Instability resulting from stratification in thermal conductivity

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There are many instances of hydrodynamic instability induced by a variation, or stratification, in either a fluid property or a flow property. In this article a new instability is presented. It is shown that when there is a variation in thermal conductivity in the fluid, instability can occur in the presence of a longitudinal gravitational field.

I. INTRODUCTION

In the majority of cases of hydrodynamic instability, there is a stratification of either a fluid property or some quantity of flow. The most obvious case of instability is that of two superposed fluids, with the upper fluid heavier than the lower one. A statically stratified fluid can be unstable if the fluid is accelerated downward with an acceleration greater than the gravitational acceleration g, as noted by Taylor and as is well known now. The stratification in density of an incompressible fluid has its counterpart in the stratification of entropy of a compressible fluid, as meteorologists who invented the concept of potential density to account for the effect of compressibility have long recognized. Stratification in density in the presence of longitudinal gravity can be unstable, as is now well known. Less well known is the instability resulting from a stratification in electric conductivity, as shown by Taylor and McEwan[1] for a steady vertical electric field and by Yih[2] for a vertical time-periodic electric field. Instability resulting from viscosity variation in shear flows (Yih[3]) is a subject that, after many years, is now enjoying a period of revival of interest. In porous media, a less viscous fluid pushing a more viscous one can induce instability and produce fingers of penetration, as Saffman and Taylor[4] showed.

But it does not necessarily need to be a fluid property that, when stratified, can induce instability. If some quantity of the *flow* of a fluid is stratified, it can be unstable too. A prominent example is the Couette flow, which can be unstable if the square of the circulation decreases outwards, resulting in the formation of Taylor vortices. The electromagnetic counterpart (Yih[5]) of Taylor vortices is the result of a radial stratification of a circular magnetic field. In two-dimensional flows the stratification of vorticity can induce instability when there is a point of inflection in the velocity profile, a famous and extreme case of which is the Helmholtz instability, where the density stratification is stabilizing and the instability results from the vortex sheet. Even when there is no point of inflection in the velocity profile of a two-dimensional flow of a viscous fluid, stratification of vorticity is still important for instability, as indicated by the stability of plane Couette flows, which has uniform vorticity. (For axisymmetric flows it is the stratification of the azimuthal vorticity divided by the radial distance that is important. When this quantity is constant, as in Poiseuille flow, the flow is stable against axisymmetric small disturbances.)

In this article, I shall show a new instability: the instability resulting from thermal-conductivity stratification. With the other instances of how a fluid or a flow can be unstable when a stratification is present, one could perhaps make the point that hydrodynamic stability is a subject within the field of stratified flows.

II. PRIMARY TEMPERATURE AND VELOCITY FIELDS

Consider two superposed fluids (Fig. 1), each of thickness d, between two plane boundaries inclined at an angle β to the horizontal. To show that the instability to be revealed results from conductivity variation alone, we shall assume the two fluids to have the same viscosity and the same dependence of density on temperature, but different thermal conductivities: k_2 for the upper fluid and k_1 for the lower fluid. That two such fluids are not easy to find is not necessarily an objection to this study, since instabilities resulting from density and viscosity variations are known, as mentioned already in the Introduction, and the new cause of instability is *in addition* to those other known causes of instability.

Let the origin of Cartesian coordinates be situated on the interface of the fluids, and let x be measured along the interface down the incline, and y be measured upward in a direction normal to the interface. The temperatures at the lower and upper boundaries will be denoted by $T_0 - \Delta T$ and $T_0 + \Delta T$, respectively.

We shall measure x and y in units of d, so that they are dimensionless. The temperature in the lower and upper fluids will be denoted by $T_1(y)$ and $T_2(y)$, respectively. Defining $\bar{h}_1$ and $\bar{h}_2$ by

$$\bar{h}_1 = \frac{T_1(y) - T_0}{\Delta T}, \quad \bar{h}_2 = \frac{T_2(y) - T_0}{\Delta T}, \tag{1}$$

one can readily solve the Laplace equation governing heat conduction, with regards to the boundary and interfacial conditions, and obtain

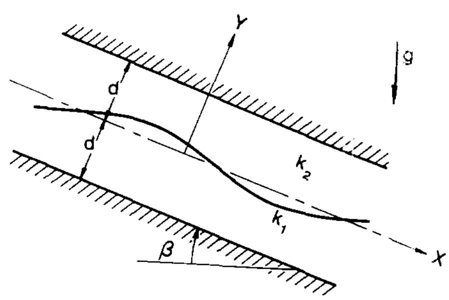

FIG. 1. Definition sketch.

