Properties of the $(:\varphi^4:)_{1+1}$ interaction Hamiltonian

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Using a convergent expansion of the resolvent of the Hamiltonian $H = H_0 + \lambda V$, $V = \int dx \times (x) : \varphi^4 : (x)$, $g(x) \in C_0^\infty$, $g(x) \geqslant 0$, we give a simple proof of (a) the self-adjointness of the Hamiltonian and (b) the volume independent lower bound of the vacuum energy per unit volume. Also, we obtain some coupling constant analyticity properties of the Hamiltonian, and the limit $(H_0 + \lambda v - z)^{-1} \to (H_0 - z)^{-1}$, $z \in \rho(H_0)$ in norm as $|\lambda| \to 0$ uniformly in $\{\lambda : |\arg\lambda| < \pi\}$.

1. INTRODUCTION

The past few years have seen the birth of a new branch of quantum field theory whose purpose is to prove rigorously the existence of model field theories satisfying certain physical and mathematical requirements (axioms). Glimm and Jaffe[1] have pushed the two space—time dimensional φ^4-interaction [abbreviated $(:\varphi^4:)_{1+1}$] to almost a theory which is known to satisfy all the Haag—Kastler axioms and many of the Wightman axioms. A basic step in the construction of the field theory is to prove the boundedness below and the self-adjointness of the Hamiltonian operator

$$H(\lambda) = H_0 + \lambda V(g) = H_0 + \lambda \int dx\, g(x) : \varphi^4 : (x), \tag{1.1}$$

where H_0 is the free Hamiltonian, λ the coupling constant, and $g(x) \in C_0^\infty(R)$ is real value. The first proof of the semiboundedness in a finite volume with periodic boundary conditions was given in the pioneer work of Nelson.[2] His method was extended by Glimm[3] who replaced the periodic box by a fixed $g(.)$ space cut-off. Later, Glimm and Jaffe[4] obtained a lower bound of the Hamiltonian proportional to the volume (i.e., a volume independent bound of the vacuum energy per unit volume). The first proof of the self-adjointness of the Hamiltonian (1.1) was given by Glimm and Jaffe.[5] Segal[6] simplified the proof of self-adjointness and developed powerful techniques which were elaborated further and systematized by Simon and Hoegh-Krohn.[7] Recently, Federbush[8] considered a convergent expansion of the resolvent for the Hamiltonian (1.1) and obtained easily the semiboundedness of the Hamiltonian.

On the other hand, there has been another trend in the rigorous study of model field theories. The second trend involves the study of coupling constant analyticity of various objects associated with the theory. It also, examines the asymptoticity of the perturbation series of quantities such as the ground state, the ground state energy, and equal time vacuum expectation values. Often, the exact objects can be recovered from the asymptotic series by proper summability methods such as Padé[12,13] or Borel.[14,15]

In this paper we use Federbush's expansion of the resolvent to give a simple proof of the self-adjointness of the Hamiltonian, and of the volume independent lower bound of the vacuum energy per unit volume. Also, we study the Hamiltonian (1.1) for complex values of the coupling constant λ. We prove that Federbush's expansion of the resolvent is uniformly convergent for λ in $\{\lambda : |\arg\lambda| < \pi\}$. This yields full cut plane analyticity of the resolvent, and analyticity of the ground state energy and vacuum vector in a neighborhood of the positive real axis. In addition, it implies that if $z \in \rho(H_0)$, $\rho(H_0)$ the resolvent set of H_0, then

$$(H_0 + \lambda V(g) - Z^{-1} \to (H_0 - z)^{-1} \tag{1.2}$$

in norm as $|\lambda| \to 0$ uniformly in $|\arg\lambda| < \pi$. (1.2) is important in the study of the asymptotic nature and the Borel summability of the Rayleigh—Schrödinger perturbation series for the ground state and the ground state energy. A corollary of (1.2 is the uniqueness of the ground state for small values of the coupling constant (the uniqueness of the ground state for any values of the coupling constant is a more difficult problem[4,7]). In a forthcoming paper,[16] using the methods of this paper, we give a simple proof of the self-adjointness of a local Lorentz generator formally given by

$$M = M_0 + M_1(g) = \tfrac{1}{2} \int :\{\pi^2(x) + \nabla\varphi(x)^2 + \mu_0^2\varphi^2(x)\} : x dx + V(xg). \tag{1.3}$$

The organization of this paper is as follows: In Sec. II, we summarize the most important ingredients of Federbush's expansion and prove the self-adjointness of the Hamiltonian. In Sec. III, we prove the volume independent lower bound of the vacuum energy per unit volume, and in Sec. IV, we prove the uniform convergence of Federbush's expansion for values of the coupling constant in a certain complex domain.

