. Assuming that the damping decreases as the lower bound approaches zero, it would be expected that the trajectories would closely approximate periodic solutions for $x_1$ close to $x_0$.

This is verified by numerical solutions. Figures 3 and 4 are graphs of solutions which exhibit this property.
Since $c_4$ is the constant most easily varied in the oscillator (being inversely proportional to the cavity $Q$), numerical results have been obtained varying only $c_4$, $c_6$, and $f(x)$. The values of the remaining constants are: $c_1 = 10^4$, $c_2 = 10$, $c_3 = 10^{-12}$, $x_0 = 2 \times 10^{10}$.

Figure 5 shows the percentage of damping, pulse-to-pulse, which results from varying $c_4$ with $c_6 = 0$. Note that the damping for realistic values of $c_4$ is of the order of 1%. Due to lack of resolution, analog or digital equipment may not detect this amount of damping, and consequently computer solutions may very easily lead to erroneous conclusions.

**EFFECT OF THE TERM $c_6 f(x)$**

It has previously been shown,\(^4\) that introducing the term $c_6 f(x)$, where $f(x) = x^n$, will make a limit cycle possible. To illustrate this effect numerically, a value of $c_4$ was selected which would produce considerable damping (50%) and $c_6$ and $n$ were adjusted to eliminate the damping. Figure 6 is a graph of the solution in the phase plane with damping corrected. Figures 7 and 8 show, in the $(y, \ell)$ plane, the damped solution, and the solution with $c_6 f(x)$ just sufficient to eliminate the damping. For these graphs a realistic value of $y_0$ was used, so that comparison might be made with the experimental results.\(^4\)

The introduction of $c_6 f(x)$ has other interesting effects on the solutions. The dashed line in Fig. 1 shows the effect of the term on the curve $dx/dt = 0$. It has been observed numerically that a solution can be initially damped and spiral in to a limit cycle. It is also possible to produce negative damping, and without Eq. (3) the solution spirals out, and the amplitude and frequency become infinite. With Eq. (3) in the system, $c_6$ and $n$ can be varied to produce identical pulses of arbitrary amplitude or frequency.

**METHOD OF NUMERICAL SOLUTION**

All computation was performed on the IBM 709. Solutions were obtained using a variable step method which utilizes the predictor-corrector equations of Adams and Moulton.\(^5\) The numerical accuracy was checked by parallel solution of many cases using the Runge–Kutta method, which is much slower, but is stable in cases where predictor-corrector methods may not be.\(^6\)

**ACKNOWLEDGMENTS**

The author wishes to thank G. Makhov, under whose guidance the analysis was performed, J. Ullman, J. Riordan, and C. Schensted for invaluable suggestions, and J. Mack for reproducing the figures.

---