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$$\bar{h}_1 = (\lambda - 1)/(\lambda + 1) + [2\lambda/(1+\lambda)]\, y, \quad (2)$$

$$\bar{h}_2 = (\lambda - 1)/(\lambda + 1) + [2/(1+\lambda)]\, y, \quad (3)$$

where

$$\lambda = k_2/k_1. \quad (4)$$

The interfacial conditions for the temperature field are the continuity of temperature and the continuity of heat flux across the interface.

The variation of density with temperature is assumed the same for both fluids only for the sake of simplicity. This assumption is not at all necessary. We make it here only to isolate the variation of thermal conductivity as the cause of instability. The dependence of the density ρ on temperature is

$$\rho = \rho_0[1 - \alpha(T - T_0)], \quad (5)$$

where ρ_0 is the density at temperature T_0, and α is the coefficient of thermal expansivity. With $\bar{u}$ denoting the velocity (in the x direction) of the primary flow and μ denoting the viscosity (assumed constant), we have

$$\mu \frac{d^2\bar{u}}{dy^2} + g\rho \sin\beta - K = 0, \quad (6)$$

where g is the gravitational acceleration, and

$$K = \frac{d\bar{p}}{dx_1}. \quad (7)$$

In (7), $\bar{p}$ is the pressure in the primary flow, and $x_1 = xd$, x being measured in units of d.

The $\bar{u}$ for the lower and upper fluids are, respectively, denoted by $\bar{u}_1(y)$ and $\bar{u}_2(y)$. The boundary conditions are

$$\bar{u}_1(-1) = 0, \quad \bar{u}_1(1) = 0 \quad (8)$$

and the interfacial conditions are

$$\bar{u}_1(0) = \bar{u}_2(0), \quad \frac{d\bar{u}_1}{dy} = \frac{d\bar{u}_2}{dy} \text{ at } y = 0. \quad (9)$$

Equation (6) gives two equations, one for $\bar{u}_1$, and one for $\bar{u}_2$. When these are solved with conditions (9) and (10), one obtains

$$\bar{u}_1 = \frac{Kd^2}{2\mu}(1 - y^2) + \frac{1}{6(\lambda+1)}[(\lambda - 1)(3y^2 - 2) - (\lambda+1)y + 2\lambda y^3]V \sin\beta, \quad (10)$$

$$\bar{u}_2 = \frac{Kd^2}{2\mu}(1 - y^2) + \frac{1}{6(\lambda+1)}[(\lambda - 1)(3y^2 - 2) - (\lambda+1)y + 2y^3]V \sin\beta, \quad (11)$$

where

$$V = \alpha g d^2 \Delta T/\nu \quad (\nu = \mu/\rho_0) \quad (12)$$

has the dimension of a velocity, and will be used as the velocity scale.

For our purpose of demonstrating instability resulting from conductivity variation, it is sufficient to take a special K. We shall take

$$Kd^2/\mu = [(\lambda - 1)/(\lambda + 1)]V \sin\beta, \quad (13)$$

because it will give us the simplest forms of $\bar{u}$, and $\bar{u}_2$. Using V as the velocity scale, and adopting (13), we have

$$U_1 = \bar{u}_1/V = [1/6(\lambda+1)] \times [(\lambda - 1) - (\lambda+1)y + 2\lambda y^3]\sin\beta, \quad (14)$$

$$U_2 = \bar{u}_2/V = [1/6(\lambda+1)] \times [(\lambda - 1) - (\lambda+1)y + 2y^3]\sin\beta. \quad (15)$$

III. FORMULATION OF THE STABILITY PROBLEM

Let the dimensionless temperature perturbations be expressed by

$$\Theta_1 = T_1'/\Delta T, \quad \Theta_2 = T_2'/\Delta T, \quad (16)$$

and let $\psi_1(x,y)$ be the stream function for the lower fluid and $\psi_2(x,y)$ that for the upper fluid. Then the velocity perturbations for the two fluid layers are given by

$$u_1' = (\psi_1)_y, \quad v_1' = -(\psi_1)_x, \qquad u_1' = (\psi_2)_y, \quad v_2' = -(\psi_2)_x. \quad (17)$$

We shall assume

$$(\theta_1,\theta_2,\psi_1,\psi_2) = [h_1(y), h_2(y), \theta(y), \chi(y)]\exp[ia(x - ct)], \quad (18)$$

where the scale for the time t is d/V, the scale for the wave number a is d^{-1}, and c is the dimensionless wave velocity:

$$c = c_r + ic_i. \quad (19)$$

The flow is stable or unstable according to whether c_i is positive or negative.

