Theory of light scattering from dense plasmas

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Projection operator techniques have been applied to derive a coupled set of exact kinetic equations describing microscopic density fluctuations in an infinite, fully-ionized, classical plasma in thermal equilibrium. The time dependence of the nonlocal collision terms in these equations is then approximated in such a way as to yield the correct short and long time behavior of density fluctuations, while allowing the explicit calculation of the dynamic structure factor $S(k,\omega)$ characterizing electron density fluctuations in classical plasmas of arbitrary density. This theory is then applied to analyze light scattering from dense, collision-dominated plasmas, and the results of these calculations are found to compare favorably with both experimental data as well as with earlier theories (at least in the regimes in which these earlier theories are valid).

I. INTRODUCTION

The utility of photon or neutron scattering as a diagnostic tool for probing the microscopic structure and dynamics of matter has long been recognized. With the development of the laser as an intense source of coherent light, light scattering has emerged as an extremely useful tool for the diagnostics of laboratory plasmas. Such scattering experiments have been used to study plasmas over a very wide range of densities, temperatures, and other conditions (turbulence, magnetic fields, etc.) An excellent summary of such experiments can be found in the review article of Evans and Katzenstein.[1]

The theory of such scattering experiments has also been the subject of considerable attention. Since the scattered light essentially measures electron density fluctuations in the plasma, most theoretical studies have used methods of plasma kinetic theory to calculate the autocorrelation of electron density fluctuations, or alternatively, its space-time Fourier transform, the dynamic form factor $S(k,\omega)$. The earliest investigations of Salpeter,[2] Fejer,[3] Dougherty and Farley,[4] and Rostoker and Rosenbluth[5] used the linearized Vlasov equations for the electrons and ions to study the role of collective motions (ion and electron oscillations) in light scattering. Subsequent investigations attempted to include the effects of Coulomb collisions, first by using models such as the Fokker–Planck[6] or BGK collision models,[7] and later by using perturbation theory calculations in the plasma parameter, $\Lambda=n\lambda_D^3$, to truncate the BBGKY heirarchy[8] or sum appropriate classes of diagrams in a Green's function approach.[9] These latter theories, while certainly valid for near-collisionless plasmas ($\Lambda\gg1$), are of doubtful validity for very dense or very cold plasmas in which $\Lambda\lesssim1$. (The calculation of higher-order contributions in Λ^{-1} would be required.)

In this paper we will apply projection operator methods to generate exact kinetic equations for the phase space analog of the dynamic form factor $S(k,\omega)$ characterizing electron density fluctuations in a classical plasma. A similar theory has been applied, with remarkable success, to study density fluctuations in single species systems such as liquids and dense gases.[10,11,12] Particular success has been achieved in applying this theory to study the large k and ω behavior of systems which characterizes neutron scattering from liquids and light scattering (Brillouin scattering) from liquids and gases.

We will first generalize this formalism to two-species systems, thereby generating exact kinetic equations describing density fluctuations in an infinite, fully-ionized plasma in thermal equilibrium. The relationship of these exact equations to existing plasma kinetic theory will be discussed. Approximations will then be introduced which will allow the application of this theory to the explicit calculation of the dynamic form factor $S(k,\omega)$ characterizing light scattering from equilibrium plasmas of arbitrary density.

The projection operator formalism we will employ, while certainly very popular in nonequilibrium statistical mechanics, has seen only a very limited application to plasma physics. Hence, we feel it is advisable to begin with a very brief introduction to the general formalism before applying it to derive the two-species kinetic equations.

II. KINETIC EQUATIONS DESCRIBING DENSITY FLUCTUATIONS IN TWO-SPECIES SYSTEMS

The projection operator formalism developed by Zwanzig[13] and Mori[14] for the study of irreversible processes has been applied to a variety of physical systems such as liquids, solids, and gases with considerable success. The theory is based upon an exact equation which can be shown to describe the time evolution of an arbitrary vector $\mathbf{a}(t)$ whose components $a_i(t)$ are dynamical variables of the coordinates

$(\mathbf{x}^1, \cdots, \mathbf{x}^N, \mathbf{p}^1, \cdots, \mathbf{p}^N)$ of a many-body system:

$$\dot{\mathbf{a}} - i\boldsymbol{\Omega}\cdot\mathbf{a}(t) + \int_0^t d\tau\, \boldsymbol{\varphi}(\tau)\cdot\mathbf{a}(t-\tau) = \mathbf{f}(t). \quad (1)$$

Here, the "frequency matrix" $\boldsymbol{\Omega}$ is defined by

$$i\boldsymbol{\Omega} \equiv \langle \dot{\mathbf{a}}\mathbf{a}^* \rangle \cdot \langle \mathbf{a}\mathbf{a}^* \rangle^{-1}, \quad (2)$$

where $\mathbf{a}$ denotes $\mathbf{a}(0)$, $\langle \cdots \rangle$ denotes an average over an equilibrium stationary ensemble [usually the canonical ensemble $\rho_0 = \exp(-\beta H)/Z$, $\beta \equiv 1/kT$], and $\mathbf{a}^*$ is the row vector adjoint to $\mathbf{a}$. The "damping matrix" $\boldsymbol{\varphi}(\tau)$ is given by

$$\boldsymbol{\varphi}(\tau) \equiv \langle \mathbf{f}(\tau)\mathbf{f}^*(0) \rangle \cdot \langle \mathbf{a}\mathbf{a}^* \rangle^{-1}, \quad (3)$$

and the "random force" $\mathbf{f}(\tau)$ is given by

$$\mathbf{f}(\tau) = \exp[i\tau(1-P)L] i(1-P)L\mathbf{a}. \quad (4)$$

Here, P is a projection operator defined by its action on an arbitrary dynamical variable vector $\mathbf{G}$ as

$$P\mathbf{G} \equiv \langle \mathbf{G}\mathbf{a}^* \rangle \cdot \langle \mathbf{a}\mathbf{a}^* \rangle^{-1} \cdot \mathbf{a}, \quad (5)$$

and L is the Liouville operator, $L = i\{H, \ \}$.

Equation (1) is known as the "generalized Langevin equation"[14] because of its similarity to the more familiar Langevin equation characterizing Brownian motion. However, unlike the Langevin equation, Eq. (1) is an exact equation for $\mathbf{a}(t)$, and hence is equivalent to the equations of motion for the many-body system.

If one now notes[14] that

$$\langle \mathbf{f}(t)\mathbf{a}^* \rangle = 0, \qquad t \geq 0, \quad (6)$$

then by post-multiplying Eq. (1) by $\mathbf{a}^* \cdot \langle \mathbf{a}\mathbf{a}^* \rangle^{-1}$, one can derive an exact equation for the correlation matrix

$$\mathbf{R}(t) \equiv \langle \mathbf{a}(t)\mathbf{a}^* \rangle \cdot \langle \mathbf{a}\mathbf{a}^* \rangle^{-1} \quad (7)$$

which takes the form

$$\dot{\mathbf{R}} - i\boldsymbol{\Omega}\cdot\mathbf{R}(t) + \int_0^t d\tau\, \boldsymbol{\varphi}(\tau)\cdot\mathbf{R}(t-\tau) = 0. \quad (8)$$

Of course, since this equation is still exact, it is only a formal identity to the equations of motion, and one must eventually resort to approximation in order to obtain useful results. The frequency matrix $\boldsymbol{\Omega}$ can usually be calculated explicitly in terms of static quantities. However, the damping matrix $\boldsymbol{\varphi}(\tau)$ requires the study of the modified propagator, $\exp[it(1-P)L]$, which, in turn, would involve solving the many-body problem directly. The attractive feature of equations such as (8) which are generated by projection operator techniques is that the "damping" terms are quite susceptible to perturbation theory treatments or to modeling. That is, the generalized Langevin equation (1) is of value primarily because it re-expresses the quantities of interest (e.g., time correlation functions) in forms involving damping terms which can then easily be approximated.

