and either

$$\mu = +1, \quad A = -B/a \tag{61}$$

or

$$\mu = -1, \quad A = 0. \tag{62}$$

Equations (46) and (47) give in either case

$$f = m/a, \tag{63}$$

so that, with the transformation $t' = t + aZ$, the metric tensor becomes diagonal. Also, with (61), $(F^{41})'$ vanishes and

$$(F^{31})' = -\frac{B}{a(-g)^{\frac{1}{2}}}, \tag{64}$$

while, with (62),

$$(F^{31})' = F^{31} = 0$$

and

$$(F^{41})' = B/(-g)^{\frac{1}{2}}. \tag{65}$$

Thus the two cases correspond, respectively, to an axial magnetic field and radial electric field in the cylindrically symmetric case. These are, of course, equivalent under a duality rotation to an axial electric field and radial magnetic field, respectively.

The metric in the two cases is

$$l = x/\xi, \quad f = \xi, \tag{66}$$

and

$$f = x/\xi, \quad l = x\xi, \tag{67}$$

where we have absorbed "a" by a suitable transformation and ψ and ξ are given by (58) and (55), respectively. These solutions already occur in the literature.[2]

[2] W. B. Bonnor, Proc. Phys. Soc. **A66**, 145 (1953).

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Local Operator Algebras in the Presence of Superselection Rules

PAUL FEDERBUSH*
Department of Mathematics, University of Michigan, Ann Arbor, Michigan

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An introductory study is made of the relation between the local algebra of observables associated to a region of space–time and the algebra of all operators (disregarding superselection rules) associated to the same region of space–time. With a reasonable axiomatization of the problem and assuming the local algebra of observables is a factor of type I, II_∞, or III, the relation between the two algebras is specified up to unitary equivalence in terms of algebraic invariants.

The objects of study in this paper are some of the operator algebras arising in quantum field theory. The general problem to be considered is the relation between the local algebra of observables associated to a region in space–time and the local algebra of all operators constructed from the fields (disregarding superselection rules) associated to the same region in space–time. The situation is abstracted to a mathematical system with several axioms, axioms that seem reasonable from a physical point of view. Some insight is gained into the factor types of the local algebras—assuming indeed that the algebras are factors.

We proceed to describe the mathematical system of interest. First we presume a discrete group, the superselection group, called G. We denote elements of G as g_i, $i \in I$; I is an indexing set, g_0 the identity. To each g_i we associate a Hilbert space H_i; H_0 is associated to g_0. The total Hilbert space H is the orthogonal direct sum of the H_i:

$$H = \oplus H_i. \tag{1}$$

O is a * algebra in $B(H)$, the algebra of bounded operators. O_i, $i \in I$, are vector subspaces of O considered as a vector space over the complex numbers:

$$O - \sum_i O_i. \tag{2}$$

This is taken to mean that O is the algebra generated by finite sums of elements from the subspaces O_i. O_0 is the algebra of local observables.

Using the notation

$$g_i g_j = g_{(i,j)},$$

we assume

$$O_i O_j \subset O_{(i,j)}, \tag{3}$$

$$O_i H_j \subset H_{(i,j)}. \tag{4}$$

* Sloan Foundation Fellow.

Observation 1: $O_i^* = O_{i^{-1}}$, with the notation, $(g_i)^{-1} = g_{i^{-1}}$.

Observation 2: The weak closures of the O_i still satisfy (3), (4), and the relation in Observation 1.

We assume from now on that O_i and O are weakly closed, reinterpreting (2) to mean that O is the weakly closed algebra generated by finite sums of the type indicated.

We further require

$$\dim H_i = N, \quad i \in I, \tag{5}$$

that is, the cardinality of bases of all the H_i are the same. The final essential property we require is the existence of a vector ψ contained in H_0 that is a separator and generator for O. In the physical system this is the vacuum vector.

Observation 3: O_0 is a sub-w^* algebra of O.

Observation 4: ψ is a separator and generator for O_0 restricted to H_0.

A system satisfying all the above conditions will be called a superselection paired local algebra system, an SPLA. We wish to study the relation between O_0 and O in an SPLA. In general, the structure is not yet rigid enough to be able to say anything simple. We will restrict ourselves to the case when O_0 is a factor, in which case a structure theorem is accessible. First we give two examples to show that it is possible for either O_0 or O to be a factor without the other being a factor.