2. A CONVERGENT EXPANSION FOR THE RESOLVENT AND SELF-ADJOINTNESS OF THE HAMILTONIAN

In this section we summarize the main points of Federbush's paper[8] and prove the self-adjointness of the Hamiltonian. Following Ref. 8, we consider

$$H_\kappa = H_0 + V_\kappa(g) = H_0 + \lambda \int dx g(x) : \varphi_\kappa^4 : (x), \tag{2.1}$$

where $g \in C_0^\infty(R)$, $0 \leqslant g(x) \leqslant 1$, H_0 is the free Hamiltonian, and φ_κ is the boson field with a momentum cut-off κ. Let P_i be the projection operator onto states with number of particles in the interval $(2^i, 2^{i+2})$, $i = -1, 0, 1, 2, \ldots$, and P_e and P_d the projection operators onto states with number of particles in the ranges

$$\bigcup_{i=\text{even}} (2^i - 4 \leqslant N \leqslant 2^i + 4), \tag{2.2a}$$

$$\bigcup_{i=\text{odd}} (2^i - 4 \leqslant N \leqslant 2^i + 4), \tag{2.2b}$$

respectively. We define

$$H_i = p_i H_\kappa P_i = P_i H_0 P_i + P_i V_\kappa P_i = H_{0,i} + V_{\kappa,i}, \tag{2.3}$$

$$H_e = \sum_{i=\text{even}} H_i = H_0 + \sum_{i=\text{even}} V_{\kappa,i}, \tag{2.4}$$

$$H_d = \sum_{i=\text{odd}} H_i = H_0 + \sum_{i=\text{odd}} V_{\kappa,i}, \quad (2.5)$$

$$W_e = V_\kappa - \sum_{i=\text{even}} V_{\kappa,i}, \quad (2.6)$$

$$W_d = V_\kappa - \sum_{i=\text{odd}} V_{\kappa,i}. \quad (2.7)$$

Then

$$H = H_e + W_e = H_d + W_d. \quad (2.8)$$

Federbush considers the expansion

$$\begin{aligned} R_\kappa(b;H_\kappa) &= R_\kappa(b;H_e) \\ &\quad - R_\kappa(b;H_d)W_eR_\kappa(b;H_e) \\ &\quad + R_\kappa(b;H_e)W_dR_\kappa(b;H_d)W_eR_\kappa(b;H_e) \\ &\quad - \ldots\ldots, \quad (2.9a) \\ &= R_\kappa(b;H_e) \\ &\quad - R_\kappa(b;H_d)P_eW_eP_eR_\kappa(b;H_e) \\ &\quad + R_\kappa(b;H_e)P_dW_dP_dR_\kappa(b;H_d)P_eW_eP_eR_\kappa(b;H_e) \\ &\quad - \ldots, \quad (2.9b) \end{aligned}$$

where b is a large positive number, and $R(Z) = (Z+A)^{-1}$ denotes the resolvent of operator A. The results of Ref. 8 can be summarized in Theorem (2.1).

Theorem 2.1: There exists a finite constant α, independent of κ, such that for $b > \alpha$, expansion (2.9) converges in the uniform operator topology and is continuous in κ for $0 \leq \kappa \leq +\infty$. $R_\infty(b)$ is the resolvent of an operator $H = H_\infty(g)$ such that $H > -\alpha$.

The basic estimates which yield theorem (2.1) are

Estimate 1:

$$H_i \geq 2^{i-1}P_i, \quad \text{for large } i; \quad (2.10)$$

Estimate 2:

$$\|P_eP_iR(b;H_i)P_d\| \leq c_1 \exp(-c_2 2^{i/2}) \quad (2.11)$$

for large i, and for some $c_1, c_2 > 0$.

To obtain Estimate 1, we choose an increasing sequence of momentum cut-offs and write

$$H_i = P_iH_0P_i + P_iV_\kappa P_i + P_i(V_\kappa - V_{\kappa_i})P_i. \quad (2.12)$$

Clearly, one has

$$P_iH_0P_i \geq m2^iP_i. \quad (2.13)$$

By undoing the Wick ordering of V_{κ_i}, we obtain the momentum cut-off dependent bound

$$V_{\kappa_i} \geq -\text{const}(\ln\kappa_i)^2. \quad (2.14)$$

By a standard N_τ estimate[1,3]

$$\|(N+I)^{-1}(V_\kappa - V_{\kappa_i})(N+I)^{-1}\| \leq 0(\kappa_i^{-1/2}). \quad (2.15)$$

Estimate 1 is obtained from (2.13–2.15) by choosing

$$\kappa_i = \exp[(M/C)^{1/2}2^{(i-1)/2}],$$

where c is the constant in (2.13).