We will now utilize the generalized Langevin equation to obtain a kinetic equation description of an infinite, fully-ionized plasma which will be assumed to be in thermodynamic equilibrium. The dynamical variables of most interest are the microscopic phase-space densities for the electrons and the ions:

$$g_e(\mathbf{x}, \mathbf{p}, t) \equiv \sum_{\alpha_e=1}^{N_e} \delta[\mathbf{x} - \mathbf{x}^{\alpha_e}(t)]\delta[\mathbf{p} - \mathbf{p}^{\alpha_e}(t)],$$

$$g_i(\mathbf{x}, \mathbf{p}, t) \equiv \sum_{\alpha_i=1}^{N_i} \delta[\mathbf{x} - \mathbf{x}^{\alpha_i}(t)]\delta[\mathbf{p} - \mathbf{p}^{\alpha_i}(t)]. \quad (9)$$

To this end, we will choose $\mathbf{a}$ to be a column vector whose components are indexed by a continuous parameter $\mathbf{p}$ and correspond to the spatial Fourier transforms of the fluctuations of $g_e(\mathbf{x}, \mathbf{p}, 0)$ and $g_i(\mathbf{x}, \mathbf{p}, 0)$ from their equilibrium values

$$\mathbf{a} = \mathrm{col}[a_e(\mathbf{p}), a_i(\mathbf{p})],$$

$$a_e(\mathbf{p}) \equiv \sum_{\alpha_e=1}^{N_e} \exp(i\mathbf{k}\cdot\mathbf{x}^{\alpha_e})\delta(\mathbf{p} - \mathbf{p}^{\alpha_e}) - n_e\delta(\mathbf{k})M_e(\mathbf{p})$$
$$\equiv \delta g_e(\mathbf{k}, \mathbf{p}, 0),$$
$$a_i(\mathbf{p}) \equiv \sum_{\alpha_i=1}^{N_i} \exp(i\mathbf{k}\cdot\mathbf{x}^{\alpha_i})\delta(\mathbf{p} - \mathbf{p}^{\alpha_i}) - n_i\delta(\mathbf{k})M_i(\mathbf{p}) \quad (10)$$
$$\equiv \delta g_i(\mathbf{k}, \mathbf{p}, 0).$$

(The $\mathbf{k}$ dependence will frequently be suppressed for convenience.) Here,

$$M_r(\mathbf{p}) \equiv (\beta_r/2\pi m_r)^{3/2} \exp(-\beta_r p^2/2m_r),$$
$$\beta_r = 1/k_B T_r, \qquad r = e, i. \quad (11)$$

The extension of the generalized Langevin equation to vectors with both discrete and continuous parameter dependence[10] is

$$\frac{\partial \mathbf{a}}{\partial t} - i\int d^3p'\, \boldsymbol{\Omega}(\mathbf{p}, \mathbf{p}')\cdot\mathbf{a}(\mathbf{p}', t)$$
$$+ \int_0^t d\tau \int d^3p'\, \boldsymbol{\varphi}(\mathbf{p}, \mathbf{p}', \tau)\cdot\mathbf{a}(\mathbf{p}', t-\tau) = \mathbf{f}(\mathbf{p}, t), \quad (12)$$

where $\boldsymbol{\Omega}(\mathbf{p}, \mathbf{p}')$ and $\boldsymbol{\varphi}(\mathbf{p}, \mathbf{p}', \tau)$ are 2×2 matrices of "frequency kernels" and "damping kernels." To calculate these matrices, we follow the earlier theory of a single species system[10] to first calculate the static correlation matrix

$$\boldsymbol{\phi}(\mathbf{p}, \mathbf{p}') = \langle \mathbf{a}(\mathbf{p})\mathbf{a}^*(\mathbf{p}') \rangle, \quad (13)$$

as

$$\phi_{rs}(\mathbf{p}, \mathbf{p}') = n_r M_r(\mathbf{p})\delta(\mathbf{p} - \mathbf{p}')\delta_{rs}$$
$$+ n_r n_s M_r(\mathbf{p}) M_s(\mathbf{p}') h_{rs}(k), \qquad r, s = e, i \quad (14)$$

where $h(k)$ is the Fourier transform of $[g(r) - 1]$, $g(r)$ being the static pair correlation function.

The inverse of this matrix is defined by

$$\int d^3p'\boldsymbol{\phi}(\mathbf{p},\mathbf{p}')\cdot\boldsymbol{\phi}^{-1}(\mathbf{p}',\mathbf{p}'')=\delta(\mathbf{p}-\mathbf{p}'')\mathbf{I}. \quad (15)$$

Substituting (14) into this equation and solving the resulting system of linear integral equations for the unknowns, $\phi_{rs}^{-1}(\mathbf{p},\mathbf{p}'')$ yields

$$\phi_{rs}^{-1}(\mathbf{p},\mathbf{p}'')=[n_rM_r(\mathbf{p})]^{-1}\delta(\mathbf{p}-\mathbf{p}'')\delta_{rs}-C_{rs}(k), \quad (16)$$

where

$$C_{ee}(k)=\frac{h_{ee}(k)[1+n_ih_{ii}(k)]-n_ih_{ei}^2(k)}{[1+n_ih_{ii}(k)][1+n_eh_{ee}(k)]-n_in_eh_{ei}^2(k)},$$

$$C_{ii}(k)=\frac{h_{ii}(k)[1+n_eh_{ee}(k)]-n_eh_{ei}^2(k)}{[1+n_ih_{ii}(k)][1+n_eh_{ee}(k)]-n_in_eh_{ei}^2(k)}, \quad (17)$$

$$C_{ei}(k)=\frac{h_{ei}(k)}{[1+n_ih_{ii}(k)][1+n_eh_{ee}(k)]-n_in_eh_{ei}^2(k)}=C_{ie}(k).$$

We can now calculate the frequency matrix

$$i\boldsymbol{\Omega}(\mathbf{p},\mathbf{p}')=\int d^3p''\langle\dot{\mathbf{a}}(\mathbf{p})\mathbf{a}^*(\mathbf{p}'')\rangle\cdot\boldsymbol{\phi}^{-1}(\mathbf{p}'',\mathbf{p}'), \quad (18)$$

by first noting

$$\dot{a}_e(\mathbf{p})=(i\mathbf{k}\cdot\mathbf{p}/m_e)a_e(\mathbf{p})+\sigma_e(\mathbf{p}),$$
$$\sigma_e(\mathbf{p})\equiv\sum_{\alpha_e=1}^{N_e}\exp(i\mathbf{k}\cdot\mathbf{x}^{\alpha_e})\mathbf{F}^{\alpha_e}\cdot\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}^{\alpha_e}), \quad (19)$$

and similarly for $a_i(\mathbf{p})$. If we further note

$$\langle\sigma_r(\mathbf{p})a_s^*(\mathbf{p}'')\rangle=-(i\mathbf{k}\cdot\mathbf{p}/m_r)M_r(\mathbf{p})M_s(\mathbf{p}'')n_rn_sh_{rs}(k), \quad (20)$$

then a straightforward calculation yields

$$i\Omega_{rs}(\mathbf{p},\mathbf{p}')=(i\mathbf{k}\cdot\mathbf{p}/m_r)\delta(\mathbf{p}-\mathbf{p}')\delta_{rs}-(i\mathbf{k}\cdot\mathbf{p}/m_r)n_rM_r(\mathbf{p})C_{rs}(k). \quad (21)$$

The damping matrix is similarly given as

$$\boldsymbol{\varphi}(\mathbf{p},\mathbf{p}',t)=\int d^3p''\langle\mathbf{f}(\mathbf{p},t)\mathbf{f}^*(\mathbf{p}'',0)\rangle\cdot\boldsymbol{\phi}^{-1}(\mathbf{p}'',\mathbf{p}'), \quad (22)$$

or noting

$$\int d^3p''\mathbf{f}(\mathbf{p}'',0)=0,$$

$$\varphi_{rs}(\mathbf{p},\mathbf{p}',t)=[n_sM_s(\mathbf{p}')]^{-1}\langle f_r(\mathbf{p},t)f_s^*(\mathbf{p}',0)\rangle. \quad (23)$$

Here,

$$f_r(\mathbf{p},0)=(1-P)\dot{a}_r(\mathbf{p})=(1-P)\sigma_r(\mathbf{p}), \quad (24)$$

This is essentially as far as one can proceed without introducing some approximate calculation of the damping kernel.

