and either
\[ \mu = +1, \quad A = -B/a \] (61)
or
\[ \mu = -1, \quad A = 0. \] (62)

Equations (46) and (47) give in either case
\[ f = m/a, \] (63)
so that, with the transformation \( t' = t + aZ \), the metric tensor becomes diagonal. Also, with (61), \((F^{31})'\) vanishes and
\[ (F^{31})' = -\frac{B}{a(-g)^{\frac{1}{2}}}, \] (64)
while, with (62),
\[ (F^{31})' = F^{31} = 0 \] and
\[ (F^{41})' = B/(-g)^{\frac{1}{2}}. \] (65)

Thus the two cases correspond, respectively, to an axial magnetic field and radial electric field in the cylindrically symmetric case. These are, of course, equivalent under a duality rotation to an axial electric field and radial magnetic field, respectively.

The metric in the two cases is
\[ l = x/\xi, \]
\[ f = \xi, \] (66)
and
\[ f = x/\xi, \]
\[ l = x/\xi. \] (67)

Where we have absorbed "a" by a suitable transformation and \( \psi \) and \( \xi \) are given by (58) and (55), respectively. These solutions already occur in the literature.²


Local Operator Algebras in the Presence of Superselection Rules

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An introductory study is made of the relation between the local algebra of observables associated to a region of space–time and the algebra of all operators (disregarding superselection rules) associated to the same region of space–time. With a reasonable axiomatization of the problem and assuming the local algebra of observables is a factor of type I, II, or III, the relation between the two algebras is specified up to unitary equivalence in terms of algebraic invariants.

The objects of study in this paper are some of the operator algebras arising in quantum field theory. The general problem to be considered is the relation between the local algebra of observables associated to a region in space–time and the local algebra of all operators constructed from the fields (disregarding superselection rules) associated to the same region in space–time. The situation is abstracted to a mathematical system with several axioms, axioms that seem reasonable from a physical point of view. Some insight is gained into the factor types of the local algebras—assuming indeed that the algebras are factors.

We proceed to describe the mathematical system of interest. First we presume a discrete group, the superselection group, called \( G \). We denote elements of \( G \) as \( g_i, i \in I; I \) is an indexing set, \( g_0 \) the identity. To each \( g_i \) we associate a Hilbert space \( H_i; H_0 \) is associated to \( g_0 \). The total Hilbert space \( H \) is the orthogonal direct sum of the \( H_i \):
\[ H = \bigoplus H_i. \] (1)

\( O \) is a \( * \) algebra in \( B(H) \), the algebra of bounded operators. \( O_i, i \in I \), are vector subspaces of \( O \) considered as a vector space over the complex numbers:
\[ O = \bigoplus_i O_i. \] (2)

This is taken to mean that \( O \) is the algebra generated by finite sums of elements from the subspaces \( O_i \). \( O_0 \) is the algebra of local observables.

Using the notation
\[ g_0 g_i = g(i,i), \]
we assume
\[ O_i O_j \subseteq O(i,j), \] (3)
\[ O_i H_j \subseteq H(i,j). \] (4)
Observation 1: $O_i^* = O_{i-1}$, with the notation, $(g_i)^{-1} = g_{i-1}$.

Observation 2: The weak closures of the $O_i$ still satisfy (3), (4), and the relation in Observation 1.

We assume from now on that $O_i$ and $O$ are weakly closed, reinterpreting (2) to mean that $O$ is the weakly closed algebra generated by finite sums of the type indicated.

We further require
\[
\dim H_i = N, \quad i \in I, \tag{5}
\]
that is, the cardinality of bases of all the $H_i$ are the same. The final essential property we require is the existence of a vector $\psi$ contained in $H_0$ that is a separator and generator for $O$. In the physical system this is the vacuum vector.

Observation 3: $O_0$ is a sub-$w^*$ algebra of $O$.

Observation 4: $\psi$ is a separator and generator for $O_0$ restricted to $H_0$.

A system satisfying all the above conditions will be called a superselection paired local algebra system, an SPLA. We wish to study the relation between $O_0$ and $O$ in an SPLA. In general, the structure is not yet rigid enough to be able to say anything simple. We will restrict ourselves to the case when $O_0$ is a factor, in which case a structure theorem is accessible. First we give two examples to show that it is possible for either $O_0$ or $O$ to be a factor without the other being a factor.