The proof of Estimate 2 is based on the following theorem:

Theorem 2.2: Let A be a positive self-adjoint operator of norm less or equal to M, and $|\alpha\rangle$ and $|\beta\rangle$ two vectors of unit length. Suppose

$$\langle\alpha|A^k|\beta\rangle = 0, \text{ for } 0 \leq k \leq N. \quad (2.16)$$

Then, for any $\mu > 0$ a real number,

$$|\langle\alpha|R(+\mu;A)|\beta\rangle| \leq \frac{4}{\mu\sqrt{2\mu/M}}(1+\sqrt{2\mu/M})^{-N}. \quad (2.17)$$

Federbush applies Theorem 2.2 to the operator $A_i = b + P_i(N + V_\kappa(g))P_i - 2^{i-1}$ which satisfies

$$A_i \geq 0, \quad (2.18)$$

$$\|b + P_i(N + V_\kappa(g))P_i\| \leq d2^{2i}. \quad (2.19)$$

Estimate 2 is obtained by using $N \leq \text{const } H_0$, and choosing $|\alpha_i\rangle = P_iP_d|\alpha\rangle$, $|\beta\rangle = P_iP_e|b\rangle$ ($|a\rangle$ and $|b\rangle$ normalized vectors), $\mu_i = 2^{i-1}$, $M_i = d2^{2i}$, and $N_i < [(2^{i+1}-4) - (2^i+4)]/4$, and i large enough.

Our main result in this section is Theorem 2.3.

Theorem 2.3: The operator H defined in Theorem 2.1 is self-adjoint. The proof of Theorem 2.3 follows from the following two lemmas.

Lemma 2.4: Let $\epsilon > 0$ be sufficiently small. Then there exists a constant $c(\epsilon)$ such that

$$\|R(b;H_e)\|, \ \|R(b;H_d)\| \leq 1/b, \quad (2.20)$$

$$\|W_eP_dR(b;H_d)P_e\|, \ \|W_eP_eR(b;H_e)P_d\| < \tfrac{1}{2}, \quad (2.21)$$

$$\|R(b;H_d)W_eR(b;H_e)\| \leq c(\epsilon)/b^{1+\epsilon}, \quad (2.22)$$

$$\|W_dR(b;H_d)W_eR(b;H_e)\| \leq c(\epsilon)/b^{+1+\epsilon}. \quad (2.23)$$

Proof: Inequality (2.20) is easily obtained from Estimate 1. Let $|a\rangle$ and $|b\rangle$ be two normalized states in the Fock space. Then, to obtain Estimate (2.21), we consider

$$\begin{aligned} &\langle a|W_dP_dR(b;H_d)P_e + b\rangle \\ &= \sum_{i=\text{odd}} \langle a|W_dP_dR(b;H_d)P_eP_i + b\rangle \\ &= \sum_{i=\text{odd}} \langle a|W_dP_dR(b;H_i)P_e|b\rangle \\ &= \sum_{i=\text{odd}} \langle a|(I-P_i)V_\kappa P_iP_dP_iR(b;H_i)P_e|b\rangle, \\ &\leq \sum_{i=\text{odd}} c_1 2^{2i}\exp(-c_2 2^{i/2}) < \tfrac{1}{2}. \quad (2.24) \end{aligned}$$

In the last step above we have used Estimate 2 and a standard N_t estimate. Similar arguments establish inequalities (2.22) and (2.23).

Remark: Estimates (2.20) through (2.23), without the b-independence of the bounds, were also used in Ref. 8.

Lemma 2.5: For b large enough, the series (2.9) converges in the uniform operator topology to an operator $R_\kappa(b)$ which is a continuous in κ pseudoresolvent and satisfies

$$\text{norm-}\lim_{b\to+\infty} bR_\kappa(b) = I, \quad (2.25)$$

$$R_\kappa(b): \mathcal{H} \to \mathcal{H} \text{ is injective}, \quad (2.26)$$

where $\mathcal{H}$ is the Fock-Hilbert space.

Proof: Estimates (2.20)–(2.23) imply that the nth term in the expansion (2.9) is bounded by $c(\epsilon)^n b^{-1-n\epsilon}$. Therefore, the series (2.9) converges, in the uniform operator topology, for $b > (\epsilon)^{1/\epsilon}$, for $0 \leq \kappa \leq +\infty$. The continuity in κ is obtained from the uniform in κ estimatês (2.20)—(2.23). The bound $c(\epsilon)^n/b^{1+n\epsilon}$ of the nth order term yields

$$\lim_{b\to+\infty} b\{\|R(b;H_e)\| + \|R(b;H_d)W_eR(b;H_e)\| + \cdots\}$$
$$= \lim_{b\to+\infty} b\|R(b;H_e\| + \lim_{b\to+\infty} b\{\|R(b;H_d)W_eR(b;H_e)\| + \cdots\}. \quad (2.27)$$

The second term above goes to zero as $b\to+\infty$ while the first term goes to one. It is easily shown that $R_\kappa(b)$ is a pseudoresolvent, and satisfies (2.26).