The linearized heat equations are then, upon use of (16)–(18),

$$ia(U_1 - c)h_1 - \frac{i2\lambda a}{\lambda+1}\phi = \frac{1}{\lambda RP}(h_1'' - a^2 h_1), \quad (20)$$

$$ia(U_2 - c)h_2 - \frac{i2a}{\lambda+1}\chi = \frac{1}{RP}(h_2'' - a^2 h_2), \quad (21)$$

where, for simplicity, we have assumed the thermal diffusivities of the fluid to have the ratio λ also (in effect ignoring the variation of specific heat, which can be accounted for without difficulty), and

$$R = Vd/\nu, \quad P = \nu/\kappa_2 \quad (22)$$

are the Reynolds number and the Prandtl number (for the upper fluid), respectively. (The thermal diffusivity of the upper fluid is denoted by κ_2.) The boundary conditions are, assuming that the boundaries are thermally much more conductive than the fluid,

$$h_1(-1) = 0 = h_2(1), \quad (23)$$

and the interfacial conditions are

$$h_1'(0) = \lambda h_2'(0), \quad (24)$$

$$h_2(0) - h_1(0) = [2(\lambda - 1)/(\lambda+1)][\phi(0)/c'], \quad (25)$$

with

$$c' = c - U_1(0).$$

The term on the right-hand side of (25) arises from the difference in slope of $\bar{h}_1$ and $\bar{h}_2$ at $y = 0$, which contributes the term when the interface is displaced from its mean position. The ratio $\phi(0)/c'$ multiplied by the exponential factor exp$[ia(x - ct)]$ is indeed equal to the interfacial displacement, as can be deduced from the kinematic condition at the interface. This term is crucial in the calculation for stability.

If p_i' denotes the pressure perturbation ($i = 1,2$), and if we write

$$p_i'/\rho V^2 = f_i(y)\exp[ia(x-ct)], \quad (26)$$

the linearized Navier–Stokes equations are, for the lower fluid,

$$ia(U_1-c)\phi - iaU_1\phi = -iaf_1 + (1/R)(\phi''' - a^2\phi') - (\sin\beta/R)h_1, \quad (27)$$

$$a^2(c-U_1)\phi = f_1' + (ia/R)(\phi'' - a^2\phi) - (\cos\beta/R)h_1. \quad (28)$$

The last term in (27) arises form the body force term

$$(d/\rho_0 V^2)(\rho_0 g\alpha\,\Delta T\, h_1 \sin\beta),$$

the multiplier $d/\rho_0 V^2$ is to make the entire equation dimensionless [similarly for the last term in (28)]. For the upper fluid, the linearized Navier–Stokes equations are

$$ia(U_2-c)\chi' - iaU_2\chi = -iaf_2 + (1/R)(\chi''' - a^2\chi') - (\sin\beta/R)h_2, \quad (29)$$

$$a^2(c-U_2)\chi = f_2' + (ia/R)(\chi'' - a^2\chi) - (\cos\beta/R)h_2. \quad (30)$$

Eliminating f_1 in (27) and (28), and f_2 in (29) and (30), we obtain the augmented Orr–Sommerfeld equations

$$\phi^{iv} - 2a^2\phi'' + a^4\phi = iaR\,[(U_1-c)(\phi'' - \alpha^2\phi) - U_1''\phi] + h_1'\sin\beta + iah_1\cos\beta, \quad (31)$$

$$\chi^{iv} - 2a^2\phi'' + a^4\chi = iaR\,[(U_2-c)(\chi'' - a^2\chi) - U_2''\chi] + h_2'\sin\beta + iah_2\cos\beta. \quad (32)$$

