The Head-On Collision of a Shock Wave and a Rarefaction Wave in One Dimension

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A procedure for investigating the strengthening of a shock which collides head-on with a rarefaction wave is suggested and is carried through for the case in which the entropy jump across the shock is small enough to be negligible.

I. INTRODUCTION

THE problem which is considered in the present paper is the strengthening and acceleration of a shock which moves in such a way that it collides with a rarefaction which approaches the shock from the low pressure side. Such a collision may be thought to take place in a semi-infinite tube as shown in Fig. 1. The x axis is taken to be the axis of the tube. A piston is initially moving toward the left with a constant velocity less than that of sound and the gas in the tube moves with the same velocity. (The restriction on the initial velocity of the piston is not essential; it is merely convenient for the purpose of describing the phenomenon.) When the piston reaches the position $x=0$ at time $t=0$, it is assumed that the piston accelerates in some manner toward the left and thus produces a simple rarefaction wave which moves toward the right. At a large positive value of x a shock wave whose low pressure side faces the origin is assumed to exist. The shock wave will move toward the origin with uniform speed and constant strength until it meets the oncoming rarefaction wave at a distance L from the origin. The shock will then accelerate and be strengthened as it meets the gas of decreasing density. Furthermore, the entropy jump across the shock, which is constant as long as the shock moves through the gas which has not yet been disturbed by the rarefaction, will increase. Thus an entropy wave which moves with the fluid will be formed on the high pressure side of the shock so that the flow on the high pressure side of the shock is no longer isentropic.

In reference 1 Courant and Friedrichs consider the interaction discussed above and conclude that the final result of such an interaction will be a shock wave moving towards the left and a rarefaction wave moving toward the right, separated by a zone of gas of varying entropy.[1] The calculations in the present paper have been made with these results kept in mind and will describe in more detail than is given in reference 1 the actual process of interaction between the shock and rarefaction waves.

It might be pointed out that the collision between the shock wave and rarefaction wave as discussed in the present paper is an idealization of processes which occur in intermittent jet engines and supersonic wind tunnels which operate by permitting air from the outside atmosphere to pass through the tunnel into a low pressure reservoir.

II. PROCEDURE

Before discussing the interaction between the rarefaction and shock, the properties of the rarefaction and shock waves will be reviewed briefly. The terminology of reference 2 will be used throughout.[2]

A simple wave in non-steady, one-dimensional flow refers to a special isentropic flow in which the fluid velocity, pressure, density, and speed of sound assume constant values along each straight line of a one-parameter family of straight lines in the $x-t$ plane. These values in general differ from line to line of this family. For general non-steady one-dimensional flows, two families of curves in the $x-t$ plane play particularly important roles. These curves are called the characteristic curves and are defined by the differential

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[1] R. Courant and K. Friedrichs, "Interaction of shock and rarefaction waves in one-dimensional motion," OSRD Report No. 1567.

[2] "Supersonic flow and shock waves (shock wave manual)," AMP Report 38.2R.

equations $dx/dt = u \pm c$, where u is the velocity of the fluid and c is the local velocity of sound. When the flow is a simple wave, one family of these characteristics is the family of straight lines discussed above. A fundamental theorem on simple waves says that flows adjacent to flows of steady state are simple waves. Hence, the rarefaction wave described above, produced by acceleration of the piston, is a simple wave, since the fluid is initially in a steady state. In the present case the family of characteristics which are straight lines is that one which has the plus sign.

In a simple wave the pressure p, the density ρ, and the velocity of sound c are related to the velocity of the fluid by the following formulas:

$$p = p_i\left[1 + \frac{\gamma-1}{2}\frac{u-u_i}{c_i}\right]^{2\gamma/(\gamma-1)}, \tag{1}$$

$$\rho = \rho_i\left[1 + \frac{\gamma-1}{2}\frac{u-u_i}{c_i}\right]^{2/(\gamma-1)}, \tag{2}$$

$$c = \left(\frac{\gamma p}{\rho}\right)^{\frac{1}{2}} = c_i\left[1 + \frac{\gamma-1}{2}\frac{u-u_i}{c_i}\right], \tag{3}$$

where subscript i refers to the initial constant state of the gas before the rarefaction has affected the flow and γ is the ratio of specific heats.