Proof of Theorem 2.3 *and of Theorem* 2.1: Let $\mathcal{N}(b)$ be the null space of $R_\infty(b)$, i.e.,

$$\mathcal{N}(b) = \{\Phi\in\mathcal{H}: R_\infty(b)\Phi = 0\}. \quad (2.28)$$

Since $R_\infty(b)$ is a pseudoresolvent, $\mathcal{N}(b)$ is independent of b, and, by (2.25), $\mathcal{N}(b)=\{0\}$. We define

$$H(b) = -b + R_\infty(b)^{-1} \quad (2.29)$$

and domain

$$D(H(b)) = R_\infty(b)\mathcal{H}. \quad (2.30)$$

Let $\Phi\in\mathcal{H}$ be orthogonal to $R_\infty(b)\mathcal{H}$. Then,

$$(\Phi, R_\infty(b)\Psi) = (R_\infty(b)\Phi, \Psi) \quad (2.31)$$

for all $\Psi\in\mathcal{H}$. Since $\mathcal{N}(b)=\{0\}$, we get $\Phi=0$, and, therefore, $D(H(b))$ is dense in $\mathcal{H}$. The pseudoresolvent property of $R_\infty(b)$ implies that $D(H(b))$ is independent of b. Therefore, for large b, $H=H(b)$ is bounded below and independent of b. The self-adjointness of this operator follows from the following lemma.[17]

Lemma 2.6: If T is an operator on the Hilbert space $\mathcal{H}$, and if T^{-1} exists and has dense domain, then $(T^*)^{-1} = (T^{-1})^*$.

3. VACUUM ENERGY PER UNIT VOLUME

The Hamiltonian $H(g)=H_0+V(g)$ has a unique ground state $\Omega(g)$ with eigenvalue $E(g)$. According to the perturbation theory, $E(g)$ is proportional to the volume of space in which the particles interact in each order of the Rayleigh—Schrödinger (≡ Feynman!) perturbation series. However, the perturbation expansion for $E(g)$ diverges.[18] Thus, we cannot conclude from perturbation theory that $E(g)$ is proportional to the volume. In this section we prove rigorously that the prediction of the perturbation theory is correct. Our main tool in the proof is the localization indices introduced by Glimm and Jaffe in a similar context.[19] We consider the Hamiltonian $H(g)=N+V(g)$.

Theorem 3.*d*: Let $g(x)\in C_0^\infty(R)$ have the following properties:

(i) $0\le g(x)\le 1$; (3.1)

(ii) for some constant $\alpha>0$,

$$\left|\frac{dg(x)}{dx}\right| \le \alpha. \quad (3.2)$$

Set $m(g)=$measure (supp. g). There exist constants $a>0$ and $c>0$, independent of g, such that

$$0\le c(H(g)+am(g)). \quad (3.3)$$

Remark 1: Inequality (3.3) implies that there exists $c_1>0$ independent of g, such that

$$-c_1 m(g) \le E(g) \le 0, \quad (3.4)$$

that $E(g)\le 0$ is trivial.

Remark 2: Estimate (3.4) is the main technical step in proving the *locally Fock* property[19] of the representation (of the algebra of local observables) associated with the Hilbert space of the physical states obtained in the limit $g\to 1$.

Proof of Theorem 3.1: Let $\eta(x)\in C_0^\infty(R)$, $0\le\eta(x)\le 1$, be such that the translates

$$\eta_j(x) = \eta(x-j),\quad j\in Z \quad (3.5)$$

define a partition of unity:

$$\sum_{j\in Z}\eta_j(x) = 1 \quad \text{for all } X\in R. \quad (3.6)$$

Let $I(g)=\{j\in Z: \mathrm{supp}g\cap\mathrm{supp}\eta_j(x)\neq\}$ and $|I(g)|$ denote the number of elements in $I(g)$. We decompose g and $V(g)$ into a sum of local parts

$$g = \sum_{j\in I(g)}\eta_j g, \quad (3.7)$$

$$V(g) = \sum_{j\in I(g)} V(\eta_j g), \quad (3.8a)$$

$$= \sum_{j\in I(g)} \int dk_1\cdots dk_4 \sum_{\alpha=0}^{4}\binom{4}{\alpha}\hat{b}_i(k_1\cdots k_4)a^*_{k_1}\cdots a^*_{k_\alpha} \times a_{-k_{\alpha+1}}\cdots a_{-k_\alpha} \quad (3.8b)$$