As we will see in later applications of this formalism, most interest will be in the electron density fluctuations. With this in mind, we define the time correlation functions involving fluctuations in the electron density as

$$\mathcal{G}_{ee}(\mathbf{k},\mathbf{p},\mathbf{p}'',t)\equiv\langle\delta g_e(\mathbf{k},\mathbf{p},t)\delta g_e^*(\mathbf{k},\mathbf{p}'',0)\rangle=\langle a_e(\mathbf{p},t)a_e^*(\mathbf{p}'')\rangle,$$
$$\mathcal{G}_{ie}(\mathbf{k},\mathbf{p},\mathbf{p}'',t)\equiv\langle\delta g_i(\mathbf{k},\mathbf{p},t)\delta g_e^*(\mathbf{k},\mathbf{p}'',0)\rangle=\langle a_i(\mathbf{p},t)a_e^*(\mathbf{p}'')\rangle. \quad (25)$$

Hence, by multiplying both of the components of the matrix equation (12) by $a_e^*(\mathbf{p}'')$, averaging over an equilibrium ensemble, and noting

$$\langle f_r(\mathbf{p},t)a_e^*(\mathbf{p}'')\rangle=0 \quad (26)$$

[in analogy with Eq. (6)], we find a coupled set of exact kinetic equations for $\mathcal{G}_{ee}(\mathbf{k},\mathbf{p},\mathbf{p}'',t)$ and $\mathcal{G}_{ie}(\mathbf{k},\mathbf{p},\mathbf{p}'',t)$:

$$\frac{\partial\mathcal{G}_{ee}}{\partial t}-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}\mathcal{G}_{ee}+\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}n_eM_e(\mathbf{p})\times\left(C_{ee}(k)\int d^3p'\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t)+C_{ei}(k)\int d^3p'\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t)\right)$$
$$+\int_0^t d\tau\int d^3p'\varphi_{ee}(\mathbf{p},\mathbf{p}',\tau)\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t-\tau)+\int_0^t d\tau\int d^3p'\varphi_{ei}(\mathbf{p},\mathbf{p}',\tau)\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t-\tau)=0, \quad (27)$$

$$\frac{\partial\mathcal{G}_{ie}}{\partial t}-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_i}\mathcal{G}_{ie}+\frac{i\mathbf{k}\cdot\mathbf{p}}{m_i}n_iM_i(\mathbf{p})\times\left(C_{ie}(k)\int d^3p'\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t)+C_{ii}(k)\int d^3p'\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t)\right)$$
$$+\int_0^t d\tau\int d^3p'\varphi_{ie}(\mathbf{p},\mathbf{p}',\tau)\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t-\tau)+\int_0^t d\tau\int d^3p'\varphi_{ii}(\mathbf{p},\mathbf{p}',\tau)\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t-\tau)=0.$$

Here, we have substituted in the explicit form of the frequency matrix $\boldsymbol{\Omega}(\mathbf{p},\mathbf{p}'')$, but left the damping matrix $\boldsymbol{\varphi}(\mathbf{p},\mathbf{p}',\tau)$ unchanged since it is yet to be explicitly determined.

These coupled equations for the general time correlation functions $\mathcal{G}_{ee}$ and $\mathcal{G}_{ie}$ are still exact—and still quite formal. It should be noted that they are nonlocal in space and time, in contrast to more standard (and approximate) kinetic equations such as the Boltzmann, Fokker–Planck, or Vlasov equations. It should also

be noted that these equations are still quite general in that they describe any two species system (e.g., two-component liquids or gases) which are in thermal equilibrium. We will now apply these equations to the specific case of a fully ionized plasma.

III. APPLICATION TO FULLY IONIZED PLASMAS

We will now consider the two species of interest to be electrons of mass m_e, charge $Z_e=-1$, and ions of mass m_i, charge $Z_i=Z$. Since the plasma must be macroscopically neutral, we also demand that $N_e=ZN_i$, $n_e=Zn_i=n$. We will further assume that both species are in thermal equilibrium at a common temperature $T_e=T_i=T$ (although this latter assumption can be relaxed to allow $T_e\neq T_i$).

It is important to recognize that our general kinetic equation description assumes a knowledge of static equilibrium quantities such as the static pair correlation function $g(r)$. There are well-known prescriptions for calculating $g(r)$ for specified potentials $V(r)$ using techniques from equilibrium statistical mechanics[15,16] Since an equilibrium plasma does not possess the short-range order characterizing liquids or dense gases, it is reasonable to assume that the equilibrium theory will not be too sensitive to the specific form chosen for $g(r)$. For this reason, we will assume $g(r)$ is given by a simple Debye–Hückel form

$$g_{rs}(r)=1-Z_rZ_sl r^{-1}\exp(-r/\lambda_D), \tag{28}$$

where $l=e^2/kT$ and $\lambda_D=[4\pi\beta ne^2(1+Z)]^{-1/2}$ is the Debye length. Hence, one finds

$$h_{rs}(k)=-4\pi Z_rZ_sl\lambda_D^2/(1+k^2\lambda_D^2). \tag{29}$$

In this case, the coefficients $C_{rs}(k)$ reduce exactly to the very simple forms

$$C_{rs}(k)=-4\pi Z_rZ_s\beta e^2/k^2=\beta V_{rs}(k), \tag{30}$$

where $V_{rs}(k)$ is the Fourier transform of the Coulomb interaction potential.

Hence, we can rewrite the coupled kinetic equations (27) in the more transparent form

$$\frac{\partial \mathcal{G}_{ee}}{\partial t}-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}\mathcal{G}_{ee}+\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}nM_e(\mathbf{p})\left(\frac{4\pi e^2\beta}{k^2}\right)$$
$$\times\int d^3p'[Z\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t)-\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t)]=\Phi_{ee}+\Phi_{ei},$$
$$\frac{\partial \mathcal{G}_{ie}}{\partial t}-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_i}\mathcal{G}_{ie}-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}nM_i(\mathbf{p})\left(\frac{4\pi e^2\beta}{k^2}\right)$$
$$\int d^3p'[Z\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',t)-\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',t)]=\Phi_{ie}-\Phi_{ii}, \tag{31}$$

where the damping terms have been formally denoted by Φ_{rs}.

It should be noted that the left-hand-sides of these equations are just the linearized Vlasov equations. The effects of collisions are evidently contained in the damping terms Φ_{rs}. At this point, however, we should stress once again that Eqs. (31) are still exact.

To proceed further, we must now introduce approximations in order to calculate an explicit expression for the damping matrix $\varphi(\mathbf{p},\mathbf{p}',\tau)$. One could proceed by using perturbation theory to calculate $\varphi(\mathbf{p},\mathbf{p}',\tau)$ to lowest order in some suitable expansion parameter. For example, if an expansion in the plasma parameter $\epsilon=1/\Lambda$ were utilized, then to lowest order, φ could be set equal to zero and one would arrive at once at the Vlasov description considered by Salpeter,[2] Fejer,[3] Dougherty and Farley,[4] and Rostoker and Rosenbluth.[5] Retaining higher-order terms would yield more complicated kinetic descriptions such as the Balescu–Lenard[17] equations.

Our immediate goal, however, is to develop a theory of electron fluctuations in dense plasmas for which Λ may be quite small ($\Lambda<1$). Hence, we will seek an alternative approximation to the damping matrix by guessing or modeling its time behavior. Such a modeling approach has proven remarkably effective in describing the large k, ω behavior of dense systems such as simple liquids. We will now develop a very similar model to describe a dense plasma.