The properties of shock waves will now be discussed. Let the subscripts 0 and 1 refer, respectively, to the low and high pressure sides of the shock. There are three shock conditions which arise from the conditions of conservation of mass, momentum, and energy across the shock. These relations can be written

$$\rho_0{}^2 v_0{}^2 = \rho_0\rho_1\frac{p_1-p_0}{\rho_1-\rho_0} = \rho_1{}^2 v_1{}^2, \tag{4}$$

$$\rho_0 u_0 v_0 + p_0 = \rho_1 u_1 v_1 + p_1, \tag{5}$$

$$\rho_1 = \rho_0\frac{\mu^2 p_0 + p_1}{\mu^2 p_1 + p_0}. \tag{6}$$

In the above equations

$$v_0 = u_0 - \dot{x}_1, \tag{7}$$

$$v_1 = u_1 - \dot{x}_1, \tag{8}$$

where x_1 indicates the position of the shock and the dot indicates differentiation with respect to time, so that $\dot{x}_1$ is the velocity of the shock. The constant μ^2 equals $(\gamma-1)/(\gamma+1)$.

It will be convenient to measure the shock strength in terms of the excess pressure ratio $\xi = (p_1 - p_0)/(p_0)$. In terms of ξ the shock conditions may be written

$$v_0 = c_0(1+\nu\xi)^{\frac{1}{2}}, \quad (\nu = 1/(1+\mu^2)), \tag{9}$$

$$\xi = (\gamma v_0/c_0{}^2)(u_0 - u_1), \tag{10}$$

and from (9) and (10)

$$v_0 = \frac{-\nu\gamma(u_1-u_0)}{2} + \left(\frac{\nu^2\gamma^2(u_1-u_0)^2}{4} + c_0{}^2\right)^{\frac{1}{2}}, \tag{11}$$

$$\dot{x}_1 = u_0 + \frac{\nu\gamma(u_1-u_0)}{2} - \left(\frac{\nu^2\gamma^2(u_1-u_0)^2}{4} + c_0{}^2\right)^{\frac{1}{2}}, \tag{11a}$$

$$u_1 - u_0 = \frac{-c_0\xi}{\gamma(1+\nu\xi)^{\frac{1}{2}}}, \tag{12}$$

$$c_1 = c_0\left(\frac{(1+\xi)(1+\nu\mu^2\xi)}{(1+\nu\xi)}\right)^{\frac{1}{2}}. \tag{13}$$

The process of interaction will now be considered. The quantities on the low and high pressure sides of the shock before the interaction will be denoted by the subscripts A and B, respectively. Thus, before the interaction of the shock wave and rarefaction wave $p_1 = p_B$, $u_1 = u_B$, $\rho_1 = \rho_B$, $c_1 = c_B$, $p_0 = p_A$, $u_0 = u_A$, $\rho_0 = \rho_A$, $c_0 = c_A$. Using this notation Eqs. (1), (2), and (3) may be rewritten as follows:

$$p_0 = p_A\left[1 + \frac{\gamma-1}{2}\frac{u_0-u_A}{c_A}\right]^{2\gamma/(\gamma-1)}, \tag{1a}$$

$$\rho_0 = \rho_A\left[1 + \frac{\gamma-1}{2}\frac{u_0-u_A}{c_A}\right]^{2/(\gamma-1)}, \tag{2a}$$

$$c_0 = c_A\left[1 + \frac{\gamma-1}{2}\frac{u_0-u_A}{c_A}\right]. \tag{3a}$$

Since in the wave $u_0 < u_A$ (u_0 and u_A are negative in our coordinates), it is seen that $p_0 < p_A$, $\rho_0 < \rho_A$,

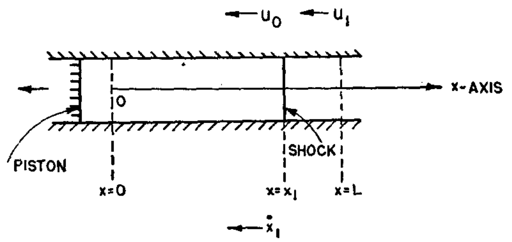

FIG. 1. Collision between shock and rarefaction wave.

$c_0 < c_A$. Furthermore, the head of the wave travels with velocity $u_A + c_A$.

Furthermore, if ξ_1 denotes the excess pressure ratio of the shock before interaction, then

$$u_B - u_A = \frac{-c_A \xi_1}{\gamma(1+\nu\xi_1)^{\frac{1}{2}}}. \tag{12a}$$

The rarefaction wave is completely determined by the velocity with which the piston is withdrawn from the tube. It will, therefore, be assumed that u_0, hence also p_0, ρ_0, c_0 are known functions of x and t.

It is the problem of the present paper to find x_1, $\dot{x}_1$, u_1, ξ_1 as a function of time as the shock wave moves along its path. If u_1 is known as a function of u_0, then the differential Eq. (11a) can be solved to give $x_1(t)$ and $\dot{x}_1(t)$. Then also $\xi(t)$ can be found from (10). Therefore, in addition to Eq. (11), another relation between v_0, u_0, u_1 is needed so that v_0, u_1 can be solved in terms of u_0 alone. Having found u_1 as a function of u_1 the procedures outlined above can be used to find the desired quantities.