where

$$\hat{b}_j(k_1\cdots k_4) = [1/(2(2\pi))^2]\widehat{(\eta_j g)}(k_1+\cdots+k_4)\prod_{l=1}^{4}\mu_{k_l}^{-1/2}, \quad (3.9)$$

$$\mu_{k_l}^2 = k_l^2 + \mu_0^2. \quad (3.10)$$

Instead of using g, we use a simplified space cutoff g_n defined by

$$g_n(x) = \sum_{|j|\le n}\eta_j(x),\quad n\in Z^+. \quad (3.11)$$

Instead of (3.3), we will prove

$$0\le H(g_n) + 0(n) = N + V(g_n) + 0(1). \quad (3.12)$$

We prove (3.12) for each translate separately, i.e., we prove

$$0\le N_{\mathrm{loc}} + V(\eta_j) + 0(1). \quad (3.13)$$

N_{loc} is a local number operator to be defined below, and then summing over all translates we obtain (3.12) and hence (3.3). To prove (3.13) we introduce localization indices in configuration space (localization indices in momentum space[4] could be used as well). First, we introduce local N_τ operators.

Let $\mu_k=(k^2+\mu_0^2)^{1/2}$ denote the one particle energy. Let a_k^* and a_k be creation and annihilation operators in momentum space, and $A^\#(x)=(2\pi)^{-1/2}\int dk\, e^{\pm ikx}A_k^\#$ annihilation and creation operators in configuration space. If $\mu_k^\tau\in O_m(R)$, $0\le\tau\le 1$, is considered as a multiplication operator on $S(R^1)$, then the configuration space operation μ_x^τ corresponding to μ_k^τ is convolution by a kernel $k_\tau(x)\in O_m'(R)$ (for notation see Ref. 20). $k_\tau(.)$ decreases exponentially at infinity. Explicitly,[21]

$$k_\tau(x)=\frac{2^{\tau/2+1}}{\Gamma(-\tau/2)}\left(\frac{\mu_0}{|x|}\right)^{\tau+1/2}\int_0^\infty \exp(-\mu_0|x|\cosh t)\times\cosh t\left(\frac{1+\tau}{2}\right)dt. \quad (3.14)$$

For $\tau\geq -1$

$$\left|\frac{d^n}{dx^n}k_\tau(x)\right|\leq O[\exp(-\mu_0|x|)],\quad \text{as } |x|\to+\infty,\quad n=0,1,2,\cdots. \quad (3.15)$$

For $\tau<-1$, (3.15) holds if μ_0 is replaced by $\mu_0-\epsilon$, for any $\epsilon>0$. For $\delta>0$

$$|k_\tau(x)|\leq C(\delta)|x|^{-1-\delta}\ \text{for all } x. \quad (3.16)$$

For $\xi\in O_M(R)$, nonnegative real, we define the local number operator

$$N_{\tau,\xi}=\int dxdy\,A^*(x)\xi(x)k_\tau(x-y)\xi(y)A(y), \quad (3.17a)$$

$$=\int dk_1dk_2\,a^*_{k_1}\left(\int dk\,\hat{\xi}(k_1-k)\mu_k^\tau\hat{\xi}(k-k_2)\right)a_{k_2}. \quad (3.17b)$$

Strictly speaking, $N_{\tau,\xi}$ is the Friedrichs extension of the positive operator defined by the right hand side of (3.17) on D_0, the set of vectors in Fock space with a finite number of particles and wavefunctions in ξ. If $\tau=1$, then $N_{1,\xi}$ provides a local energy operator H_0^{loc}. For $0\leq\tau<\frac{1}{4}$, one can replace ξ by the characteristic function $E_B(x)$ of an interval B, to obtain a sharply localized operator $N_{\tau,B}$. N_{loc} in (3.13) is of this type. Using techniques of Fourier analysis, Glimm and Jaffe[19] proved that

$$E_B(x)\mu_x^\tau E_B(x)\leq \text{const}\,\xi(x)\mu_x^{2\tau+\epsilon}\xi(x), \quad (3.18)$$

if $\tau<\frac{1}{2}$, and if $\xi(x)\equiv1$ on a neighborhood of B^-. Furthermore, if $\xi_j(x)=\xi(x-j)$, then

$$\sum_{j\in Z}\xi_j(x)\mu_x^\tau\xi_j(x)\leq\text{const}\,\mu_x^\tau. \quad (3.19)$$

These inequalities on the single particle space sym $L_2(R)$ lead to the estimate

$$\sum_{j\in Z}N_{\tau,B+j}\leq\text{const}N_\tau\leq\text{const}H_0 \quad (3.20)$$

of the sum of local number operators by (global) N_τ operators. Estimate (3.20) holds as an inequality between positive self-adjoint operators in Fock space. It shows that in order to prove (3.12) [or (3.3)] it is enough to prove (3.13), or equivalently

$$0\leq N_{\text{loc}}+H_I(g)+O(1), \quad (3.21)$$

where N_{loc} is a localized number operator, and g is supported in a fixed interval B.