IV. MODELED KINETIC EQUATIONS

The damping matrix $\varphi(\mathbf{p},\mathbf{p}',t)$ is composed of four damping kernels, $\varphi_{rs}(\mathbf{p},\mathbf{p}',t)$. A direct calculation of these kernels is formidable (amounting to a solution of the many-body problem), and hence approximations will be necessary if we are to proceed further in our analysis of density fluctuations.

Before we introduce these approximations, however, it is important to note that the initial values of these kernels can be calculated exactly and explicitly (see Appendix A for details) as

$$\varphi_{rs}(\mathbf{p},\mathbf{p}',0)=\varphi_{rs}{}^s(\mathbf{p},\mathbf{p}',0)+\varphi_{rs}{}^d(\mathbf{p},\mathbf{p}',0), \tag{32}$$

where

$$\int d^3p'\varphi_{rs}{}^s(\mathbf{p},\mathbf{p}',0)a_s(\mathbf{p}')$$
$$=-\delta_{rs}D_s(0)\left(\frac{\partial}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\beta_s}{m_s}\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{p}\right)a_s(\mathbf{p}), \tag{33}$$

$$\int d^3p'\varphi_{rs}{}^d(\mathbf{p},\mathbf{p}',0)a_s(\mathbf{p}')=\mathbf{p}M_r(\mathbf{p})$$
$$\cdot\mathfrak{D}_{rs}(k)\cdot\int d^3p'\mathbf{p}'a_s(\mathbf{p}'), \tag{34}$$

and

$$D_s(0)=\tfrac{1}{3}\langle \mathbf{F}^{1s}\cdot\mathbf{F}^{1s}\rangle \tag{35}$$

$$\mathfrak{D}_{rs}(k)=\sum_{\alpha_r=1}^{N_r}\sum_{\alpha_s=1}^{N_s}{}'\ \langle \mathbf{F}^{\alpha_r}\mathbf{F}^{\alpha_s}\exp(i\mathbf{k}\cdot(\mathbf{x}^{\alpha_r}-\mathbf{x}^{\alpha_s}))\rangle .$$

It is also demonstrated in Appendix B that for a Coulomb potential, these coefficients reduce to

$$(\beta_e/m_e)D_e(0)=(1+Z)\omega_{pe}^2/3,$$

$$(\beta_i/m_i)D_i(0)=(1+Z^{-1})\omega_{pi}^2/3,$$

$$\mathfrak{D}_{rs}(k)=\mathbf{kk}(n_rn_s/\beta_r\beta_s)h_{rs}(k)-n_rD_{rs}(k), \tag{36}$$

$$D_{rs}{}^{ij}(k)=\delta_{ij}D_{rs}(0)-\delta_{ij}\frac{n_s}{\beta_s}\times\int d^3R\ h_{rs}(R)\frac{\partial^2V_{rs}}{\partial R_i\partial R_j}(1-\cos k R_z).$$

It should be noted that the damping kernels separate very naturally into "self" and "distinct" parts which essentially characterize test particle or collective motions in the plasma, respectively.[12]

We now introduce our principal approximation by modeling the time dependence of each of these kernels such that

$$\varphi_{rs}{}^{s,d}(\mathbf{p},\mathbf{p}',t)=\varphi_{rs}{}^{s,d}(\mathbf{p},\mathbf{p}',0)\exp[-\alpha_{rs}{}^{s,d}(k)t]. \tag{37}$$

The exact form for $\varphi_{rs}(\mathbf{p},\mathbf{p}',0)$ will be used, while the k-dependent relaxation parameters $\alpha_{rs}(k)$ will be chosen to satisfy various constraints demanded by the known short- and long-time behavior of $\mathcal{G}_{ee}$ and $\mathcal{G}_{ie}$. Such an exponential time dependence is motivated by the recognition that such damping kernels can usually be shown to be rapidly decaying functions of time,[13] and hence provided one has used an accurate description of their initial values, the solutions to the corresponding modeled kinetic equations should be relatively insensitive to the choice of the $\alpha_{rs}(k)$. And, as we will see momentarily, such modeling has the additional advantage of yielding kinetic equations for $\mathcal{G}_{ee}$ and $\mathcal{G}_{ie}$ which can be solved exactly to obtain a closed form expression for $S(k,\omega)$ (a feature not characteristic of most kinetic equations).

It should be noted here that we have introduced different relaxation parameters not only for each of the four damping kernels $\varphi_{rs}(\mathbf{p},\mathbf{p}',t)$, but also for their self and distinct parts.[12] Furthermore, we have allowed these parameters to be k dependent. We will see later that this generality is essential for an adequate description of electron and ion density fluctuations in a plasma.

If we substitute the modeled damping kernels (37) into the exact kinetic equations (31) and then Laplace transform in time, we find the coupled set of modeled kinetic equations:

$$\left(s-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}\right)\mathcal{G}_{ee}+\frac{i\mathbf{k}\cdot\mathbf{p}}{m_e}nM_e(\mathbf{p})\left(\frac{4\pi e^2\beta}{k^2}\right)\times\int d^3p'[Z\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',s)-\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',s)]$$

$$=\mathcal{G}_{ee}(k,\mathbf{p},\mathbf{p}'',0)+\frac{D_e(0)}{s+\alpha_{ee}{}^s(k)}\times\left(\frac{\partial}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\beta}{m_e}\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{p}\right)\mathcal{G}_{ee}(k,\mathbf{p},\mathbf{p}'',s)$$

$$-\frac{\mathbf{p}M_e(\mathbf{p})\cdot\mathfrak{D}_{ee}(k)}{s+\alpha_{ee}{}^d(k)}\cdot\int d^3p'\ \mathbf{p}'\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',s)$$

$$-\frac{\mathbf{p}M_e(\mathbf{p})\cdot\mathfrak{D}_{ei}(k)}{s+\alpha_{ei}{}^d(k)}\cdot\int d^3p'\ \mathbf{p}'\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',s), \tag{38}$$

(s is the Laplace transform variable),

$$\left(s-\frac{i\mathbf{k}\cdot\mathbf{p}}{m_i}\right)\mathcal{G}_{ie}+\frac{i\mathbf{k}\cdot\mathbf{p}}{m_i}nM_i(\mathbf{p})\left(\frac{4\pi e^2\beta}{k^2}\right)\times\int d^3p'[\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',s)-Z\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',s)]$$

$$=\mathcal{G}_{ie}(k,\mathbf{p},\mathbf{p}'',0)+\frac{D_i(0)}{s+\alpha_{ii}{}^s(k)}\times\left(\frac{\partial}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\beta}{m_i}\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{p}\right)\mathcal{G}_{ie}(k,\mathbf{p},\mathbf{p}'',s)$$

$$-\frac{\mathbf{p}M_i(\mathbf{p})\cdot\mathfrak{D}_{ie}(k)}{s+\alpha_{ie}{}^d(k)}\cdot\int d^3p'\mathbf{p}'\mathcal{G}_{ee}(k,\mathbf{p}',\mathbf{p}'',s)$$

$$-\frac{\mathbf{p}M_i(\mathbf{p})\cdot\mathfrak{D}_{ii}(k)}{s+\alpha_{ii}{}^d(k)}\cdot\int d^3p'\ \mathbf{p}'\mathcal{G}_{ie}(k,\mathbf{p}',\mathbf{p}'',s).$$

Here, the initial values of the correlation functions are

$$\mathcal{G}_{ee}(k,\mathbf{p},\mathbf{p}'',0)=n_eM_e(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'')+n_e{}^2M_e(\mathbf{p})M_e(\mathbf{p}'')h_{ee}(k), \tag{39}$$

$$\mathcal{G}_{ie}(k,\mathbf{p},\mathbf{p}'',0)=n_en_iM_i(\mathbf{p})M_e(\mathbf{p}'')h_{ei}(k).$$