The boundary conditions are

$$\phi(-1) = 0 = \phi'(-1), \quad \chi(1) = 0 = \chi^1(1), \quad (33)$$

expressing the no-slip condition. The interfacial conditions are

$$\phi(0) = \chi(0), \ \phi'(0) = \chi'(0), \quad (34)$$

expressing the continuity of velocity, and

$$\phi''(0) = \chi''(0),$$

$$(\phi''' - 3a^2\phi') - (\chi''' - 3a^2\chi') = ia^3 S\phi/c' \quad \text{at } y=0, \quad (35)$$

expressing the continuity of shear and normal stresses. In formulating the second condition in (35), one needs to evaluate f_1 and f_2 from (27) and (29), because the normal stress involves the pressure (and another term involving the viscosity). The S in (35) is defined by

$$S = \hat{S}/\rho V^2 d, \quad (36)$$

$\hat{S}$ being the surface tension.

The stability problem is thus governed by four simultaneous differential equations, two of which are of the second order and the other two of the fourth order, and twelve boundary or interfacial conditions. Given the parameters R, P, λ, a, β, and S, one seeks to determine c.

IV. SOLUTION

We consider long waves, and adopt the method of solution given by Yih.[6] First, we expand the unknowns in power series of a:

$$h_1 = H_0 + aH_1 + a^2H_2 + \cdots,$$
$$h_2 = G_0 + aG_1 + a^2G_2 + \cdots,$$
$$\phi = \phi_0 + a\phi_1 + a^2\phi_2 + \cdots,$$
$$\chi = \chi_0 + a\chi_1 + a^2\chi_2 + \cdots,$$
$$c = c_0 + ac_1 + a^2c_2 + \cdots.$$

Substituting these into the governing differential system, and collecting terms of order a only, we obtain

$$H_0'' = 0, \quad G_0'' = 0, \quad (37)$$

with the boundary conditions

$$H_0(-1) = 0 = G_0(1), \quad H_0'(0) = \lambda G_0'(0), \quad (38)$$

and

$$G_0(0) - H_0(0) = [2(\lambda-1)/(\lambda+1)][\phi_0(0)/c_0']. \quad (39)$$

Leaving (39) alone for the moment, one solves (37) and (38) and obtains

$$H_0 = 1 + y, \quad G_0 = (1/\lambda)(-1+y). \quad (40)$$

The equations (31) and (32) yield, upon use of (40),

$$\phi_0''' = \sin\beta, \quad \chi_0''' = \sin\beta/\lambda, \quad (41)$$

for which the boundary conditions are

$$\phi_0(-1) = 0 = \phi_0'(-1), \quad \chi_0(1) = 0 = \chi_0'(1), \quad (42)$$

$$(\phi_0,\phi_0',\phi_0'',\phi_0''') = (\chi_0,\chi_0',\chi_0'',\chi_0''') \quad \text{at } y=0. \quad (43)$$

Solution of (41)–(43) gives

$$\phi_0 = A + By + Cy^2 + Dy^3 + (\sin\beta/24)y^4, \quad (44)$$

$$\chi_0 = A + By + Cy^2 + Dy^3 + (\sin\beta/24\lambda)y^4, \quad (45)$$

in which

$$A = [(\lambda+1)/48\lambda]\sin\beta,$$
$$B = -[(\lambda-1)/96\lambda]\sin\beta, \quad (46)$$
$$C = -[(\lambda+1)/24\lambda]\sin\beta, \quad D = [(\lambda-1)/32\lambda]\sin\beta.$$

With ϕ_0 given by (44), one returns to (39) and obtains

$$c_0' = -[(\lambda-1)/24(\lambda+1)]\sin\beta,$$

or

$$c_0 = [(\lambda-1)/8(\lambda+1)]\sin\beta. \quad (47)$$

We now proceed to the next approximation. Collecting terms of order a in (20) and (21), we have

$$H_1'' = i\lambda RP\{(U_1 - c_0)H_0 - [2\lambda/(\lambda+1)]\phi_0\}, \quad (48)$$

$$G_1'' = i\lambda RP\{(U_2 - c_0)G_0 - [2/(\lambda+1)]\chi_0\}. \quad (49)$$

The conditions (23) and (24) give

$$H_1(-1) = 0 = G_1(-1), \quad H_1'(0) = \lambda G_1'(0), \quad (50)$$

and (25) gives

$$G_1(0) - H_1(0) = \frac{2(\lambda-1)}{\lambda+1}\left(\frac{\phi_1(0)}{c_0'} - \frac{\phi_0(0)c_1}{(c_0')^2}\right). \quad (51)$$