Example 1: G contains only two elements, g_0 and g_1. H_0 and H_1 are two-dimensional with bases (a_1, a_2) and (b_1, b_2), respectively. In the basis (a_1, a_2, b_1, b_2) for H we choose

$$\psi = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

O_0 consists of diagonal matrices of the form

$$\left(\begin{array}{cc|cc} A & & & \\ & B & & O \\ \hline & & A & \\ O & & & B \end{array}\right), \tag{7}$$

and O_1 the matrices of the form

$$\left(\begin{array}{cc|cc} & O & & A \\ & & B & \\ \hline & A & & \\ B & & & O \end{array}\right). \tag{8}$$

This provides an SPLA with O a factor and O_0 not.

Example 2: Let G be a group with N elements and each H_i a Hilbert space of dimension M^2. We view H as the tensor product $H_A \times H_B$, H_A and H_B of dimensions M^2 and N, respectively. We associate each element of some basis $\{h_i\}$ of H_B with an element of G, $h_i \leftrightarrow g_i$, so that each g_i acts as a unitary operator in H_B, $g_i h_j = h_{(i,j)}$. We let R be a factor, algebraically isomorphic with the full $M \times M$ matrix algebra, in H_A with ϕ as separator and generator. Finally, we identify

$$\begin{aligned} O_i &\sim R \times g_i, \\ H_i &\sim H_A \times h_i, \\ \psi &\sim \phi \times h_0. \end{aligned} \tag{9}$$

This is an SPLA with O_0 a factor and O not a factor, unless G has only one element. Note that this construction generalizes in an obvious way to the case when N and M may be infinite, and R may be any factor with a separator and generator. We will see later that this is a canonical situation and will be referred to as a trivial SPLA.

The following result provides the basic device employed in this paper to study an SPLA.

Lemma: Let $a_1, a_2, \cdots, a_s$ be in $B(H)$, F a factor of infinite type in $B(H)$. Suppose

$$a_i^* a_j \in F \quad \text{all} \quad i \text{ and } j. \tag{10}$$

Then there exists a partial isometry U, with initial projector onto the span of the ranges of the a_i, such that

$$U a_i \in F \quad \text{all } i. \tag{11}$$

Proof: By induction, assume the lemma true for $s - 1$ (the case $s = 0$ is trivial), so that

$$V a_i \in F, \quad 1 \le i \le s - 1, \tag{12}$$

with V a partial isometry onto the span of the ranges of a_i, $1 \le i \le s - 1$. We claim $V a_s \in F$, as

$$a_i^* V^* V a_s = a_i^* a_s \in F. \tag{13}$$

Therefore

$$\frac{1}{\epsilon + \sum_{i=1}^{s-1} (V a_i a_i^* V^*)} \sum_1^{s-1} (V a_i a_i^* V^*) V a_s \in F. \tag{14}$$

In the limit $\epsilon \xrightarrow[st.]{} 0^+$ we see that $Va_s \in F$, since F is strongly closed, and the initial projector of V is onto the ranges of the a_i, $i \leq s - 1$:

$$\begin{aligned} a_s &= V^*Va_s + (1 - V^*V)a_s, \\ Va_s &= V(V^*V)a_s, \qquad (15) \\ a_s^*a_s &= a_s^*V^*Va_s + a_s^*(1 - V^*V)a_s, \end{aligned}$$

and therefore

$$a_s^*(1 - V^*V)a_s \in F. \qquad (16)$$

We write the polar decomposition of $(1 - V^*V)a_s$:

$$(1 - V^*V)a_s = V'^*h. \qquad (17)$$

We note that $h \in F$. Therefore

$$V'(1 - V^*V)a_s \in F. \qquad (18)$$

Using the fact that F is a factor of infinite type, we can move the final projectors of V and V' to be orthogonal and construct a new U, so that $Ua_i \in F$ for all i and its intial projector is onto the span of the ranges of the a_i, $1 \leq i \leq s$. ($U = W_1V + W_2V'$ for suitable isometries W_1 and W_2 in F.)

We see that the above lemma is also true for a countably infinite number of a_i instead of a finite number. A variation of the above argument will prove the result for factors of type I_N with a separator and generator. We do not consider factors of type II_1.

