The Casimir energy of the twisted string loop: Uniform and two segment loops

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We calculate the Casimir energy of a two segment loop of string with one normal boundary point and one twisted boundary point. The energy is renormalized relative to the twisted uniform loop. The use of the twisted loop in simplifying untwisted loop calculations is discussed. © *1996 American Institute of Physics.* [S0022-2488(96)01908-1]

I. INTRODUCTION

Oscillating loops of cosmic strings have been suggested as a possible seed for galaxy formation in the early universe.[1] In these models, matter condenses about the oscillating string loops which radiate into the surrounding matter as they decay. String loops have been reviewed by Vilenkin[2] who discusses their radial oscillations and gives examples of the kinds of density fluctuations that could be seeded by decaying loops of various structures. In early models loops were created in a symmetry breaking phase transition[2] but more recently, Basu, Guth, and Vilenkin[3] have discussed spontaneous nucleation processes that create loops and other defects during the early universe inflationary period and the effects of these more persistent defects on galaxy seeding. A theory relating quantum fluctuations to the evolution of loops and other defects has been developed by Garriga and Vilenkin[4] and Basu and Vilenkin.[5]

Several kinds of radiation can be associated with an oscillating string, such as electromagnetic, gravitational or goldstone-boson emission. The importance of the latter type of radiation in galaxy formation models has been emphasized by Davis[6] and Vilenkin and Vachasspati.[7] Oscillations associated with strings have also been discussed by Rosenzweig and Srivastava[8] who consider the bulk vibrations associated with the Higgs field surrounding a string and point out that this contribution to the radiation emission might be dominant in string decay processes. They also point out an interesting aspect of twisted strings and string loops; one of the models discussed in this paper is a straight string threading a string loop and it is found that the interaction energy is absent when the configuration is twisted. Other complex loop arrangements are also possible.

One possibility for a more complex loop is a loop with segments. Davis,[6] for example, mentions that when formed a vortex loop could be made of many segments coming from previously casually disconnected regions so that oscillating segmented string loops could be considered. Pagels,[9] in a calculation of loop correlation functions, assumed that the initial loop in a loop decay process was segmented. There has also been considerable interest in the Casimir or zero point energy of oscillation segmented loops.[10–13] This particular aspect of loop behavior is interesting and important, given the possible importance of oscillating loops in galaxy formation processes and the role of fluctuations in the evolution of loop dynamics.[14] The segmented loop is also a relatively simple system on which to perform Casimir calculations and thus provides a useful testing ground for various renormalization methods. It may also be applicable to the vacuum states of two dimensional field theories.[10] The models so far discussed in the literature have had two building blocks, ϕ_i $(i=1,2)$ segments, each of a different density and tension but

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with the same wave speed $v=c$. The segments are joined alternatingly at points $x_{\bar{N}}$ to create segmented loops. The boundary condition is

$$\Phi_1(x_N)=\Phi_2(x_{\bar{N}}), \tag{1}$$

with a similar tension relation for the derivatives. $\bar{N}$ is the number of segments. In the literature, the energies for the segmented string have been renormalized by the energy of the uniform string joined to itself, a system with one boundary point. The boundary conditions at this point for the uniform string are analogous to (1).

In this paper, we will consider a string loop in which one of the boundary points, x_1, is twisted, where the fields obey the boundary conditions

$$\Phi_1(x_1)=-\Phi_2(x_1), \quad \Phi_1(x_{\bar{N}})=\Phi_2(x_{\bar{N}}) \;\; \bar{N}\neq 1. \tag{2}$$

The tension relations for the derivatives at the boundary points are similar. For the uniform loop with one boundary point only, we are obviously creating a Möbius loop.