As explained previously, the fluid on the high pressure side of the shock wave is *not* isentropic. Consider the shock at a given time. The region on the high pressure side can be divided into small regions in which the pressure, density, velocity, and entropy are considered constant. In particular consider the small region immediately adjacent to the shock. The fluid particles which have just passed through the shock will move toward the left with a velocity greater (i.e., more negative in the coordinate system chosen) than the fluid in the small region being considered. Hence the particles which have just passed through the shock may be considered the front of a rarefaction wave which will pass through the small region. Inasmuch as the fluid is isentropic in the region considered, Eq. (1) describes the relation between pressure and velocity.

In Eq. (1) we shall write

$$\begin{aligned} &p = p_1, \quad u = u_1, \quad c = c_1, \\ &u_i = u_1 + \Delta u_1, \quad c_i = c_1 + \Delta c_1, \\ &p_i = p_1 + \Delta p_1, \end{aligned} \tag{14}$$

passing to the limit, the following differential equation is obtained:

$$dp_1/du_1 = \gamma p_1/c_1. \tag{15}$$

By means of Eqs. (12), (13), and (1a), Eq. (15) can be converted into a differential equation with ξ and u_0 as variables. By integrating this differential equation, a new relation which gives ξ as a function of u_0 is obtained. By substituting for ξ in (12), $u_1 - u_0$ is found as a function of u_0 alone, and the procedure outlined above may be used to find $x_1(t)$, $\dot{x}_1(t)$, $\xi(t)$, and finally $u_1(t)$.

A particularly simple and interesting case is the one in which the shock is weak. As shown in reference 2, the entropy jump across a sufficiently weak shock is proportional to ξ^3. Thus if the shock is so weak that all powers of ξ higher than the second can be neglected the fluid on the high pressure side of the shock can be considered isentropic.

When the value of c_1 corresponding to isentropic flow is substituted in (15), we obtain

$$p_1 = p_B\left[1 + \frac{\gamma-1}{2}\frac{u_1 - u_B}{c_B}\right]^{2\gamma/(\gamma-1)}. \tag{1b}$$

This result was to be expected from the manner of derivation of (15). It is also to be expected from the fact that we have a non-uniform isentropic state adjacent to the constant state given by p_B, u_B, ρ_B, therefore the wave on the high pressure side of the shock must be a simple wave and the simple wave relations must hold.

From (1b)

$$\xi = -1 + (1+\xi_1)\frac{\left[1 + \frac{\gamma-1}{2}\frac{u_1 - u_B}{c_B}\right]^{2\gamma/(\gamma-1)}}{\left[1 + \frac{\gamma-1}{2}\frac{u_0 - u_A}{c_A}\right]^{2\gamma/(\gamma-1)}},$$

and (13) may be replaced by its isentropic counterpart

$$c_0/c_1=(1+\xi)^{(\gamma-1)/2\gamma}. \quad (17)$$

From (17)

$$c_A/c_B=(1+\xi_1)^{(\gamma-1)/2\gamma}. \quad (17a)$$

For the special case of weak shocks, which is to be treated, the general procedure outlined above will be modified as follows: From the nature of the problem it is expected that u_1 can be expanded in powers of ξ_1.

$$u_1(\xi_1, u_0) \doteq f_0(u_0)+\xi_1 f_1(u_0)+\xi_1^2 f_2(u_0)+\cdots. \quad (18)$$

The functions f_0, f_1, $\cdots$ etc. will be sought. Using (18), (17a), and (12a) in (16), ξ is obtained as a power series in ξ_1 which involves the f_i's as coefficients. This series is not valid beyond the second power of ξ_1 because of the assumption of isentropic flow. The expression for ξ so obtained is substituted in Eq. (9) to obtain v_0 in an expansion in ξ_1, which again will not be valid beyond the second power of ξ_1. By substituting u_1 as given by Eq. (18) into Eq. (11) an alternative expansion of v_0 in powers of ξ_1 is obtained. Comparison of the coefficients of powers of ξ_1 of both expansions yields the functions f_0, f_1, and f_2.

Having found these functions, u_1 is known as a function of u_0; and x_1, $\dot{x}_1$, ξ, u_1, u_0 can be found as functions of time as the shock moves along its path, as explained previously.

III. RESULTS

A. Infinitesimal Shocks

For initially infinitesimal shocks $\xi_1=0$. It is easily verified that $f_0=u_0$ and that $\xi=0$. Therefore, an infinitesimal shock remains an infinitesimal shock when passing through a rarefaction wave.

Furthermore, $\dot{x}_1=u_0-c_0$, which is the equation of a backward characteristic of the simple wave. Hence, infinitesimal shocks move along the characteristics of the simple wave. This result was to have been expected from the role played by characteristics as propagators of small disturbances.