We are primarily interested in the electron density correlation function

$$G_{ee}(k,s)\equiv n_e{}^{-1}\int d^3p\int d^3p''\mathcal{G}_{ee}(k,\mathbf{p},\mathbf{p}'',s), \tag{40}$$

since the dynamic form factor $S(k,\omega)$ is given by

$$S(k,\omega)=(1/\pi)\ \mathrm{Re}\{\lim_{s\to i\omega}G_{ee}(k,s)\}. \tag{41}$$

The coupled kinetic equations (38) can be solved exactly for $G_{ee}(k,s)$ by using Fourier transforms in the momentum variable p. Since the details of such a solution have been discussed elsewhere for a single species system,[11,12] we will merely indicate the results of

such a calculation

$$G_{ee}(k,s)=\frac{(1+a_{ii})\mathcal{S}_{ii}-a_{ei}\mathcal{S}_{ie}}{(1+a_{ii})(1+a_{ee})-a_{ei}a_{ie}}, \tag{42}$$

$$G_{ie}(k,s)=\frac{(1+a_{ee})\mathcal{S}_{ie}-a_{ie}\mathcal{S}_{ee}}{(1+a_{ii})(1+a_{ee})-a_{ei}a_{ie}}, \tag{43}$$

where

$$\begin{aligned}
a_{ee}\equiv&[(m_e\beta_e s/k^2)g_{ee}(k,s)I(\kappa_e^2,z_e-1)-1]\\
&\times[n_eC_{ee}(k)-(m_e\beta_e s/k^2)f_{ee}(k,s)],\\
a_{ei}\equiv&[(m_e\beta_e s/k^2)g_{ee}(k,s)I(\kappa_e^2,z_e-1)-1]\\
&\times[n_eC_{ei}(k)-(m_i\beta_e^2/k^2)f_{ei}(k,s)],\\
a_{ii}\equiv&[(m_i\beta_i s/k^2)g_{ii}(k,s)I(\kappa_i^2,z_i-1)-1]\\
&\times[n_iC_{ii}(k)-(m_i\beta_i s/k^2)f_{ii}(k,s)],\\
a_{ie}\equiv&[(m_i\beta_i s/k^2)g_{ii}(k,s)I(\kappa_i^2,z_i-1)-1]\\
&\times[n_iC_{ie}(k)-(m_e\beta_i s/k^2)f_{ie}(k,s)];
\end{aligned} \tag{44}$$

$$\begin{aligned}
\mathcal{S}_{ee}\equiv&n_eS_{ee}(k)[(m_e\beta_e/k^2)g_{ee}(k,s)I(\kappa_e^2,z_e-1)]\\
&-[(m_e\beta_e s/k^2)g_{ee}(k,s)I(\kappa_e^2,z_e-1)-1]\\
&\times[n_eS_{ee}(k)(m_e\beta_e/k^2)f_{ee}(k,s)\\
&+(m_i\beta_e/k^2)n_in_eh_{ie}(k)f_{ei}(k,s)],
\end{aligned} \tag{45}$$

$$\begin{aligned}
\mathcal{S}_{ie}\equiv&n_in_eh_{ie}(k)[(m_i\beta_i/k^2)g_{ii}(k,s)I(\kappa_i^2,z_i-1)]\\
&-[(m_i\beta_i s/k^2)g_{ii}(k,s)I(\kappa_i^2,z_i-1)-1]\\
&\times[n_in_eh_{ie}(k)(m_i\beta_i/k^2)f_{ii}(k,s)\\
&+(m_e\beta_i/k^2)n_eS_{ee}(k)f_{ie}(k,s)];
\end{aligned} \tag{46}$$

$$z_e\equiv s/g_{ee}(k,s),\qquad z_i\equiv s/g_{ii}(k,s); \tag{47}$$

$$\kappa_e\equiv k/(m_e\beta_e)^{1/2}g_{ee}(k,s),\qquad \kappa_i\equiv k/(m_i\beta_i)^{1/2}g_{ii}(k,s); \tag{48}$$

$$g_{ee}(k,s)\equiv\frac{(\beta_e/m_e)D_e(0)}{s+\alpha_{ee}{}^s(k)}, \tag{49}$$

$$g_{ii}(k,s)\equiv\frac{(\beta_i/m_i)D_i(0)}{s+\alpha_{ii}{}^s(k)};$$

$$\begin{aligned}
f_{ee}(k,s)\equiv&[s+\alpha_{ee}{}^d(k)]^{-1}\\
&\times[(k^2/m_e\beta_e)n_eC_{ee}(k)-(\beta_e/m_e)D_{ee}(k)],\\
f_{ei}(k,s)\equiv&[s+\alpha_{ei}{}^d(k)]^{-1}\\
&\times[(k^2/m_i\beta_e)n_eC_{ei}(k)-(\beta_i/m_i)(n_e/n_i)D_{ei}(k)],\\
f_{ii}(k,s)\equiv&[s+\alpha_{ii}{}^d(k)]^{-1}\\
&\times[(k^2/m_i\beta_i)n_iC_{ii}(k)-(\beta_i/m_i)D_{ii}(k)],\\
f_{ie}(k,s)\equiv&[s+\alpha_{ie}{}^d(k)]^{-1}\\
&\times[(k^2/m_e\beta_i)n_iC_{ie}(k)-(\beta_e/m_e)(n_i/n_e)D_{ie}(k)];
\end{aligned} \tag{50}$$

$$I(\kappa^2,z)\equiv\exp(\kappa^2)\kappa^{-2(z+\kappa^2)}\int_0^{\kappa^2}du\exp(-u)u^{z+\kappa^2}. \tag{51}$$

The specification of the k-dependent relaxation parameters $\alpha_{rs}(k)$ is the only task remaining to complete the model. To do this, one can utilize the known behavior of $S(k,\omega)$ for limiting values of k and ω to place constraints upon the $\alpha_{rs}(k)$. Since our objective here is merely to illustrate the modeling approach to approximating the damping kernels, we will defer the detailed choice of the $\alpha_{rs}(k)$ to a future publication, and only summarize the resulting constraints here:

(i) The large k behavior of $S(k,\omega)$ requires $\alpha_{rs}(k)=O(k^2)$ as $k\to\infty$. We will therefore choose the simplest k dependence consistent with this limiting behavior:

$$\begin{aligned}
\alpha_{rr}{}^s(k)&=\alpha_{rr}{}^s(0)[1+(k/k_{rr}{}^s)^2],\\
\alpha_{rs}{}^d(k)&=\alpha_{rs}{}^d(0)[1+(k/k_{rs}{}^d)^2].
\end{aligned} \tag{52}$$

(ii) The $k=0$ behavior of the correlation functions demands

$$\begin{aligned}
\alpha_{ee}{}^d(0)&=\alpha_{ee}{}^s(0)\\
\alpha_{ii}{}^d(0)&=\alpha_{ii}{}^s(0)Z
\end{aligned} \tag{53}$$

(iii) Symmetry properties require

$$\alpha_{ei}{}^d(k)=[\alpha_{ee}{}^d(k)\alpha_{ii}{}^d(k)]^{1/2}=\alpha_{ie}{}^d(k). \tag{54}$$

(iv) In analogy with the single species formalism,[12] $\alpha_{rr}{}^s(0)$ can be identified as

$$\alpha_{rr}{}^s(0)=[(\beta_r/m_r)D_r(0)][m_r\beta_rD_r], \tag{55}$$

where D_r is the coefficient of self-diffusion for the rth species. If we recall $D_r=(m_r\beta_r\nu_{rr})^{-1}$, then

$$\alpha_{rr}{}^s(0)=\frac{\beta_r}{m_r}\frac{D_r(0)}{\nu_{rr}}, \tag{56}$$

where ν_{rr} is the collision frequency for the rth species.