We prove (3.21) by the method of Sec. II. Our discussion in that section shows that we only need to prove (2.15), i.e.,

$$\|(N_{\text{loc}}+I)^{-1}(V_\kappa-V_{\kappa i})(N_{\text{loc}}+I)^{-1}\|\leq O(\kappa_i^{-1/2}). \quad (3.22)$$

Of course, one has to check also (2.25), but this is not hard. To prove (3.22), we note that $V'_{\kappa_i}=V_\kappa-V_{\kappa_i}$ is a sum of five monomials in creation and annihilation operators

$$V'_{\kappa_i}(g)=\int dk_1\cdots dk_4\hat{b}'_{\kappa_i}(k_i\cdots k_4)\sum_{\alpha=0}^4\binom{4}{\alpha}a^*_{k_1}\cdots a^*_{k_\alpha}\times a_{-k_{\alpha+1}}\cdots a_{-k_4}, \quad (3.23)$$

$$=\int dx_1\cdots dx_4 b'_{\kappa_i}(x_i\cdots x_4)\sum_{\alpha=0}^4\binom{4}{\alpha}A^*(x_1)\cdots A^*(x_\alpha)A(-x_{\alpha+1})\cdots A(-x_4), \quad (3.24)$$

where

$$\hat{b}'_{\kappa_i}(k_1,\ldots,k_4)=\hat{b}_\kappa(k_1\cdots k_4)-\hat{b}_{\kappa_i}(k_1\cdots k_4) \quad (3.25)$$

and $b'_{\kappa_i}(x_1,\ldots,x_4)$ is the Fourier transform of $\hat{b}'_{\kappa_i}(k_1,\ldots,k_4)$.

Without loss of generality, we consider the case where B is an open interval whose closure is contained in (0,1). Let

$$X_j=(j,j+1) \quad (3.26)$$

and N_{X_j} the corresponding local number operators, $j=0,\pm1,\pm2,\ldots$, which are commuting self-adjoint operators. We define N_{loc} by

$$N_{\text{loc}}=\sum_{j=-\infty}^{+\infty}N_{X_j}\exp[-\mu_0(|j|/2)] \quad (3.27)$$

and localize the operator $V'_{\kappa_i}(g)$ as follows. Define

$$b'_{\kappa_j}j_1\cdots j_4(x_1,\ldots,x_4)=b'_{\kappa_i}(x_1\cdots x_4)\prod_{l=1}^4E_{j_l}(x_l), \quad (3.28)$$

where $E_{j_l}(x)$ is the characteristic function of X_j. Equation (3.28) localizes each coordinate X_l in the interval X_j. Using (3.28) we will prove that $b'_{\kappa_i}(x_1,\ldots,x_4)$ is exponentially small at infinity in each of the variables X_l. We prove

Lemma 3.1: $b'_{\kappa_i;j_1\cdots j_4}(x_1,\ldots x_4)$ is a bounded operator on $L_2(R^4)$ satisfying

$$\|b'_{\kappa_i;j_1\cdots j_4}\|_2\leq\|b'_{\kappa_i}\|_2\exp(-\mu_0|j_1|\cdots-\mu_0|j_4|), \quad (3.29a)$$

$$\leq O(\kappa_i^{-1/2})\exp(-\mu_0|j_1|\cdots-\mu_0|j_4|). \quad (3.29b)$$

Proof: The transition from (3.29a) to (3.29b) has been established by Glimm[3] (see also Ref. 1). Thus, we need only prove (3.29a). The crux of the proof is a representation of $b'_{\kappa_i;j_1\cdots j_4}(x_1,\ldots,x_4)$ developed by Glimm and Jaffe (Ref. 19, pp. 84–95). Let $\xi(x)$ be a $C_0^\infty(R)$ with support in χ_0 and which equals one on a neighborhood of B. We define operators

$$K_l=\mu_{x_l}^{-1/2}\xi(x_l)\mu_{x_l}^{1/2}\quad l=1,\ldots,4, \quad (3.30)$$

where kernels $k(x_lz_l)$ are tempered distributions. Estimate (3.15) of k_τ implies that $k(x_l,z_l)$ is a C^∞ function of x_l and z_l such that