B. Weak Shocks

Weak shocks are defined as those shocks for which the powers of ξ higher than the first can be neglected.

It is found that $f_1=-c_A/\gamma$. Furthermore, to this approximation

$$u_1-u_B=u_0-u_A, \quad (19)$$

which shows that for weak shocks the increase in the velocity of the fluid behind the shock is equal to the increase in velocity which occurs ahead of the shock, or to put the result in different words, the velocity of the fluid behind the shock differs from the velocity ahead of the shock only by a constant.

The differential equation for the position of the shock is

$$\dot{x}_1=u_0+(\nu\gamma/2)(u_B-u_A)-c_0. \quad (20)$$

The path followed by a weak shock is, therefore, no longer a characteristic of the rarefaction wave. Moreover, since u_B-u_A is negative, the shock wave will travel faster than the velocity of an infinitesimal disturbance (i.e., a sound wave) through the rarefaction region.

The strength of the shock as measured by its excess pressure ratio increases as it passes through the rarefaction region. The expression for ξ is

$$\xi=\xi_1(c_A/c_0). \quad (21)$$

From (3a) it is clear that c_0/c_A decreases so that ξ increases. An initially weak shock may, therefore, become strong when interacting with the rarefaction.

In order to present a specific example of the method of finding the velocity of shock and of the fluid before and behind the shock, a special rarefaction wave will be considered, namely, the rarefaction wave which results when the piston undergoes infinite acceleration in changing its initial velocity u_A to some final constant velocity. Such a rarefaction wave is called a centered rarefaction wave. The velocity u_0 is given by

$$u_0-u_A=(1-\mu^2)[(x/t)-c_A-u_A]. \quad (22)$$

For simplicity, c_A will be taken equal to unity, u_A will be taken equal to zero, and the distance from the origin to L at which the shock and rarefaction interact will also be taken as unity. Likewise the time interval required by the head of the rarefaction wave to move from the origin to the position $x=L$ will be taken as unity. These simplifications correspond merely to a choice of units and a frame of reference from which the phenomenon is viewed.

Then for $t>1$

$$x_1 = t\left[\frac{4(1-\mu^2)-u_B}{4\mu^2(1-\mu^2)}t^{-2\mu^2} - \frac{4(1-\mu^2)^2-u_B}{4\mu^2(1-\mu^2)}\right], \quad (23)$$

$$\dot{x}_1 = \left[\frac{4(1-\mu^2)(1-2\mu^2)-u_B(1-2\mu^2)}{4\mu^2(1-\mu^2)}t^{-2\mu^2} - \frac{4(1-\mu^2)^2-u_B}{4\mu^2(1-\mu^2)}\right], \quad (24)$$

$$u_0(t) = (1-\mu^2)\left[\frac{4(1-\mu^2)-u_B}{4\mu^2(1-\mu^2)}t^{-2\mu^2} - \frac{4(1-\mu^2)^2-u_B}{4\mu^2(1-\mu^2)} - 1\right], \quad (25)$$

$$u_1(t) = u_0(t) + u_B, \quad (26)$$

and the various quantities which are desired are completely solved for.

C. Moderately Strong Shocks

A moderately strong shock is one in which all powers of higher than the second can be neglected. For such shocks it is found that

$$f_2 = (\nu/2\gamma)c_A. \quad (27)$$

Therefore, $u_1 - u_B = u_0 - u_A$ as for weak shocks. This result is somewhat surprising, inasmuch as one might expect the relation between u_1 and u_0 to be non-linear for moderately strong shocks. It therefore appears from this point of view that instead of using the excess pressure ratio as a criterion of shock strength it would be more satisfactory to use the non-dimensional velocity difference $(u_1-u_0)/c_0$. Thus, weak disturbances might better be defined as those such that all powers of $(u_1-u_0)/c_0$ beyond the first may be neglected. However, we shall continue to use our original definition.

For moderately strong shocks the excess pressure ratio is given by

$$\xi = \xi_1\frac{c_A}{c_0}\left[1+\frac{\gamma+1}{4\gamma}\xi_1\left(\frac{c_A}{c_0}-1\right)\right]. \quad (28)$$

Since $c_A/c_0>1$, the strength of the moderately strong shock increases at a faster rate than that of a weak shock in terms of c_A/c_0. Also the equation of motion of the shock is

$$\dot{x}_1 = u_0 + \frac{\nu\gamma}{2}(u_B-u_A) - c_0 - \frac{\gamma^2\nu^2(u_B-u_A)^2}{8c_0}, \quad (29)$$

which shows that the moderately strong shock travels faster through the rarefaction region than the weak shock.

As before, when u_0 is given as a function of x and t, (29) may be integrated to give the position and velocity of the shock as a function of the time and then x_1, $\dot{x}_1$, u_0, u_1, ξ may be found as functions of the time or position of the shock.

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