Setting (51) aside for the moment and solving (48)–(50), we have

$$H_1 = \frac{i\lambda RP\sin\beta}{48(\lambda+1)}\left(\frac{15\lambda-39}{60}y - 2y^2 - \frac{5\lambda+11}{6}y^3 - \frac{\lambda+1}{3}y^4 + \frac{13\lambda+3}{20}y^5 + \frac{2\lambda}{5}y^6\right). \quad (52)$$

The term of zeroth power in y is deliberately dropped to keep $h_1(0) = 1$, since the amplitude of the disturbance is immaterial and already $H_0(0) = 1$. The result for G_1 is

$$G_1 = \frac{iRP\sin\beta}{48\lambda(\lambda+1)}\left(\frac{-5\lambda^2+26\lambda-5}{20} + \frac{15\lambda^2-39\lambda}{60}y - 2\lambda y^2 + \frac{11\lambda+5}{6}y^3 - \frac{\lambda+1}{3}y^4 - \frac{3\lambda+13}{20}y^5 + \frac{2}{5}y^6\right). \quad (53)$$

Equations (31) and (32) give

$$\phi_1''' = iR\,[(U_1 - c_0)\phi_0'' - U_1''\phi_0] + H_1'\sin\beta + iH_0\cos\beta, \quad (54)$$

$$\chi_1''' = iR\,[(U_2 - c_0)\chi_0'' - U_2''\chi_0] + G_1'\sin\beta + iG_0\cos\beta. \quad (55)$$

The boundary conditions are

$$\phi_1(-1) = 0 = \phi_1'(-1), \quad \chi_1(1) = 0 = \chi_1'(1), \quad (56)$$

and the four interfacial conditions are obtained from the continuity of ϕ_1 and χ_1 and of their first three derivatives, at $y = 0$. A straightforward solution gives

$$\phi_1 = F_1 + A_1 + B_1 y + C_1 y^2 + D_1 y^3, \quad F_1 = \phi_{11} + \phi_{12} + \phi_{13}, \quad (57)$$

$$\phi_{11} = \frac{iR\sin^2\beta}{6\lambda(\lambda+1)}\left(-\frac{\lambda^2-1}{48}\frac{y^4}{4!} - \frac{23\lambda^2+34\lambda-25}{192}\frac{y^5}{5!} + \frac{\lambda^2-4\lambda+3}{8}\frac{y^6}{6!} - \frac{\lambda(\lambda+1)y^7}{7!} + \frac{60\lambda^2 y^9}{9!}\right), \quad (58)$$

$$\phi_{12} = \frac{i\lambda RP\sin^2\beta}{48(\lambda+1)}\left(\frac{15\lambda-39}{60}\frac{y^4}{4!} - \frac{4y^5}{5!} - \frac{(5\lambda+11)y^6}{6!} - \frac{8(\lambda+1)y^7}{7!} + \frac{6(13\lambda+3)y^8}{8!} + \frac{288\lambda y^9}{9!}\right), \quad (59)$$

$$\phi_{13} = i\cos\beta\left(\frac{y^4}{4!} + \frac{y^5}{5!}\right), \quad (60)$$

and

$$\chi_1 = F_2 + A_1 + B_1 y + C_1 y^2 + D_1 y^3, \quad F_2 = \chi_{11} + \chi_{12} + \chi_{13}, \quad (61)$$

$$\chi_{11} = \frac{iR\sin^2\beta}{6(\lambda+1)}\left(-\frac{\lambda^2-1}{48\lambda}\frac{y^4}{4!} + \frac{25\lambda^2-34\lambda-23}{192\lambda}\frac{y^5}{5!} - \frac{3\lambda^2-4\lambda+1}{8\lambda}\frac{y^6}{6!} - \frac{\lambda+1}{\lambda}\frac{y^7}{7!} + \frac{60y^9}{9!\lambda}\right), \quad (62)$$

$$\chi_{12} = \frac{iRP\sin^2\beta}{48\lambda(\lambda+1)}\left(\frac{15\lambda^2-39\lambda}{60}\frac{y^4}{4!} - \frac{4\lambda y^5}{5!} + \frac{(11\lambda+5)y^6}{6!} - \frac{8(\lambda+1)y^7}{7!} - \frac{6(3\lambda+13)y^8}{8!} + \frac{288y^9}{9!}\right), \quad (63)$$

$$\chi_{13} = -\frac{i\cos\beta}{\lambda}\left(\frac{y^4}{4!} - \frac{y^5}{5!}\right). \quad (64)$$