From each O_i select a sequence of elements a_j^i, $j = 1, 2, \cdots, N$, N the dimension of the H_i, such that $a_j^i\psi$ is a basis for H_i. We assume all our Hilbert spaces are separable at this point. Denote O_0 restricted to H_0 by F, O_i restricted to H_0 by F_i, and a_j^i restricted to H_0 by b_j^i. We note that

$$b_j^{i*}b_k^i \in F, \quad \text{all} \quad j \text{ and } k, \qquad (19)$$

as

$$O_{(i)^{-1}}O_i \subset O_0. \qquad (20)$$

Therefore we are in the situation of the lemma. Find a partial isometry V_i with

$$V_ib_j^i \in F. \qquad (21)$$

If one is in the I_N factor situation, it is trivial that V_i is unitary. We proceed to show that if the factor is of infinite type, V_i may be picked unitary.

The final projector of V_i is a projection in F. If this projection is equivalent to 1, then it is clear that a new V_i may be picked to be unitary. In general, let P_i be this projection in F. Using V_i, we identify H_i with P_iH_0, and in this new basis for each H_i:

$$b_j^i \in P_iF, \quad \text{all} \quad i \text{ and } j. \qquad (22)$$

(We carelessly identify operators in F_i and F that have identical matrix form in the present basis system for H_0 and H_i.) Also

$$F_i \subset P_iF.$$

Now $O_iO_0 \subset O_i$ implies $F_iF \subset F_i$, and therefore F_i is a right ideal of F. Since the b_j^i span P_iH_0 for fixed i, the weak closure of F_i, F_i^{cL}, equals P_iF. Calling C_i, O_0 restricted to H_i, it follows from $O_iO_{(i)^{-1}} \subset O_0$ that C_i contains P_iFP_i, and from $O_{(i)^{-1}}O_0O_i \subset O_0$ that $C_i \subset F$ and therefore $C_i = P_iFP_i$. (C_i is weakly closed as it is $E'O_0E'$ with E' a projection commuting with O_0.) F is represented by C_i, uniformly continuously, and thus either F is isomorphic (as a C^* algebra) to P_iFP_i or else F modulo a nontrivial uniformly closed two-sided ideal is isomorphic to P_iFP_i. The second situation is impossible and the first situation implies $P_i \sim 1$. Thus we may pick V_i unitary and work in a basis with $F_i \subset F$, $F_i^{cL} = F$.

Using the axiom $O_iO_j \subset O_{(i,j)}$, we easily show that, in the present bases, O_i restricted to H_i (and weakly closed, for every i and j) is equal to F.

If we now let X be the restriction of an operator X in O_0 to H_0, then the image of X in H_i is $R_iXR_i^{-1}$ with R_i unitary, $R_iFR_i^{-1} = F$, and R_i is independent of X. To establish this we observe that since F has a separator and generator, all isomorphisms are spatial; and the image of O_0 in each H_i is faithful, since $F/I = F$ with I a nontrivial uniformly closed two-sided ideal is impossible. F and the R_i completely describe O_0.

It is important now to establish that each F_i is weakly closed. Consider an increasing sequence of projectors in F_i, weakly approaching 1. Since $O_0O_iO_0 \subset O_i$, such a sequence is easy to come by. We claim the corresponding sequence of elements in O_i (call them T_1^i, $T_2^i, \cdots$) converges weakly, and therefore converges to an element 1_i in O_i. From the fact that $O_iO_{(i)^{-1}} \subset O_0$ and $O_{(i)^{-1}}O_i \subset O_0$, it follows that each of the elements T_k^i restricted to each H_i is a partial isometry. Further, from $O_iO_0 \subset O_i$ it follows that the partial isometries in the sequences are increasing; that is, T_r^i restricted to the range of the initial projector of T_{r-1}^i restricted to H_k, agrees with T_{r-1}^i restricted to H_k. It easily follows that T_r^i converges to an element 1_i and F_i is weakly closed. We may now study O_i.

The essential observation here is that knowledge of the single element 1_i in O_i that restricts to the identity in H_0 is sufficient to characterize O_i, as all of O_i is generated by multiplying 1_i by the algebra O_0. It must be recalled that since $\psi \in H_0$ separates O_i, the restriction of an operator in O_i to H_0 determines the operator.