(v) Earlier work on single-species systems[12] has identified $k_{rr}{}^s$ as characterizing the transition from collective behavior for individual particle behavior with increasing k. For a plasma, this parameter is naturally identified as the Debye wave-number

$$k_{rr}{}^s=k_{D_r}=1/\lambda_{D_r}. \tag{57}$$

(vi) The only remaining parameter to be specified is $k_{rr}{}^d$. A detailed analysis of the $k=0$ behavior of the density fluctuations demands that we choose

$$k_{rr}{}^d=1.08(\nu_{rr}/\omega_{pr})k_{D_r}. \tag{58}$$

It is interesting to note that although our model (37) introduced six k-dependent relaxation parameters, $\alpha_{rs}{}^{s,d}(k)$, various constraints have considerably restricted our freedom in choosing these parameters. Indeed, the model has been completely specified in terms of known plasma parameters.

At this juncture, it is of interest to compare our modeled kinetic equations with some of the more standard kinetic equations used in plasma physics.

Of course, if we had set the damping kernels

$$\varphi_{rs}(\mathbf{p}, \mathbf{p}', t) = 0,$$

we would have immediately arrived at the linearized Vlasov equations. This feature can also be seen directly from our solutions (42)–(51) if one notes that the "collisionless" limit corresponds to $g(k, s) \to 0$, and also that

$$\lim_{g(k,s)\to 0} [g(k, s) I(\kappa^2, z-1)] = \frac{ik}{(2m\beta)^{1/2}} Z[-(is/k)(\tfrac{1}{2}m\beta)^{1/2}], \quad (59)$$

where $Z(\xi)$ is the plasma dispersion function

$$Z(\xi) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} dx \exp(-x^2)/(x-\xi). \quad (60)$$

Another approximate kinetic equation can be obtained if we ignore the distinct damping contributions $\varphi_{rs}{}^{d}(\mathbf{p}, \mathbf{p}', t)$ and use only the "Markovian limit," $k \to 0$ and $s \to 0$, of $g(k, s)$ in the self term, noting that

$$\lim_{s\to 0, k\to 0} g_r(k, s) = \frac{\beta_r D_r(0)}{m_r \alpha_{rr}(0)} = \nu_{rr}. \quad (61)$$

The self term in this limit reduces to the Fokker–Planck operator familiar from the theory of Brownian motion.[19]

$$J_{FP}[\ ] \equiv \nu_{rr} \left(\frac{\partial}{\partial \mathbf{p}} \cdot \frac{\partial}{\partial \mathbf{p}} + \frac{\beta_r}{m_r} \frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{p} \right). \quad (62)$$

The resulting system of equations has been used by Grewal[6] to analyze light scattering from electron density fluctuations in a plasma. One can readily see the limitations of such a Fokker–Planck description, since it ignores the nonlocal behavior of the damping or collision terms (as well as any collective behavior present in them), and, moreover, does not yield the correct short time behavior as $t \to 0$.

One can obtain the collision-dominated behavior of such a model by noting that this corresponds to

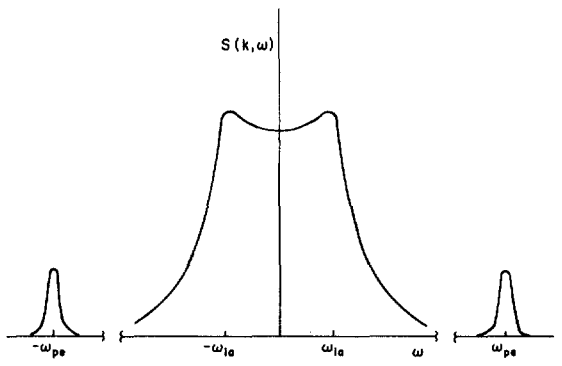

FIG. 1. A qualitative sketch of $S(k, \omega)$ for $\alpha = (k\lambda_D)^{-1} \gg 1$.

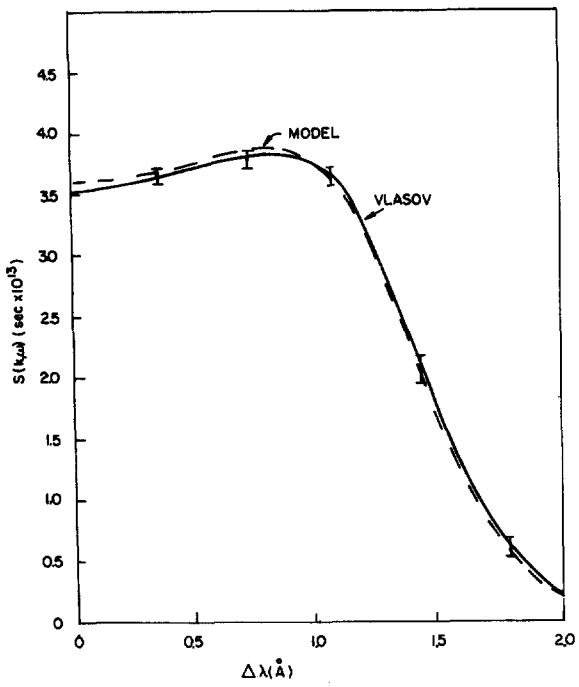

FIG. 2. The ion feature of $S(k, \omega)$ for $n = 5 \times 10^{17}$ cm^{-3}, $T_e = T_i = 4.5$ eV. The dashed curve indicates the modeled kinetic equation predictions, the solid curve corresponds to the Vlasov equation, while the data points are those of Rohr (Ref. 20).

$g(k, s) \to \infty$ in our solutions, while using

$$\lim_{g(k,s)\to\infty} [g(k, s) I(\kappa^2, z-1)] = \frac{g(k, s)}{[sg(k, s)(m\beta/k^2)+1]}. \quad (63)$$

More systematic approximations to the damping matrix can be obtained by applying standard perturbation theory methods directly to (23). This approach will be discussed in a future publication.

V. APPLICATION TO LIGHT SCATTERING EXPERIMENTS

As an illustration of the utility of this theory of density fluctuations in plasmas, we will use the modeled kinetic equations (38) to calculate the dynamic form factor $S(k, \omega)$ characterizing light scattering from dense, low temperature (and hence collision dominated) plasmas.

It should be recalled that the key parameter in conventional Vlasov calculations of $S(k, \omega)$ is the scattering parameter $\alpha \equiv (k\lambda_D)^{-1}$, where k is the wavevector change of the scattering light. For the large values of α characteristic of 90° scattering from dense plasmas ($\alpha > 5$), the dynamic form factor $S(k, \omega)$ is characterized by a large central peak with two small side peaks corresponding to ion-acoustic oscillations, and two very weak satellite peaks characterizing electron plasma oscillations (see Fig. 1). Our modeled

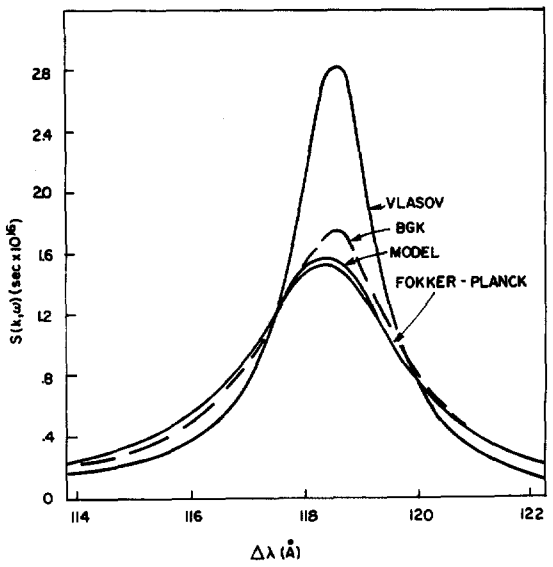

FIG. 3. The electron satellite peak of $S(k, \omega)$ for $n=5\times10^{17}$ cm^{-3}, $T_e=T_i=4.5$ eV as predicted by several alternative kinetic theories.

kinetic equations will clearly indicate the effects of collisions upon each of these features.