$$|k(x_l,z_l)|\leq O[\exp(-\mu_0|x_l|-\mu_0|Z_l|)]. \quad (3.31)$$

We now define localized operators K_{l,i_l} by

$$K_{l,jl}=E_{jl}(x_l)\mu_{x_l}^{-1/2}\xi(x_l)\mu_{x_l}^{1/2},\quad l=1,\ldots,4, \quad (3.32)$$

whose kernels $k_{i_l}(x_l,z_l)$ satisfy

$$k_{j_l}(x_l,z_l)=\begin{cases}k(x_l,z_l), & x_l\in X_{j_l}\\ 0, & \text{otherwise}\end{cases}. \quad (3.33)$$

Using (3.31), one can prove that for $j_l\neq0$ the operator norm of K_{l,j_l} is bounded by

$$\|K_{l,j_l}\| \leq O[\exp(-\mu_0|j_l|)]. \quad (3.34)$$

It is not hard to show that (3.34) remains true for j_l, even for $j_l=0$. The representation of $b'_{\kappa_i;j_1\cdots j_4}$ that we are after is

$$b'_{\kappa_i;j_1\cdots j_4}=(\hat{\prod_{l=1}} K_{l,j_l})b'_{\kappa_i}. \quad (3.35)$$

Representation (3.35) together with estimate (3.34) proves *Lemma* 3.1.

We now return to the proof of (3.22). From (3.24) we get for $\Phi \in D_0$.

$$\|V'_{\kappa_i}(g)\| \leq \sum_{j_1\cdots j_4} \|V'_{\kappa_i;j_1\cdots j_4}\Phi\| \quad (3.36a)$$

$$\leq \text{const} \sum_{j_1\cdots j_4} \|b'_{\kappa_i;j_1\cdots j_4}\|_2 \,\|\hat{\prod_{l=1}}(N_{X_{j_l}}+I)^{1/2}\Phi\| \quad (3.36b)$$

$$\leq 0(\kappa_i^{-1/2}) \sum_{j_1\cdots j_4} \exp(-\mu_0|j_1|\cdots\mu_0|j_4|) \times \|\prod_{l=1}^{4}(N_{X_{j_l}}+I)^{1/2}\Phi\| \quad (3.36c)$$

$$\leq 0(\kappa_i^{-1/2}) \sum_{j_1\cdots j_4} \exp(-\mu_0|j_1|\cdots\mu_0|j_4|) \times \exp\left(\mu_0\frac{j_1}{2}+\cdots+\mu_0\left|\frac{j_4}{2}\right)\right. \times \|(N_{\text{loc}}+I)^2\Phi\| \quad (3.36d)$$

$$\leq 0(\kappa_i^{-1/2})\|(N_{\text{loc}}+I)^2\Phi\|. \quad (3.36e)$$

Inequality (3.36b) is an elementary local N_τ estimate. In (3.36c) we used (3.29b). In (3.36d–e) we used the estimate $\|(N_{X_j}+I)^{1/2}(N_{\text{loc}}+I)^{-1/2}\| \leq \exp(\mu_0|j/2|)$.

Estimate (3.36) implies (3.22). QED

4. COUPLING CONSTANT ANALYTICITY

In this section we study $H_\lambda=H_0+\lambda V(g)$ for λ in the complex λ plane cut along the negative real axis. We prove that Federbush's expansion for the resolvent is valid for values of λ in the domain

$$\{\lambda: |\arg\lambda| \leq \pi-\epsilon\}, \quad (4.1)$$

where $\epsilon>0$. The basic theorem used in the proof is a generalization of Theorem 2.2 for bounded sectorial operators[22].

Theorem 4.1: Let A be a bounded sectorial operator of norm less or equal to M, i.e., the numerical range $\theta(A)$ of A is the subset of the sector

$$|\arg(z-\gamma)| \leq \theta, \quad 0 \leq \theta < \pi/2, \quad \gamma \text{real} \quad (4.2)$$

lying in a circle with center γ and radius M. Let $|\alpha\rangle$ and $|\beta\rangle$ be two vectors of unit length. Suppose

$$\langle\alpha|A^k|\beta\rangle=0, \quad \text{for } 0 \leq k \leq N, \quad (4.3)$$

then for any $\mu>0$ a real number:

$$|\langle\alpha|R(+\mu;A)|\beta\rangle| \leq 0[\exp(-N\sqrt{2\mu/M})],$$

$$N \text{ and } M \text{ large}. \quad (4.4)$$

The proof of this theorem is similar to the proof of Theorem 2.2, see Ref. 8. It is based on a theorem by Bernstein (Ref. 23, p. 84, and Ref. 24, pp. 130–31 and pp. 280–81) in the theory of the best approximation of analytic functions, stating that if $f(z)$ is defined in $\{Z: |\arg z|<\pi/2, |z|\leq 1\}=D$ and it is analytic in an ellipse with foci at -1 and 1 and with sum of its semiaxes equal to $r>1$, then $f(z)$ may be approximated on D by a Fourier–Chebyshev polynomial series of degree n within

$$(2f_{\max}/r-1)(1/r^n) \quad (4.5)$$

in the uniform norm. $f_{\max}$ is the supremum of the absolute value of $f(z)$ in the ellipse.