Twisted fields have been discussed generally by Isham.[15] In Minkowski space, DeWitt[16] has considered the Casimir energy of both twisted and untwisted scalar fields with a periodicity along the z-axis. Using cylindrical polar coordinates in Minkowski space, Ford[17] has treated the Casimir energy of a twisted string. Both DeWitt and Ford found the twisted scalar field to have a positive energy compared to a negative energy for the untwisted string. The twisted loop is an interesting problem because one can compare the energy to that of the untwisted loop to discover if it is energetically more favorable for a straight string to join in a simple or a twisted topology, because the differing loop topologies make Casimir energy comparisons quite valuable and because the twisted loop may offer a way of simplifying the untwisted string with a large number of segments.

The Casimir energy for the twisted uniform loop is calculated in the next section. A comparison is made to the untwisted uniform string. Twisted strings with two segments are discussed in the third part of the paper. The energies in these sections are renormalized using the Euler-MacLaurin sum formula and a cut off function. In the fourth section we compare this method to regularizing using point splitting and the generalized zeta function. It is potentially instructive to calculate the result by all three methods. There are many regularization methods, each with its advantages and problems. A comparison insures that a problem specific to one method has not influenced the result of the calculation. There are situations, usually involving a boundary, where the three methods do not agree and a study of the divergence behavior causing the disagreement can be insightful.[18] A comparison between the two segment twisted string and the four segment untwisted loop is discussed in the last section.

II. UNIFORM LOOPS

A. Twisted loop-anti periodic boundary condition

Let the field on the loop be described by both clockwise and counterclockwise moving waves,

$$\Phi(x)=\xi e^{i\omega x}+\eta e^{-i\omega x}. \tag{3}$$

The boundary conditions are

$$\Phi(x)=-\Phi(x+L), \quad \Phi'(x)=-\Phi'(x+L). \tag{4}$$

Using (3) and (4), the allowed frequencies are

$$\cos(\omega L)=-1, \quad \omega=\frac{2\pi}{L'}\left(n+\frac{1}{2}\right). \tag{5}$$

The energy associated with the zero point twisted string oscillations is found from the $\alpha\rightarrow 0$ limit of the sum with cutoff function

$$E_t=2\times\frac{1}{2}\sum_{n=0}^{\infty}\frac{2\pi}{L}\left(n+\frac{1}{2}\right)\exp\left(\frac{-2\pi\alpha(n+\frac{1}{2})}{L}\right). \tag{6}$$

Evaluating the sum using the Euler-McLaurin sum formula and taking the limit one finds

$$E_t=\frac{L}{2\pi\alpha^2}+\frac{\pi}{12L}+O(\alpha^2). \tag{7}$$

This could be compared to the energy for the untwisted uniform string given by Brevik and Nielsen,[10]

$$E_u=\frac{L}{2\pi\alpha^2}-\frac{\pi}{6L}+O(\alpha^2). \tag{8}$$

The first term in both Eqs. (7) and (8) can be identified as the energy of the string with no boundary.[19] Subtracting off the energy of the unbounded string one finds the renormalized energies,

$$E_{t,r}=\frac{\pi}{12L},\quad E_{u,r}=-\frac{\pi}{6L}. \tag{9}$$

One sees that, relative to the unbounded string, the untwisted string represents the lower energy state. This is similar to the results of DeWitt[16] and Ford.[17]

B. Another boundary condition for the uniform string

More complex twists can be constructed using the boundary conditions

$$\begin{aligned}\Phi(x)&=\phi(x+L)\exp\left[\frac{2\pi i}{N}\right],\\ \Phi'(x)&=\Phi'(x+L)\exp\left[\frac{2\pi i}{N}\right],\end{aligned} \tag{10}$$

where N is the number of circuits needed to complete the field. $N=1$ is the uniform untwisted string, $N=2$ is the uniform twisted string treated in the previous section.