At densities of 5×10^{17} cm^{-3}, $T_e=T_e=4.5$ eV, the Vlasov description of the ion feature seems adequate (Fig. 2), even though $\Lambda n=\lambda_D^3=5$. This was first noted experimentally by Rohr.[20] By way of contrast, the electron satellite lines are predicted to be broadened significantly by collisions at this density (Fig. 3). For comparison, we have also plotted the Fokker-Planck model of Grewal[6] which agrees very favorably with our more sophisticated model at these low densities.

At a higher density of 1×10^{19}, $T=4.5$ eV (corresponding, incidentally, to the critical density for reflection of 10.6 μ light), the various formalisms give quite different results. Figure 4 shows the ion feature for this cold, high-density plasma as predicted by several collision models (the Fokker–Planck,[6] BGK,[21] and our nonlocal model). All of the collision models predict narrowing of the ion feature (as first pointed out by Dubois and Gilinsky[9] using a hydrodynamic analysis). The model developed in our work predicts somewhat sharper ion-acoustic peaks.

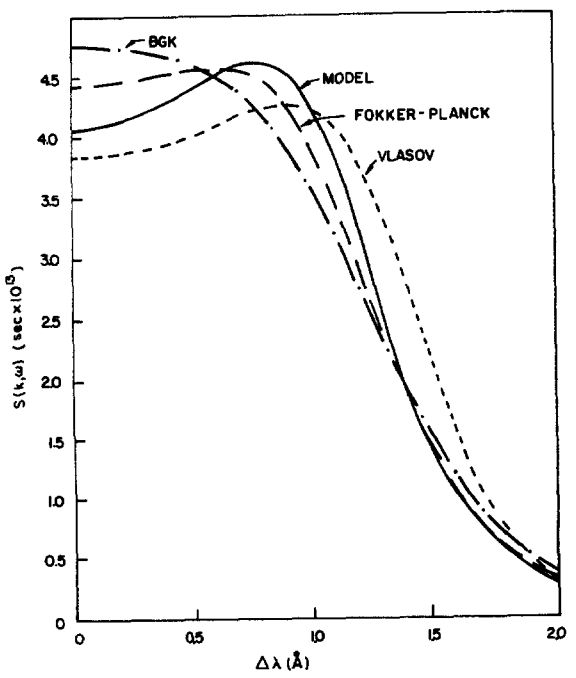

FIG. 4. The ion feature of $S(k, \omega)$ for $n=1\times10^{19}$ cm^{-3}, $T_e=T_i=4.5$ eV.

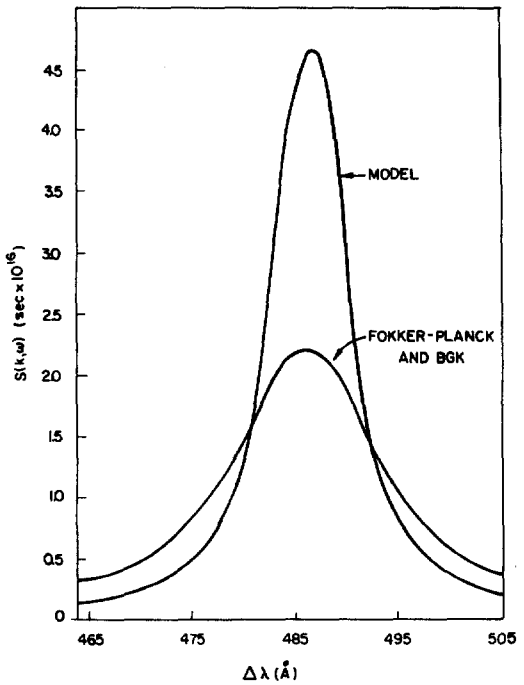

FIG. 5. The electron satellite peak of $S(k, \omega)$ for $n=1\times10^{19}$ cm^{-3}, $T_e=T_i=4.5$ eV.

The electron line at 1×10^{19}, $T=4.5$ eV is shown in Fig. 5. The Vlasov description gives essentially a delta function for the line at this high density. There is a significant difference between our model and the Fokker–Planck prediction, with the former yielding a much sharper electron line. This difference can be attributed to the significance of the non-Markovian behavior of the collision term for large frequencies $\omega\sim\omega_{pe}$. For the lower frequencies characterizing the ion peaks, $\omega\ll\alpha_{rs}(k)$, and the model becomes essentially Markovian.

We should hasten to point out that the particular model developed in this work depends upon the various collision frequencies, ν_{ee}, ν_{ei}, and ν_{ii}, just as do the more traditional Fokker–Planck or BGK models, as well as the many body theory of Kivelson and Du Bois.[22] We have used the collision frequencies prescribed by them in this particular calculation. A more detailed study of which collision frequencies are most appropriate for this model is certainly desirable.

VI. CONCLUSIONS

To summarize, we have applied projection operator techniques to derive a coupled set of exact kinetic equations describing microscopic density fluctuations in a two-species system. When applied to a plasma in thermal equilibrium, these equations reduce to the coupled linearized Vlasov equations augmented by additional ("damping") terms which characterize short-range collisions. To proceed further, one must introduce approximations which allow the explicit calculation of these terms.

Although perturbation theory could be utilized at this point (and will be examined in future work), we have chosen an alternative approach by modeling the time dependence of the nonlocal damping terms. Such modeling yields both the correct short- and long-time behavior, while permitting the explicit calculation of the damping terms and the analytical solution of the resulting coupled kinetic equations for the dynamic form factor $S(k,\omega)$ characterizing electron density fluctuations. Furthermore, such modeling is not limited to low-density (near collisionless) plasmas.

To illustrate the utility of such modeling, we have applied the theory to the analysis of light scattering from dense, collision-dominated plasmas. Our results agree with experiments, as well as with earlier theories (in the regimes in which these theories are valid).

The particular model studied was presented only to illustrate the potential of the more general theory for analyzing density fluctuations in plasmas. More refined models will be developed in a future publication which will allow for unequal temperatures ($T_e \neq T_i$) and for nonequilibrium static correlations (such as encountered in laser heated plasmas). Such a theory can then be applied more generally to analyze the interaction of electromagnetic radiation with plasmas (including light absorption and emission as well as scattering).

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APPENDIX A

We wish to calculate the $t=0$ form of the damping kernel.[10] First, note that

$$\langle f_s(\mathbf{p},0) f_r^*(\mathbf{p}',0)\rangle = \langle [(1-P)\sigma_s(\mathbf{p})][(1-P)\sigma_r^*(\mathbf{p}')]\rangle = \langle \sigma_s(\mathbf{p})(1-P)\sigma_r^*(\mathbf{p}')\rangle. \tag{A1}$$

Using the definition of the projection operator P,

$$P\mathbf{G}(\mathbf{p}) = \int d^3p' \int d^3p'' \langle \mathbf{G}(\mathbf{p})\mathbf{a}^*(p')\rangle \cdot \boldsymbol{\phi}^{-1}(\mathbf{p}',\mathbf{p}'') \cdot \mathbf{a}(\mathbf{p}''), \tag{A2}$$

we can calculate

$$P\begin{bmatrix}\sigma_e\\ \sigma_i\end{bmatrix} = -\begin{Bmatrix}(i\mathbf{k}\cdot\mathbf{p}/m_e) n_e M_e(\mathbf{p})[C_{ee}(k)\rho_e + C_{ei}(k)\rho_i]\\ (i\mathbf{k}\cdot\mathbf{p}/m_i) n_i M_i(\mathbf{p})[C_{ii}(k)\rho_i + C_{ie}(k)\rho_e]\end{Bmatrix} \tag{A3}$$

and

$$\langle \sigma_s(\mathbf{p}) P\sigma_r^*(\mathbf{p}')\rangle = n_r M_r(\mathbf{p}')(\mathbf{k}\cdot\mathbf{p}'/m_r) n_s M_s(\mathbf{p}) \times (\mathbf{k}\cdot\mathbf{p}/m_s)[n_s C_{rs}(k) h_{ss}(k) + n_r C_{rr}(k) h_{sr}(k)]. \tag{A4}$$