The coefficients A_1, B_1, C_1, and D_1 are determined from the boundary conditions (56), which demand

$$F_1(-1) + A_1 - B_1 + C_1 - D_1 = 0, \quad F_1'(-1) + B_1 - 2C_1 + 3D_1 = 0,$$
$$F_2(-1) + A_1 + B_1 + C_1 + D_1 = 0, \quad F_2'(1) + B_1 + 2C_1 + 3D_1 = 0. \quad (65)$$

In particular

$$A_1 = -\tfrac{1}{2}[F_1(-1) + F_2(1)] + \tfrac{1}{4}[F_2'(1) - F_1'(-1)]. \quad (66)$$

The value of $\phi_1(0)$ is A_1. When (66) is substituted into (51) together with the quantities determined in the first approximation, we have

$$c_1 = \frac{i(1-\lambda)}{12(\lambda+1)^3}\left[\frac{RP\sin^2\beta}{80640}\left(55\lambda^3 + 1437\lambda^2 - 1239\lambda + 85 + \frac{248\lambda(\lambda^2-1)}{P}\right) + \frac{7(\lambda^2-1)}{20}\cos\beta\right]. \quad (67)$$

V. DISCUSSION

One can proceed further with the systematic procedure of approximation. But (67) is sufficient as a criterion for instability against long waves. Examination of (67) shows that the term containing P in the denominator in the bracket and the term containing $\cos\beta$ are always stabilizing. The term containing $\cos\beta$ arises from gravity normal to the boundaries, and its stabilizing effect is well recognized. The term containing P^{-1} in the bracket of (67) arises from the convective terms in (54) and (55), so that these convective terms are stabilizing. This isolates the longitudinal body-force terms in (54) and (55) as the cause of instability. But this instability would not have a chance to manifest itself without the conductivity discontinuity at the interface, which gives rise to the term on the right-hand side of (25). That term is crucial, for without it the calculation could not be started, and long-wave instability would not exist.

Examination of (67) further reveals:

(a) For vertical boundaries the term containing $\cos\beta$ drops out, and if the Prandtl number P is not extremely small the flow is unstable for λ small. In this case the "lower fluid" is the colder fluid.

(b) For $\lambda < 1$ and $1 - \lambda$ small, the flow is unstable, if P and $\sin\beta$ are not very small.

(c) For $\lambda > 1$, the flow is stable.

(d) There is a range of λ within $1 < \lambda < \infty$, for which the flow is stable.

(e) For given values of P and β, and a given λ less than one, if the multiplier of R in (67) is positive the critical R is obtained by setting the quantity within the brackets in (67) equal to zero.

Observations (b) and (c) show that for small $|\lambda - 1|$ the flow is unstable if the less conductive fluid is on top and stable if it is at the bottom. This rather intriguing point, together with observations (a) and (d), indicates the rather complex effect of conductivity variation on the stability of the flow.

Finally, it may not be entirely irrelevant to mention that convective stability of two superposed horizontal layers of immiscible fluids has been studied by Yuriko Renardy,[7] who discussed the effects of thermal conductivities in her work. But the stratification in thermal conductivities never *causes* any instability in her problem, as it does here. The present work brings to light an entirely new cause of hydrodynamic instability.

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[1] G. I. Taylor and A. D. McEwan, J. Fluid Mech. **22**, 1 (1965).
[2] C. S. Yih, Phys. Fluids **11**, 1447 (1968).
[3] C. S. Yih, J. Fluid Mech. **27**, 337 (1967).
[4] P. G. Saffman and G. I. Taylor, Proc. Soc. London Ser. A **245**, 312 (1958).
[5] C. S. Yih, J. Fluid Mech. **5**, 436 (1959).
[6] C. S. Yih, Phys. Fluids **6**, 321 (1963).
[7] Y. Renardy, Phys. Fluids **29**, 356 (1986).