Applying the boundary conditions one finds that the allowed frequencies are given by

$$\cos\left(\omega L+\frac{2\pi}{N}\right)=1, \tag{11}$$

$$\omega=\frac{2\pi}{L}\left(n+\frac{1}{N}\right). \tag{12}$$

The associated Casimir energy calculated with the sum formula and a cut off is

$$E_t-\frac{L}{2\pi\alpha^2}=\frac{2\pi}{L}\left(\frac{-1}{12}+\frac{1}{2N}-\frac{1}{2N^2}\right). \tag{13}$$

For integer N this is always greater than the untwisted string energy E_u given in (9). For $0<N<1$ the twisted uniform string energy can be less than that of the untwisted string.

III. THE TWO SEGMENT TWISTED STRING

A. Allowed frequencies

The two segment string has two scalar fields, one moving on each of the two kinds of string,

$$\Phi_1(x)=\xi_1 e^{i\omega x}+\eta_1 e^{-i\omega x},$$
$$\Phi_2(x)=\xi_2 e^{i\omega x}+\eta_2 e^{-i\omega x}. \tag{14}$$

We will take the twisted boundary point to be $x=0(L)$ and the normal boundary point to be $x=L_1$, where we allow for the possibility that the segments are of different lengths.

The boundary conditions on the fields gives

$$O(L):\xi_1 e^0+\eta_1 e^{-0}=\xi_2 e^{i\omega L}+\eta_2 e^{-i\omega L},$$
$$L_1:\xi_1 e^{i\omega L_1}+\eta_1 e^{-i\omega L_1}=\xi_2 e^{i\omega L_1}+\eta_2 e^{-i\omega L_1}. \tag{15}$$

The derivative conditions are

$$O(L):R(\xi_1 e^0-\eta_1 e^{-0})=\xi_2 e^{i\omega L}-\eta_2 e^{-i\omega L},$$
$$L_1:R(\xi_1 e^{i\omega L_1}-\eta_1 e^{-i\omega L_1})=\xi_2 e^{i\omega L_1}-\eta_2 e^{-i\omega L_1}, \tag{16}$$

where $R=T_1/T_2$, the ratio of the tensions in the two segments. Brevik and Nielsen[10] use x for this ratio.

From the coefficient determinent we can find the dispersion relation giving the allowed frequencies,

$$(1-R)^2\cos(\omega(L-2L_1))-(1+R)^2\cos(\omega L)-4R=0, \tag{17a}$$

which can be written as

$$(1-R)^2\cos(\omega(s-1)L_1)-(1+R)^2\cos(\omega(s+1)L_1)-4R=0. \tag{17b}$$

Another equivalent but useful form is

$$\sin(s\omega L_1)\sin(\omega L_1)=\frac{4R}{(1-R)^2}\cos^2\left(\frac{(s+1)\omega L_1}{2}\right), \tag{18}$$

where $s=L_2/L_1$, the ratio of the lengths of the two possible segments and $L=L_1+L_2$.

B. Some special cases

1. $R=1$ (equal tensions)

This is the same as the uniform twisted string treated in the second part of the paper.

2. $R=0$ ($T_2=\infty$ or $T_1=0$)

From the dispersion relation (18) we find

$$\omega_{n1}=\frac{\pi n_1}{L},\quad \omega_{n2}=\frac{\pi n_2}{L}. \tag{19}$$

This is the same result as for the $R=0$ untwisted string and has an identical Casimir energy.[10] Since we started with a twisted string, this is an interesting result but may be understood by realizing that the infinite T_2, implies infinite density since the wave speed is unity so that the second string blocks passage of the wave in the first string segment, effectively turning the boundary points of passage into points of reflection. The fact that the $R=0$ point is the same in both the twisted and untwisted loops may also identify this point as a common or double point between the two different loop topological phases. The zero tension limit has been discussed by Lindström.[20]

C. General energies and frequencies-odd s

1. The Casimir energy

The dispersion relation can be written as a polynomial in $\sin^{(s+1)}(\omega L_1)$. There will be $((s+1)/2)$ double branches, i.e., solutions to the equation for $\sin^2(\omega L_1)$. Each root is an allowed frequency. Each double branch has roots $\pi\beta_i$. There are $((s+1)/2)$ values of β_i. For each β_i the frequency spectrum is

$$\omega L_1=\begin{cases}\pi(\beta+n), \\ \pi(1-\beta+n),\end{cases} \quad 0\leq\beta\leq\tfrac{1}{2}. \tag{20}$$

There is no degeneracy to consider.