Furthermore,

$$\begin{aligned}\langle \sigma_s(\mathbf{p})\sigma_r^*(\mathbf{p}')\rangle &= \sum_{\alpha_s=1}^{N_s}\sum_{\beta_r=1}^{N_r}\left\langle \exp(i\mathbf{k}\cdot\mathbf{x}^{\alpha_s})\mathbf{F}^{\alpha_s}\cdot\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}^{\alpha_s}) \times \exp(-i\mathbf{k}\cdot\mathbf{x}^{\beta_r})\mathbf{F}^{\beta_r}\cdot\frac{\partial}{\partial\mathbf{p}}\delta(\mathbf{p}'-\mathbf{p}^{\beta_r})\right\rangle\\ &= \delta_{sr}\frac{\partial}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{p}}:\delta(\mathbf{p}-\mathbf{p}')M_s(\mathbf{p}')\sum_{\alpha_s=1}^{N_s}\langle \mathbf{F}^{\alpha_s}\mathbf{F}^{\alpha_s}\rangle\\ &\quad + \frac{\partial}{\partial\mathbf{p}}\frac{\partial}{\partial\mathbf{p}'}:M_s(\mathbf{p})M_r(\mathbf{p}')\\ &\quad \times \sum_{\alpha_s=1}^{N_s}\sum_{\beta_r=1}^{N_r}{}' \langle \mathbf{F}^{\alpha_s}\mathbf{F}^{\beta_r}\exp[i\mathbf{k}\cdot(\mathbf{x}^{\alpha_s}-\mathbf{x}^{\beta_r})]\rangle\end{aligned} \tag{A5}$$

But, if we note that

$$\sum_{\alpha_s=1}^{N_s}\langle \mathbf{F}^{\alpha_s}\mathbf{F}^{\alpha_s}\rangle = n_s \mathbf{I} D_s(0),$$

$$\begin{aligned}\sum_{\alpha_s=1}^{N_s}\sum_{\beta_r=1}^{N_r}{}' \langle \mathbf{F}^{\alpha_s}\mathbf{F}^{\beta_r}\exp[i\mathbf{k}\cdot(\mathbf{x}^{\alpha_s}-\mathbf{x}^{\beta_r})]\rangle &= \mathfrak{D}_{sr}(k)\\ &= \mathbf{kk}(n_s n_r/\beta_s\beta_r) h_{sr}(k) - n_s \mathbf{D}_{sr}(k),\end{aligned} \tag{A6}$$

where the prime denotes $\alpha_s \neq \beta_r$ if $s=r$, then it is a straightforward calculation to combine (A4) and (A5),

using (A6) to find

$$\varphi_{ee}(\mathbf{p},\mathbf{p}',0)=-D_e(0)\left(\frac{\partial}{\partial\mathbf{p}}\cdot\frac{\partial}{\partial\mathbf{p}}+\frac{\beta_e}{m_e}\frac{\partial}{\partial\mathbf{p}}\cdot\mathbf{p}\right)\delta(\mathbf{p}-\mathbf{p}')$$

$$+\mathbf{p}M_e(\mathbf{p})\cdot\left(\mathbf{kk}\frac{n_eC_{ee}(k)}{m_e^2}-\frac{\beta_e^2}{m_e^2}\mathbf{D}_{ee}(k)\right)\cdot\mathbf{p}' \quad \text{(A7)}$$

and

$$\varphi_{ei}(\mathbf{p},\mathbf{p}',0)=\mathbf{p}M_e(\mathbf{p})\cdot\left(\mathbf{kk}\frac{n_eC_{ei}(k)}{m_em_i}-\frac{\beta_e\beta_i}{m_em_i}\frac{n_e}{n_i}\mathbf{D}_{ei}(k)\right)\cdot\mathbf{p}' \quad \text{(A8)}$$

with similar expressions for φ_{ii} and φ_{ie}.

APPENDIX B

In this appendix we will sketch the calculation of $D_s(0)$. Recall

$$n_sD_s(0)\equiv\tfrac{1}{3}\sum_{\alpha_s=1}^{N_s}\langle\mathbf{F}^{\alpha_s}\cdot\mathbf{F}^{\alpha_s}\rangle=\tfrac{1}{3}N_s\langle\mathbf{F}^{1s}\cdot\mathbf{F}^{1s}\rangle, \quad \text{(B1)}$$

where

$$\mathbf{F}^{1s}=-\frac{\partial}{\partial\mathbf{x}^{1s}}\Big[\sum_{\beta_s=2}^{N_s}V^{ss}(|\mathbf{x}^{1s}-\mathbf{x}^{\beta_s})+\sum_{\beta_r=1}^{N_r}V^{sr}(|\mathbf{x}^{1s}-\mathbf{x}^{\beta_r}|)\Big]. \quad \text{(B2)}$$

Making use of

$$\mathbf{F}^{1s}=(\beta\rho_0)^{-1}\frac{\partial\rho_0}{\partial\mathbf{x}^{1s}}, \quad \text{(B3)}$$

where ρ_0 is the ensemble density, (B1) becomes

$$n_sD_s(0)=-\frac{N_s(N_s-1)}{3\beta}\int d[N]\frac{\partial\rho_0}{\partial\mathbf{x}^{1s}}\cdot\frac{\partial V^{ss}}{\partial\mathbf{x}^{1s}}-\frac{N_sN_r}{3\beta}\int d[N]\frac{\partial\rho_0}{\partial\mathbf{x}^{1s}}\cdot\frac{\partial V^{sr}}{\partial\mathbf{x}^{1s}} \quad \text{(B4)}$$

$$=-\frac{n_s^2}{3\beta}\int d^3R\frac{\partial g_{ss}(R)}{\partial\mathbf{R}}\cdot\frac{\partial V^{ss}(R)}{\partial\mathbf{R}}-\frac{n_sn_r}{\beta}\int d^3R\frac{\partial g_{sr}(R)}{\partial\mathbf{R}}\cdot\frac{\partial V^{sr}(R)}{\partial\mathbf{R}}. \quad \text{(B5)}$$

If we now consider an electron-ion plasma, then

$$D_e(0)=D_{ee}(0)+D_{ei}(0)=(1+Z)D_{ee}(0), \quad \text{(B6)}$$

where

$$D_{ee}(0)=-\frac{n_e}{3\beta}\int d^3R\frac{\partial g_{ee}(R)}{\partial\mathbf{R}}\cdot\frac{\partial V^{ee}(R)}{\partial\mathbf{R}}, \quad \text{(B7)}$$

$$D_{ei}(0)=-\frac{n_i}{3\beta}\int d^3R\frac{\partial g_{ei}(R)}{\partial\mathbf{R}}\cdot\frac{\partial V^{ei}(R)}{\partial\mathbf{R}}=ZD_{ee}(0).$$

For a Coulomb potential we can calculate

$$D_{ee}(0)=\frac{4\pi n_ee^2}{3\beta}\int_0^\infty dr\frac{dg_{ee}(r)}{dr}, \quad \text{(B8)}$$

and making use of the boundary condition for $g_{ee}(r)$, i.e., $g_{ee}(0)=0$ and $g_{ee}(\infty)=1$, we find

$$(\beta_e/m_e)D_{ee}(0)=\tfrac{1}{3}\omega_{pe}^2, \qquad \omega_{ps}^2\equiv4\pi n_sZ_s^2e^2/m_s. \quad \text{(B9)}$$

A similar calculation for the ions gives

$$D_i(0)=(1+Z^{-1})D_{ii}(0), \quad \text{(B10)}$$

where

$$(\beta_i/m_i)D_{ii}(0)=\tfrac{1}{3}\omega_{pi}^2. \quad \text{(B11)}$$

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