Example 1: $G$ contains only two elements, $g_0$ and $g_1$. $H_0$ and $H_1$ are two-dimensional with bases $(a_1, a_2)$ and $(b_1, b_2)$, respectively. In the basis $(a_1, a_2, b_1, b_2)$ for $H$ we choose
\[
\psi = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{6}
\]

$O_0$ consists of diagonal matrices of the form
\[
\begin{pmatrix} 1 & B \\ B & 0 \end{pmatrix}, \tag{7}
\]
and $O_1$ the matrices of the form
\[
\begin{pmatrix} O & A \\ A & O \end{pmatrix}. \tag{8}
\]
This provides an SPLA with $O$ a factor and $O_0$ not.

Example 2: Let $G$ be a group with $N$ elements and each $H_i$ a Hilbert space of dimension $M^2$. We view $H$ as the tensor product $H_A \times H_B$, $H_A$ and $H_B$ of dimensions $M^2$ and $N$, respectively. We associate each element of some basis $\{h_i\}$ of $H_B$ with an element of $G$, $h_i \leftrightarrow g_i$, so that each $g_i$ acts as a unitary operator in $H_B$, $g_i h_j = h_{i,j}$. We let $R$ be a factor, algebraically isomorphic with the full $M \times M$ matrix algebra, in $H_A$, and $\phi$ as separator and generator. Finally, we identify
\[
O_i \sim R \times g_i, \\
H_i \sim H_A \times h_i, \tag{9}
\]
\[
\psi \sim \phi \times h_0.
\]
This is an SPLA with $O_0$ a factor and $O$ not a factor, unless $G$ has only one element. Note that this construction generalizes in an obvious way to the case when $N$ and $M$ may be infinite, and $R$ may be any factor with a separator and generator. We will see later that this is a canonical situation and will be referred to as a trivial SPLA.

The following result provides the basic device employed in this paper to study an SPLA.

Lemma: Let $a_1, a_2, \ldots, a_s$ be in $B(H)$, $F$ a factor of infinite type in $B(H)$. Suppose
\[
a_i^* a_j \in F \quad \text{all } i \text{ and } j. \tag{10}
\]
Then there exists a partial isometry $U$, with initial projector onto the span of the ranges of the $a_i$, such that
\[
U a_i \in F \quad \text{all } i. \tag{11}
\]
Proof: By induction, assume the lemma true for $s - 1$ (the case $s = 0$ is trivial), so that
\[
V a_i \in F, \quad 1 \leq i \leq s - 1, \tag{12}
\]
with $V$ a partial isometry onto the span of the ranges of $a_i$, $1 \leq i \leq s - 1$. We claim $V a_s \in F$, as
\[
a_i^* V^* V a_s = a_i^* a_s \in F. \tag{13}
\]
Therefore
\[
\frac{1}{\epsilon + \sum_{i=1}^{s-1} (V a_i a_i^* V^*)} \sum_{i=1}^{s-1} (V a_i a_i^* V^*) V a_s \in F. \tag{14}
\]
In the limit \( \epsilon \to 0^+ \) we see that \( V\alpha \in F \), since \( F \) is strongly closed, and the initial projector of \( V \) is onto the ranges of the \( a_i \), \( i \leq s - 1 \):

\[
\begin{align*}
\alpha_i &= V^*V\alpha_i + (1 - V^*V)a_i, \\
V\alpha_i &= V(V^*V)a_i, \\
a^*_i a_i &= a^*_i V^*V\alpha_i + a^*_i (1 - V^*V)a_i,
\end{align*}
\]

and therefore

\[
a_i^*(1 - V^*V)a_i \in F. \tag{15}
\]

We write the polar decomposition of \( (1 - V^*V)a_i \):

\[
(1 - V^*V)a_i = V'^*h. \tag{17}
\]

We note that \( h \in F \). Therefore

\[
V'(1 - V^*V)a_i \in F. \tag{18}
\]

Using the fact that \( F \) is a factor of infinite type, we can move the final projectors of \( V \) and \( V' \) to be orthogonal and construct a new \( U \), so that \( Ua_i \in F \) for all \( i \) and its initial projector is onto the span of the ranges of the \( a_i \), \( 1 \leq i \leq s \). (\( U = W_1V + W_2V' \) for suitable isometries \( W_1 \) and \( W_2 \) in \( F \).)

We see that the above lemma is also true for a countably infinite number of \( a_i \) instead of a finite number. A variation of the above argument will prove the result for factors of type \( I_N \) with a separator and generator. We do not consider factors of type \( I_1 \).