Using the Euler MacLaurin Sum formula with a cut off as in the uniform string case, the Casimir energy for a single β is

$$E(\beta)=\frac{L_1}{2\pi\alpha^2}-\frac{\pi}{2L_1}\left(\frac{\beta^2}{2}+\frac{(1-\beta)^2}{2}\right)+\frac{\pi}{6L_1}. \tag{21}$$

Summing over the $((s+1)/2)$ double branches we have

$$E=\frac{L}{2\pi\alpha^2}-\frac{\pi(s+1)}{4L}\sum_{i=1}^{(s+1)/2}(\beta_i^2+(1-\beta_i)^2)+\frac{\pi(s+1)^2}{12L}. \tag{22}$$

Once s is chosen, β can be determined from the dispersion relation (17) or (18) and numerical values for the energies found.

The energy in Eq. (22) is not yet renormalized. The prescription that we shall use is analogous to that used by Brevik and Nielsen for the untwisted string loop:

$$E_r=E\ \text{(renormalized)}=E-E\ \text{(uniform twisted loop)}. \tag{23}$$

Using this prescription and Eq. (22) we find the energy as

$$E_r=\frac{\pi(s^2+2s)}{12L}-\frac{\pi(s+1)}{4L}\sum_{i=1}^{(s+1)/2}(\beta_i^2+(1-\beta_i)^2). \tag{24}$$

We will now consider some special cases.

2. $s=1$ $(L_1=L_2)$

From the dispersion relation (18) we find one double branch for $s=1$,

$$\sin^2(\omega L_1)=\frac{4R}{(1+R)^2}. \tag{25}$$

TABLE I. Some energies for the twisted string loop renormalized to the unform twisted loop.

R	$E_r(s=1)$	$E_r(s=2)$	$E_r(s=3)$
0	$\frac{-\pi}{4L}$	$\frac{-\pi}{3.69L}$	$\frac{-\pi}{3.27L}$
$\frac{1}{2}$	$\frac{-\pi}{85.5L}$	$\frac{-\pi}{77.32L}$	$\frac{-\pi}{66.52L}$
1	0	0	0

The inverse sine of the square root of this equation for a given R is the single value of β for this case. From Eq. (23) the renormalized energy is

$$E_r=\frac{\pi}{4L}-\left(\frac{\pi}{2L}\right)(\beta^2+(1-\beta)^2). \tag{26}$$

Some values are tabulated in Table I.

3. $s=3$ ($L_2=3L_1$)

Using Eq. (19) again we find for the allowed frequencies,

$$\sin^4(\omega L_1)-\frac{3\sin^2(\omega L_1)(R+\frac{1}{3})(R+3)}{4(1+R)^2}+\frac{R}{(1+R)^2}=0. \tag{27}$$

There will be two values for β for each R. Some of the renormalized energies for $s=3$ are tabulated in Table I.

D. Casimir energies—even s

For even s we will use both Eqs. (17) and (18). We will first show that $\cos(\omega L_1)=-1$, $\omega L_1=\pi$, is a root for any even s.

From Eq. (18) we have

$$\begin{aligned}\sin(s\pi)\sin(\pi)&=\left(\frac{4R}{(1-R)^2}\right)(1+\cos(\pi\langle s+1\rangle)),\\ 1+\cos(\pi\langle s+1\rangle)&=0.\end{aligned} \tag{28}$$

So, for all even s, the polynomial has a degenerate root at $\omega L_1=\pi$.