Let

$$A_\lambda=b+H_i(\lambda)-2^{i-1}$$
$$=b+P_iH_0P_i+\lambda P_iVP_i-2^{i-1}$$
$$=b+H_{0,i}+\lambda V_i-2^{i-1}. \quad (4.6)$$

By realizing V as a multiplication operator on the probability space $L_2(Q,dq)$, we can write $V=V^+-V^-$ with $V^+V^-=0$ (Refs. 1,7). Using (2.24) and

$$V^- \leq \text{const}\, N+\text{const} \quad (4.7)$$

implied by Theorem (2.1), we see that A_λ is a sectorial operator, i.e., there exists a γ such that

$$|\text{Im}(\Phi, A_\lambda\Phi)| \leq \tan(\arg\lambda/2)\{\text{Re}(\Phi,(A_\lambda-\gamma)\Phi\} \quad (4.8)$$

for λ such that $|\text{avg}\lambda| \leq \pi-\epsilon, \epsilon>0$. Furthermore, if $A'_\lambda=b+P_iNP_i+\lambda P_i\, VP_i-2^{i-1}$, then $\|A'_\lambda\| \leq e2^{2i}$. Therefore, Theorem 4.1 is applicable to the operator A'_λ. We make the transition from A'_λ to A_λ by using $N \leq \text{const}\, H_0$. Thus, as in the case of real λ, we obtain the convergence of Federbush's expansion (2.9) for λ in (4.1). This convergence implies that $H(\lambda)$ is a family of analytic operators of type (B) in the sense of Kato (Ref. 22, pp. 345–397). From the general theory of analytic perturbations, there follows

Theorem 4.2: (1) Let H_λ be the self-adjoint operator defined in Sec. II for $\lambda>0$. Then H_λ has a resolvent analytic continuation to the cut λ-plane.

(2) The ground state Ω_λ, normalized by $\|\Omega_\lambda\|=1$, $\Omega_\lambda \geq 0$ and the ground state energy E_λ have an analytic continuation to a neighborhood of the real axis.

Another property implied by the uniform convergence of (2.9) in the domain (4.1) is

Theorem 4.3: Let $\rho(H_0)$ the resolvent set of H_0. For $-b \in \rho(H_0)$

$$R_\lambda(b)=(H_0+\lambda V+b)^{-1} \rightarrow R_0(b)+(H_0+b)^{-1} \quad (4.9)$$

in norm as $|\lambda| \rightarrow 0$ in $|\arg\lambda|<\pi$.

Proof: Let $R^{(n)}(b;H_\lambda)$ be the nth term in the expansion (2.9). Then, one easily sees that for $n \leq 1$

$$R^{(n)}(b;H_\lambda) \xrightarrow{\text{norm}} 0 \quad (4.10)$$

as $|\lambda| \longrightarrow 0$, $\lambda \epsilon \{\lambda: |\arg\lambda| \leq \pi-\epsilon\}$; and for $n=0$

$$R^{(0)}(b;H_\lambda) \xrightarrow{\text{norm}} R_0(b)=(b+H_0)^{-1} \quad (4.11)$$

limits (4.10) and (4.11) imply (4.9)

Using standard "stability theorems" (Ref. 22, pp. 206–07, and Theorem 1.7, p. 368), we obtain

Corollary 4.4: For $\epsilon>0$, there is a $\Lambda>0$ such that if $z \in \rho(H_0)$, then $z \in \rho(H_\lambda)$ for $\lambda \epsilon \{\lambda: |\text{avg}\lambda| \leq \pi-\epsilon,\ |\lambda| \leq \Lambda\}$.

Therefore, H_λ has only one eigenvalue near zero. This eigenvalue is analytic in $\{\lambda: |\arg\lambda| \leq \pi-\epsilon,\ |\lambda| \leq \Lambda\}$.

Remark: Theorem (4.3) is useful in proving[7,15,25] that the perturbation series for $E(\lambda)$, $\Omega(\lambda)$, and $W(\lambda)$, the equal time vacuum expectation valves, are asymptotic series which are Borel summable to the exact solutions.

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