From each \( O_i \) select a sequence of elements \( \alpha_j, j = 1, 2, \cdots, N, N \) the dimension of the \( H_i \), such that \( \alpha_j^* \eta \) is a basis for \( H_i \). We assume all our Hilbert spaces are separable at this point. Denote \( O_i \) restricted to \( H_0 \) by \( F_i \), \( O_i \) restricted to \( H_0 \) by \( F_i \), and \( \alpha_j \) restricted to \( H_0 \) by \( b_j \). We note that

\[
b_j^* b_k \in F, \quad \text{all } j \text{ and } k, \tag{19}
\]

as

\[
O_i (\alpha_j^{-1} O_i) \subset O_0. \tag{20}
\]

Therefore we are in the situation of the lemma. Find a partial isometry \( V_i \) with

\[
V_i b_j^* \in F. \tag{21}
\]

If one is in the \( I_N \) factor situation, it is trivial that \( V_i \) is unitary. We proceed to show that if the factor is of infinite type, \( V_i \) may be picked unitary.

The final projector of \( V_i \) is a projection in \( F \). If this projection is equivalent to 1, then it is clear that a new \( V_i \) may be picked to be unitary. In general, let \( P_i \) be this projection in \( F \). Using \( V_i \), we identify \( H_i \) with \( P_i H_0 \), and in this new basis for each \( H_i \):

\[
b_j^* \in P_i F, \quad \text{all } i \text{ and } j. \tag{22}
\]

(We carelessly identify operators in \( F_i \) and \( F \) that have identical matrix form in the present basis system for \( H_0 \) and \( H_i \).) Also

\[
F_i \subset P_i F.
\]

Now \( O_i O_0 \subset O_i \) implies \( F_i F \subset F_i \), and therefore \( F_i \) is a right ideal of \( F \). Since the \( b_j^* \) span \( P_i H_0 \) for fixed \( i \), the weak closure of \( F_i \), \( F_i^w \), equals \( P_i F \). Calling \( C_1 \), \( O_0 \) restricted to \( H_i \), if follows from \( O_i O_0^{-1} \subset O_0 \) that \( C_1 \) contains \( P_i F \), and from \( O_0 O_i^{-1} \subset O_0 \) that \( C_1 \subset F \) and therefore \( C_1 = P_i F \). (\( C_i \) is weakly closed as it is \( E'O_E E' \) with \( E' \) a projection commuting with \( O_0 \).) \( F \) is represented by \( C_i \) uniformly continuously, and thus either \( F \) is isomorphic (as a \( C^* \) algebra) to \( P_i F \) or else \( F \) modulo a nontrivial uniformly closed two-sided ideal is isomorphic to \( P_i F \). The second situation is impossible and the first situation implies \( P_i \sim 1 \). Thus we may pick \( V_i \) unitary and work in a basis with \( F_i \subset F, F_i^w = F \).

Using the axiom \( O_i O_j \subset O_{i+j} \), we easily show that, in the present bases, \( O_i \) restricted to \( H_i \) (and weakly closed, for every \( i \) and \( j \)) is equal to \( F \).

If we now let \( X \) be the restriction of an operator \( X \) in \( O_0 \) to \( H_0 \), then the image of \( X \) in \( H_i \) is \( R_i X R_i^{-1} \) with \( R_i \) unitary, \( R_i F R_i^{-1} = F \), and \( R_i \) independent of \( X \). To establish this we observe that since \( F \) has a separator and generator, all isomorphisms are spatial; and the image of \( O_0 \) in each \( H_i \) is faithful, since \( F / I = F \) with \( I \) a nontrivial uniformly closed two-sided ideal is impossible. \( F \) and the \( R_i \) completely describe \( O_0 \).

It is important now to establish that each \( F_i \) is weakly closed. Consider an increasing sequence of projectors in \( F_i \), weakly approaching 1. Since \( O_i O_0 \subset O_i \), such a sequence is easy to come by. We claim the corresponding sequence of elements in \( O_i \) (call them \( T_1', T_2', \cdots \)) converges weakly, and therefore converges to an element \( 1_i \) in \( O_i \). From the fact that \( O_i O_0^{-1} \subset O_0 \) and \( O_0^{-1} O_i \subset O_0 \), it follows that each of the elements \( T_k' \) restricted to each \( H_i \) is a partial isometry. Further, from \( O_i O_0 \subset O_i \) it follows that the partial isometries in the sequences are increasing; that is, \( T_k' \) restricted to the range of the initial projector of \( T_k' \), restricted to \( H_k \), agrees with \( T_k' \) restricted to \( H_k \). It easily follows that \( T_k' \) converges to an element \( 1_i \) and \( F_i \) is weakly closed. We may now study \( O_i \).

The essential observation here is that knowledge of the single element \( 1_i \) in \( O_i \) that restricts to the identity in \( H_0 \) is sufficient to characterize \( O_i \), as all of \( O_i \) is generated by multiplying \( 1_i \) by the algebra \( O_0 \). It must be recalled that since \( \psi \) in \( H_0 \) separates \( O_i \), the restriction of an operator in \( O_i \) to \( H_0 \) determines the operator.
A simple calculation is required to determine the conditions on $S_i^j$, the restriction of $1_i$ to $H_j$.