The equation for the allowed frequencies is

$$(1-R)^2\cos(\omega L_1\langle s-1\rangle)-(1+R)^2\cos(\omega L_1\langle s+1\rangle)-4x=0,$$

with $1+\cos(\omega L_1)$ a factor. We will work out the $s=2$ case.

The allowed frequencies for $s=2$ are given in by

$$\cos^2(\omega L_1)-\cos(\omega L_1)+\frac{R}{(1+R)^2}=0 \quad \text{and} \quad \cos(\omega L_1)=-1. \tag{29}$$

Some values of the energy are tabulated in Table I.

IV. OTHER RENORMALIZATION METHODS

In the first part of the paper, the Casimir energies have been evaluated by considering the energy as a sum over possible modes of vibrations and using a limit of the Euler-MacLaurin sum formula with a cut-off function. There are several other regularization methods that could be applied to this problem.

A. Covariant point-splitting method-uniform twisted string

The result presented in Eq. (7) for the uniform twisted string can also be derived by the point-splitting method. This method is not only covariant but also cut-off independent.[21–23] To use this method the metric is assumed to be

$$ds^2=dt^2-(R_0)^2d\phi^2, \tag{30}$$

where R_0 is a constant and $\phi\epsilon[0,2\pi]$. This method, as in the previous one, begins with the possible modes of vibration. The wave equation in this metric is

$$D^{\mu}D_{\mu}\Phi(t,\phi)=0, \tag{31}$$

where D_μ is the covariant derivative operator for the metric (30).

As before the general solution is

$$\Phi_m(t,\phi)=Ae^{-i\omega_m t}\cos(m\phi), \quad \Phi_n(t,\phi)=Be^{-i\omega_n t}\sin(n\phi), \tag{32}$$

where A,B are constants and $\omega_m=m/R_0$, $\omega_n=n/R_0$. Now the time dependence is explicit and we use ϕ instead of x. Applying the antiperiodic boundary condition,

$$\Phi(t,\phi)=-\Phi(t,\phi+2\pi), \tag{33}$$

we obtain the eigenfunctions and eigenfrequencies, respectively, as

$$\begin{aligned}\Phi_n(t,\phi)&=\frac{1}{\sqrt{\pi(2n+1)}}\exp(-i\omega_n t)\cos\left(\frac{(2n+1)\phi}{2}\right),\\ \omega_n&=\frac{2n+1}{2R_0},\quad n=0,1,2,3,\ldots,\end{aligned} \tag{34}$$

and

$$\begin{aligned}\Phi_m(t,\phi)&=\frac{1}{\sqrt{\pi(2n+1)}}e^{-i\omega_m t}\sin\left(\frac{(2m+1)}{2}\phi\right),\\ \omega_m&=\frac{2m+1}{2R_0},\quad m=0,1,2,\ldots\ .\end{aligned} \tag{35}$$

The allowed frequencies are the same as in Eq. (5) with $L=2\phi R_0$. Rather than going directly to the energy as in the mode sum method, the modes are used to construct the Hadamard Green's function $G^{(1)}(x,x')$ which is written as[23]

$$G^{(1)}(x,x')=\frac{1}{2\pi}\ln\left[\frac{\cos(\Delta t/2R_0)+\cos(\delta)}{\cos(\Delta t/2R_0)-\cos(\delta)}\right], \tag{36}$$

where $\Delta t=t-t'$, and $\delta=(\phi-\phi')/2$. To renormalize this Green's function we express it as an image sum and write it as[24,25]

$$\frac{1}{2}G^{(1)}(x,x')=\frac{1}{4\pi}\sum_{n-\infty}^{\infty}\ln\left[\frac{1}{(2n\pi+\delta)^2-(\Delta t/2R_0)^2}\right]+\frac{1}{4\pi}\ln\left[\cos\left(\frac{\Delta t}{2R_0}\right)+\cos(\delta)\right]$$
$$-\frac{1}{4\pi}\ln\left[\frac{1}{2}\right]-\frac{1}{\pi}\sum_{n=1}^{\infty}\ln\left[\frac{1}{2n\pi}\right]. \tag{37}$$