A simple calculation is required to determine the conditions on S^i_j, the restriction of 1_i to H_j.

The first relation we obtain is

$$1_i X^{\sim} = (R_i^{-1} X R_i)^{\sim} 1_i, \quad (23)$$

where $(R_i^{-1} X R_i)^{\sim}$ is the element of O_0 that restricts to $R_i^{-1} X R_i$ in H_0. This relation holds because these two elements have the same restriction to H_0. From (23) upon examining the restriction of each side of the equation to H_k it follows that

$$X = (R_k)^{-1}(S^i_k)^{-1} R_{(i,k)}(R_i)^{-1} X R_i (R_{(i,k)})^{-1} S^i_k R_k. \quad (24)$$

Since this holds for all X in F, we arrive at

$$S^i_k \cong R_{(i,k)}(R_i)^{-1}(R_k)^{-1}, \quad (25)$$

where the equivalence means that the two sides of the equation, both unitary operators, induce the same automorphism of F. There is only one other set of conditions on the S^i_k to guarantee that the F, R_i, and S^i_k determine an SPLA. This set of relations is obtained by requiring that

$$1_i 1_j \in O_0 1_{(i,j)}. \quad (26)$$

This equation becomes

$$R_{(k,l)}(R_k)^{-1}(S^i_j)^{-1} R_k (R_{(k,l)})^{-1} S^i_{(j,l)} S^j_l = S^k_l. \quad (27)$$

Equations (25) and (27) are the algebraic relations to study for constructing an SPLA from F and the group G. Equation (25) merely states that the map $g_i \to (R_i)^{-1}$ induces a homomorphism from G into the group of outer automorphisms of F. Equation (27) gives the conditions on the S^i_j in terms of the homomorphism just mentioned. Equation (25) determines S^i_k up to a complex number of modulus one. Thus (27) may be looked upon as conditions on these complex factors, since this equation automatically holds up to a complex factor. [It is easily checked that the two sides of (27) induce the same automorphism of F.] Unfortunately, we have not been able to find whether every homomophism of G into Out (F) has at least one solution to (27). However, if there is one solution to (27), and calling this solution S^i_j, we look for all other solutions by writing $(S^i_j)' = \lambda^i_j S^i_j$ with λ^i_j a complex number of modulus one, $(S^i_j)'$ any other solution. The λ^i_j must satisfy the algebraic relations given by (27). Interestingly enough, these state that λ^i_j is a two cocycle of the usual cochain complex of the group G with coefficients in the unimodular complex numbers, the circle group. By changing the basis for the H_i by a complex factor (multiplying each basis element by a complex factor, the factors being constant in each H_i), λ^i_j is changed by a coboundary. Therefore the solutions of (27)—if they exist at all—are in one-to-one correspondence with $H^2(G, T^1)$. Unfortunately, this correspondence is not canonical as we have derived it.

Theorem: In an SPLA with $O_0 = F$, F a factor of type I, II_∞, or III, and group G, O is determined up to unitary equivalence (under basis changes in each H_i separately) by a homomorphism of G into Out (F) and an element of $H^2(G, T^1)$.

It is attractive to conjecture that if, instead of assuming F to be a factor, it is assumed that the restrictions of O_0 to all the H_i are isomorphic and faithful, the same result follows with $H^2(G, T^1)$ replaced by $H^2(G, Z)$, Z the center of F.

Corollary: If F is of type I and G is Abelian with one generator, then the SPLA is trivial; and if G contains more than one element, O is therefore not a factor.

The corollary follows from the fact that, under the stated conditions, Out (F) and $H^2(G, T^1)$ are both trivial.

The corollary may have some implications for the factor-type problem in quantum field theory.

We conclude by remarking that the results in this paper are clearly far from definitive. The obvious examples of several free boson and fermion fields with various choices of superselection rules should be computed to relate the O_0 and O in these examples. The free boson field is studied in detail elsewhere.[1] The result here should also be related to the deeper results of Borchers[2] that also relate the rings of different regions of space to each other.

[1] H. Araki, J. Math. Phys. 5, 1 (1964); 4, 1343 (1963).

[2] H. J. Borchers, Commun. Math. Phys. 1, 281 (1965).