The first relation we obtain is

$$1_i X = (R_i^{-1}XR_i)^{-1} 1_i,$$  \hspace{1cm} (23)

where $(R_i^{-1}XR_i)^{-1}$ is the element of $O_0$ that restricts to $R_i^{-1}XR_i$ in $H_0$. This relation holds because these two elements have the same restriction to $H_0$. From (23) upon examining the restriction of each side of the equation to $H_k$ it follows that

$$X = (R_k)^{-1}(S_k^i)^{-1}R_{(i,k)}(R_i)^{-1}XR_k(R_{(i,k)})^{-1}S_i^kR_k.$$  \hspace{1cm} (24)

Since this holds for all $X$ in $F$, we arrive at

$$S_k^i \simeq R_{(i,k)}(R_i)^{-1}R_k^{-1},$$  \hspace{1cm} (25)

where the equivalence means that the two sides of the equation, both unitary operators, induce the same automorphism of $F$. There is only one other set of conditions on the $S^i_k$ to guarantee that the $F_i, R_i$, and $S_i^k$ determine an SPLA. This set of relations is obtained by requiring that

$$1_i 1_j \in O_0 1_{(i,j)}.$$  \hspace{1cm} (26)

This equation becomes

$$R_{(k,i)}(R_k)^{-1}(S_k^i)^{-1}R_k(R_{(k,i)})^{-1}S_i^kS_i^k = S_i^i.$$  \hspace{1cm} (27)

Equations (25) and (27) are the algebraic relations to study for constructing an SPLA from $F$ and the group $G$. Equation (25) merely states that the map $g_i \rightarrow (R_i)^{-1}$ induces a homomorphism from $G$ into the group of outer automorphisms of $F$. Equation (27) gives the conditions on the $S^i_k$ in terms of the homomorphism just mentioned. Equation (25) determines $S^i_k$ up to a complex number of modulus one. Thus (27) may be looked upon as conditions on these complex factors, since this equation automatically holds up to a complex factor. [It is easily checked that the two sides of (27) induce the same automorphism of $F$.] Unfortunately, we have not been able to find whether every homomorphism of $G$ into $\text{Out}(F)$ has at least one solution to (27). However, if there is one solution to (27), and calling this solution $S^i_k$, we look for all other solutions by writing $(S^i_j)^{-1} = \lambda_j^i S^i_j$ with $\lambda_j^i$ a complex number of modulus one, $(S^i_j)'$ any other solution. The $\lambda_j^i$ must satisfy the algebraic relations given by (27). Interestingly enough, these state that $\lambda_j^i$ is a two cocycle of the usual cochain complex of the group $G$ with coefficients in the unimodular complex numbers, the circle group. By changing the basis for the $H_i$ by a complex factor (multiplying each basis element by a complex factor, the factors being constant in each $H_i$), $\lambda_j^i$ is changed by a coboundary. Therefore the solutions of (27)—if they exist at all—are in one-to-one correspondence with $H^2(G, T^1)$. Unfortunately, this correspondence is not canonical as we have derived it.

**Theorem:** In an SPLA with $O_0 = F$, $F$ a factor of type I, II$_\infty$, or III, and group $G$, $O$ is determined up to unitary equivalence (under basis changes in each $H_i$ separately) by a homomorphism of $G$ into $\text{Out}(F)$ and an element of $H^2(G, T^1)$.

It is attractive to conjecture that if, instead of assuming $F$ to be a factor, it is assumed that the restrictions of $O_0$ to all the $H_i$ are isomorphic and faithful, the same result follows with $H^2(G, T^1)$ replaced by $H^2(G, Z)$, $Z$ the center of $F$.

**Corollary:** If $F$ is of type I and $G$ is Abelian with one generator, then the SPLA is trivial; and if $G$ contains more than one element, $O$ is therefore not a factor.

The corollary follows from the fact that, under the stated conditions, $\text{Out}(F)$ and $H^2(G, T^1)$ are both trivial.

The corollary may have some implications for the factor-type problem in quantum field theory.

We conclude by remarking that the results in this paper are clearly far from definitive. The obvious examples of several free boson and fermion fields with various choices of superselection rules should be computed to relate the $O_0$ and $O$ in these examples. The free boson field is studied in detail elsewhere.\textsuperscript{1} The result here should also be related to the deeper results of Borchers\textsuperscript{2} that also relate the rings of different regions of space to each other.