It is clear that the $n=0$ term corresponds to the infinite unbounded string result and hence the Casimir renormalization relative to the infinite string could be accomplished by dropping this term.[21] This is analogous to the energy renormalization of Eq. (9). Expressing the expectation value of the stress energy tensor as

$$\langle T_{\mu\upsilon}\rangle_r=\tfrac{1}{2}\lim_{x'\to x}(D_\mu D_{\nu'}-\tfrac{1}{2}g_{\mu\upsilon}D^\lambda D_\lambda)G_r^{(1)}(x,x'). \tag{38}$$

Using the covariant derivatives,

$$\lim_{x'\to x}\frac{1}{2}D_0D_{0'}G_r^{(1)}(x,x')=\frac{-1}{16\pi^3R_0^2}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{32\pi R_0^2},$$
$$\lim_{x'\to x}\frac{1}{2}D_\phi D_{\phi'}G_r^{(1)}(x,x')=\frac{-1}{16\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{32\pi}, \tag{39}$$

we obtain

$$\langle T_0^0\rangle_r=\frac{1}{48\pi R_0^2},\quad \langle T_\phi^\phi\rangle_r=-\langle T_0^0\rangle_r. \tag{40}$$

The point splitting method will produce a density rather than an energy. Using this density the energy is

$$E_r=\frac{\pi}{12L}. \tag{41}$$

Equation (41) agrees with the results obtained with the sum formula and cut off, Eq. (9).

B. Regularization with the Zeta function

Another method that has been used to obtain the renormalized zero-point energies is Zeta function renormalization.[24–27] The use of Zeta function methods is interesting because the twisted string is such a simple example of the procedures. The history of Zeta function regularization applied to string loops has been reviewed by Elizalde.[27] The Zeta function method begins with a sum over an operator spectrum. We will use the Hurwitz Zeta function,[26]

$$\zeta(s^*,a)=\sum_{m=0}^{\infty}\frac{1}{(m+a)^{s^*}}, \tag{42}$$

with $\mathrm{Re}(s^*)>1$, $0<a\leq 1$. We use s^* rather than s to avoid confusion with the string length parameter.

One of the possible integral representations is

$$\zeta(s^*,a)=\frac{1}{\Gamma(s^*)}\int_0^\infty \frac{dt\; t^{(s^*-1)}e^{-at}}{1-e^{-t}}, \tag{43}$$

with $\mathrm{Re}(s^*)>1$ and $\mathrm{Re}(a)>0$. Transforming into a contour integral[26] we get

$$\zeta(s^*,a)=\frac{e^{-i\pi s^*}\Gamma(1-s^*)}{2\pi i}\oint \frac{dz\; z^{(s^*-1)}e^{-az}}{1-e^{-z}}. \tag{44}$$

This provides the analytic continuation of $\zeta(s^*,a)$ over the plane: It is regular everywhere except for a simple pole at $s^*=1$ with residue 1. In the special case where s^* is an negative integer $s^*=-m$, $m=0,1,2...$, we have

$$\zeta(-m,a)=\frac{-B_{m+1}(a)}{m+1}, \tag{45}$$

where $B_m(a)$ is the Bernoulli polynomial. The Zeta function method, just as in the mode sum method, starts with an expression for the energy as the sum over possible modes. The energy is given by Eq. (6) with $\alpha=0$, eliminating the cut off. That sum is clearly of the form (43) with $s^*=-1$. The Zeta function of interest is

$$\zeta(-1,a)=\frac{-a^2+a-\frac{1}{6}}{2}. \tag{46}$$

The Zeta function procedure regularizes the energy by assigning an analytically continued Zeta function to the energy sum.[23,27] It will be necessary to explicitly renormalize to the uniform twisted string energy at the end of the calculation. We now reconsider some of the cases considered in Sec. II using Zeta function regularization.

1. The uniform twisted string

From Eq. (5) we have $\omega_n=(2\pi/L)(n+1/2)$, $n=0,1,2,\ldots$.
The Casimir energy is

$$E_t=2\times\frac{1}{2}\sum_{n=0}^{\infty}\frac{2\pi}{L}\left(n+\frac{1}{2}\right)=\frac{2\pi}{L}\zeta\left(-1,\frac{1}{2}\right)=\frac{\pi}{12L}, \tag{47}$$

where the factor 2 takes into account that the modes are degenerate. The simplicity of this procedure, compared to the more lengthly calculations associated with the mode sum/cut-off and point splitting methods is startling. Its very simplicity is, however, sometimes considered a drawback since the analytic continuation which removes the divergences somewhat hides the divergence behavior which could be of interest.

2. R=0

The $R=0$ case implies $T_1\to 0$ or $T_2\to\infty$.[20]
From Eq. (19) we get the allowed modes just as in the mode sum/cut-off method,

$$\omega_n=\begin{cases}\dfrac{n\pi}{L_1}, \\ \dfrac{n\pi}{L_2},\end{cases} \quad n=0,1,2,\ldots \, . \tag{48}$$

The zero frequency mode is not of interest. The Casimir energy is

$$E_{1+2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{n\pi}{L_1}+\frac{n\pi}{L_2}. \tag{49}$$

Regularizing by the Riemann-Zeta function method we get

$$E_{1+2}=\left(\frac{\pi}{2L_1}+\frac{\pi}{2L_2}\right)\zeta(-1,1)=-\frac{\pi}{24}\left(\frac{1}{L_1}+\frac{1}{L_2}\right). \tag{50}$$

Using $s=L_2/L_1$ as before we can rewrite this equation as

$$E=\frac{-\pi}{24L}\left(s+\frac{1}{s}+2\right). \tag{51}$$

Renormalizing to the twisted string, using Eq. (23),

$$E_r=\frac{-\pi}{24L}\left(s+\frac{1}{s}+4\right). \tag{52}$$

This second renormalization must be included explicitly and is not performed by the Zeta function.

3. *s* an odd integer

Again, in agreement with the mode sum calculation we have from Eq. (20) the frequency spectrum

$$\omega L_1=\begin{cases}\pi(\beta+n), \\ \pi(1-\beta+n),\end{cases}$$

with $0\leqslant\beta\leqslant 1/2$, $n=0,1,2,\ldots$.

Just as in Eqs. (21) and (22) for the mode sum/cut-off method the Casimir energy for general β_i can be written as

$$E_{1+2}=\frac{\pi}{2L_1}\sum_{i=1}^{(s+1)/2}\zeta(-1,\beta_i)+\zeta(-1,1-\beta_i). \tag{53}$$

Using the special form for the Zeta function given in Eq. (46) we have

$$E_{1+2}=\frac{\pi(s+1)^2}{12L}-\frac{\pi(1+s)}{4L}\sum_{i=1}^{(s+1)/2}\beta_i^2+(1-\beta_i)^2 \tag{54}$$

or renormalizing to the uniform twisted string we have

$$E_r=E_{1+2}-E \text{ (Uniform twisted string)},$$

$$E_r = \frac{\pi(s^2+2s)}{12L} - \frac{\pi(1+s)}{4L} \sum_{i=1}^{(s+1)/2} \beta_i^2 + (1-\beta_i)^2. \tag{55}$$

V. RELATING THE TWISTED AND UNTWISTED STRING WITH $S=1$

One of the effects of once twisting the string with two segments is to compel the field to cross four string segments before returning to its initial condition. A uniform string with four segments also has this effect and the question arises of how the two cases are related.

A. The uniform string

First compare the uniform twisted string to the uniform untwisted string. Generalize the frequency conditions to allow the twisted and untwisted loops to have different radii. L_t will be the circumference of one loop of the twisted string and L_u will be the circumference of one loop of the untwisted string.

The frequency conditions are

$$\begin{aligned} &\text{twisted string: } \cos(\omega L_t) = -1, \\ &\text{untwisted string: } \cos(\omega L_u) = 1. \end{aligned} \tag{56}$$

To make the strings similar one could clearly require that the path length traveled by a wave on each type of loop be the same,

$$2L_t = L_u. \tag{57}$$

Substituting into the frequency conditions one finds

$$\text{twisted string: } \cos\left(\frac{\omega L_u}{2}\right) = -1, \tag{58}$$

$$\text{untwisted string: } \cos\left(\frac{\omega L_u}{2}\right) = \pm 1. \tag{59}$$

The untwisted string contains the twisted string and the additional frequency $\omega L_u/2 = 2\pi, 4\pi, \ldots$. For the uniform loops, there is no calculational advantage to treating one loop topology over the other but for higher numbers of segments, there is possibly a reduction in the dimensionality of the problem that needs to be solved for some of the parameter range. The next most complicated cases to compare are the two segment twisted string to the four segment untwisted string.

B. The two segment twisted string, the four segment untwisted string, $s=1$

The frequency conditions for the two segment twisted loop follows from Eq. (17b),

$$\cos\left(\frac{\omega L_u}{2}\right) = \frac{R^2 - 6R + 1}{(1+R)^2}, \tag{60}$$

where we have made the same arguments about circumference and lengths as for the uniform string. The frequency condition for the four segment untwisted string was given in Ref. 13, Eq. (17).

$$-\frac{(1+R)^2}{8R^2}(3 - 10R + 3R^2) + \frac{(1-R)^2}{2R^2}\cos\left(\frac{\omega L_u}{2}\right) - \frac{(1+R)^4}{8R^2}\cos(\omega L_u) = 0. \tag{61}$$

The roots of this equation are

$$\cos\left(\frac{\omega L_u}{2}\right)=\frac{R^2-6R+1}{(1+R)^2} \text{ and } +1. \tag{62}$$

Comparing (34) and (37) to (32) and (33) we see the same pattern. The untwisted string of 4 segments contains the twisted string plus a root of $+1$. Whether or not this pattern is more general is under investigation.

VI. CONCLUSIONS

We have calculated the energy of a uniform twisted loop of string and compared it to the uniform untwisted loop. We find that the untwisted loop has the lower energy for a pure antiperiodic boundary condition. More complex boundary conditions could reverse this.

The energy of a twisted two segmented loop was also calculated and, just as for the untwisted string, a non zero negative Casimir energy is observed. We found that the twisted and untwisted phases of the loop share a common point of their energy spectrum at zero tension. The double point in the energy spectrum of the twisted and untwisted loops may be important in loop dynamics. The loop tensions are part of the loop stress energy content and related to the behavior of the space time through the field equations. A process in which the loop tensions change with variations in the metric could create conditions for the loop phase transition to occur as either a one time transfer or as an oscillation between phases. A study of the behavior of the loop tension as a function of curvature is clearly an important question. It would be especially pertinent in the context of the types of radiation emitted by an oscillating loop and the question of loop stability. If loop stability is affected by the possibility of a phase transition or phase oscillation, this could have an effect on the galactic seeding process.[2,8,14]

A calculation involving loop tension and the curvature of the embedding cosmology opens up the possibility of a much more varied loop stress-energy structure. Cosmic string interiors with heat flow and vorticity,[28] and torsion[29] and spin density[30] have all been discussed in the literature. This larger array of possible stress-energy contents with both untwisted and twisted loop analogs might provide a richer loop thermodynamics with associated phases, such as spin-up/spin-down.[31] Casimir energy calculation on these more elaborate loops would be interesting. The existence of two accessible loop phases has suggested that more detailed calculations involving curvature would be valuable.

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